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CONTINUITY PROPERTIES OF THE INJECTIVITY RADIUS FUNCTION

Paul E. Ehrlich

Let M^n be a smooth manifold and let $R(M)$ be the space of smooth Riemannian metrics for M. Fix a complete metric g_0 in $R(M)$ and an arbitrary point p in M. Much of the Riemannian geometry of (M, g_0) is determined by the configuration of g_0-radial geodesics at p, that is, with the set of all g_0 'half geodesics'

$$c : [0, \infty) \to M$$

with $c(0) = p$ and $g_0(\dot{c}(0), \dot{c}(0)) = 1$. Let $C_{g_0}(p)$ be the g_0-cut locus at p, let $i_{g_0}(p) := \operatorname{dist}_{g_0}(p, C_{g_0}(p))$ be the g_0-injectivity radius of M at p, and let the g_0-injectivity radius of M be

$$i_{g_0}(M) := \inf \{i_{g_0}(p); p \text{ in } M\}.$$

In understanding the global geometry of M, most notably in the proof of the sphere theorem, it has been necessary to find lower bounds on $i_{g_0}(M)$ for positively curved manifolds and to understand the map $p \mapsto i_{g_0}(p)$ from $M \to \mathbb{R}$ for a fixed complete metric $g_0 \in R(M)$. However, no explicit study of the map

$$(g, p) \mapsto i_g(p)$$

from $R(M) \times M \to \mathbb{R}$ has been made.

Our study of this map presented in this paper was motivated by our study of metric deformations of curvature in [4]. We needed to know that for M compact, the convexity radius function on $R(M)$ was C^2 locally minorized. That is, if $g_0 \in R(M)$ was given we can find constants $\delta(g_0) > 0$ and $C(g_0) > 0$ such that if g in $R(M)$ is $\delta(g_0)$ close to g_0 in the C^2 topology on $R(M)$, then any g-metric ball of g-radius $\leqq C(g_0)$ would be g-convex. In order to obtain this local minorization, we used Klingenberg's minorization for $i_{g_0}(M)$ in terms of an upper bound for the sectional curvature of (M, g_0) and the length of the shortest smooth closed non-trivial g_0-geodesic. The first step was to show there exist constants $\delta(g_0) > 0$ and $L(g_0) > 0$ such that $g \in R(M)$ and g C^2 $\delta(g_0)$ close to g_0 implies that the length of the shortest smooth closed g-geodesic is greater than $L(g_0)$. This we did by applying a result of J. Cheeger, [2], minorizing the length of the shortest smooth closed geodesic for families of Riemannian n-manifolds (M^n, g) with diameter less than d, volume greater than V, and sectional curvature greater than H, for fixed constants d, V, and H. The lower bound H on the sectional curvature evidently forced us to use the C^2-topology on $R(M)$ to apply this local minorization. However we will see in this paper that to prove the local minorization of the length of the shortest smooth closed non-trivial geodesic, we need only C^1-closeness in $R(M)$. Also there is no way to prove any of the lower semi-continuity theorems using a result such as Cheeger's theorem. It is necessary to study the behavior of the radial geodesic configuration at a point p in M for all metrics in a C^1 neighborhood of a given metric.

Let M_1 be a non-compact manifold, $g_0 \in R(M_1)$ complete, and let C be a compact subset of M_1. Then

$$F_{C,g_0}(M_1) := \{g \in R(M_1);\ g = g_0 \text{ in } TM_1|_{M_1 - \operatorname{Int}(C)}\}$$

is a family of complete metrics in $R(M_1)$. In order to prove a result in [4], we needed to know that $g \mapsto i_g(M_1)$ was C^2 locally minorized on families of the form $F_{C,g_0}(M_1)$. Since M_1 is non-compact, the result of Cheeger mentioned above does not apply and his proof cannot be modified to apply to $F_{C,g_0}(M)$. Hence the geometry of compact and non-compact manifolds would be different if $g \mapsto i_g(M)$ was not C^2 locally minorized for families of complete metrics $F_{C,g_0}(M)$, M non-compact. But it seemed intuitively clear that $R(M)$ and $F_{C,g_0}(M)$ should not seem different to the injectivity radius functional $g \mapsto i_g(M)$. Once we take the point of view of this paper that the local minorization of the injectivity radius functional on $R(M)$ for M compact should be derived by considering the radial geodesic configuration, the local minorization for families of metrics of the form $F_{C,g_0}(M)$ is immediate. The geometry of $R(M)$ and $F_{C,g_0}(M_1)$ from the point of view of the minorization of the injectivity radius functional is identical.

In Section 1, we review some basic facts from Riemannian geometry that relate $i_g(p)$ to the behavior of the configuration of g-radial geodesics from p. In Section 2 we prove an estimate for systems of first order O.D.E.'s which enables us in Section 3 to study the behavior of the configuration of radial geodesics from p for all metrics in a C^1 δ-ball about a given metric in $R(M)$. In particular, if g is sufficiently close to g_0 in the C^1 topology and $i_{g_0}(p) > R_0$, there is no smooth closed g-geodesic through p of g-length less than R_0. In Section 4 for compact M we uniformize this result to prove the C^1 local minorization of the length of the shortest smooth non-trivial closed geodesic. In section 5 we prove for M compact and p in M fixed that $g \mapsto i_g(p)$ from $R(M) \to \mathbb{R}$ is lower semicontinuous. In Section 6 we use the continuity of $p \mapsto i_{g_0}(p)$

for $g_0 \in R(M)$ fixed and the lower semicontinuity of $g \mapsto i_g(p)$ for p in M fixed to prove the lower semicontinuity of $(g, p) \mapsto i_g(p)$ from $R(M) \times M \to \mathbb{R}$ for M compact. We also discuss briefly the problem in extending these results to the complete non-compact case. The basic problem is just the technical difficulty involved in defining a C^2 topology for $R(M)$ that is independent of the choice of Riemann normal coordinates when M is non-compact. In Section 7 we prove that for M compact the map $g \mapsto i_g(p)$ is upper semicontinuous and hence continuous from $R(M) \to \mathbb{R}$ with the C^2 topology on $R(M)$. We note that for the upper semicontinuity we need the C^2 topology on $R(M)$ to control (B) of Basic Lemma I of Section 1 whereas for the lower semicontinuity the C^1 topology on $R(M)$ suffices to control (B). It is then possible to see that for M compact, the map $(g, p) \mapsto i_g(p)$ from $R(M) \times M \to \mathbb{R}$ is upper semicontinuous and hence continuous. Finally in Section 8 we show that the map $g \mapsto i_g(M)$ from $R(M) \to \mathbb{R}$ is continuous with the C^2 topology on $R(M)$ for M compact.

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Notational Conventions

Fix a smooth n-manifold M, $n \geqq 2$. Let $\pi : TM \to M$ be the tangent bundle of M. Let $R(M)$ be the space of smooth Riemannian metrics for M. Given g in $R(M)$ and a sectionally smooth curve $c : [a, b] \to M$, define the *g-length* of c, written $L_g(c)$ by

$$L_g(c) := \int_a^b (g(\dot{c}(t), \dot{c}(t))^{\frac{1}{2}} \mathrm{d}t.$$

Then let $\mathrm{dist}_g : M \times M \to [0, \infty)$ be the distance function for M defined in the usual way by

$\mathrm{dist}_g(p, q) := \inf \{L_g(c);\ c$ is a sectionally smooth curve from p to $q\}$.

Given $g \in R(M)$ and $R > 0$, let

$$B_{g,R}(p) := \{q \in M;\ \mathrm{dist}_g(p, q) < R\},$$

$$U_{g,R}(p) := \{v \in M_p;\ g(v, v) < R^2\},$$

and

$$S_1(M, g) := \{v \in TM;\ g(v, v) = 1\}$$

which is the g-unit sphere subbundle of TM with fiber at p

$$S_1(M, g)|_p := \{v \in M_p;\ g(v, v) = 1\}.$$

Define the injectivity radius function

$$i : R(M) \times M \to [0, \infty]$$

written

$$(g, p) \mapsto i_g(p)$$

by

$$i_g(p) := \sup \{R > 0;\ \exp_p : U_{g,R}(p) \to B_{g,R}(p) \text{ is a diffeomorphism}\}$$

where $\exp : TM \to M$ is the exponential map determined by g. We call $i_g(p)$ the *g-injectivity radius at p*. Define the *g-injectivity radius of M*, written $i_g(M)$, by

$$i_g(M) := \inf \{i_g(p);\ p \in M\}.$$

Given a chart $(U, x_1, \cdots, x_n)$ with $x = (x_1, \cdots, x_n)$ smooth in $\bar{U}$ and $g \in R(M)$, define the Christoffel symbols

$$q \mapsto \Gamma_{ij}^k(g, x, q)$$

from the functions

$$q \mapsto g(\partial/\partial x_i, \partial/\partial x_j)|_q$$

in the usual way. When a chart (U, x) is fixed as in section 1, we will sometimes write $\Gamma_{ij}^k(g, q)$ for $\Gamma_{ij}^k(g, x, q)$. Let

$$\|\Gamma\|(g, U) := \sup \{|\Gamma_{ij}^k(g, x, q)|;\ q \in \bar{U},\ 1 \leqq i, j, k \leqq n\}$$

and

$$\|\partial\Gamma\|(g, U) := \sup \left\{ \left| \frac{\partial}{\partial x_p} (\Gamma_{ij}^k(g, x, q)) \right| ;\ q \in \bar{U},\ 1 \leqq i, j, k, p \leqq n \right\}.$$

1. A review of local Riemannian geometry of geodesics

References for this section are [1] or [6]. Since this material is standard no explicit further references will be given. Fix a complete metric g_0 for M. For all $v \in S_1(M, g_0)$ let

$$c_v : [0, \infty) \to M$$

be the unique 'half geodesic' determined by g_0 with $c_v(0) = \pi(v)$ and $\dot{c}_v(0) = v$. If $\exp : TM \to M$ is the exponential map determined by g_0 then $c_v(t) = \exp_{\pi(v)} tv$. Define

$$s : S_1(M, g_0) \to [0, \infty]$$

by

$$\begin{aligned} s(v) : &= \sup \{t > 0; \operatorname{dist}_{g_0}(c_v(t), \pi(v)) = t\} \\ &= \sup \{t > 0; c_v : [0, t] \to M \text{ is the unique minimal} \\ &\qquad\qquad g_0\text{-connection between } c_v(0) \text{ and } c_v(t)\}. \end{aligned}$$

If $p := \pi(v)$ and $q := \exp_p s(v)v$, then q is said to be the cut point of p along the radial geodesic $c_v : [0, \infty) \to M$. For instance, if $s(v) = \infty$, then c_v is a ray and if in addition $s(-v) = \infty$, then the geodesic $c : \mathbb{R} \to M$ with $\dot{c}(0) = v$ is a line. Given p in M, let $C(p) := \{s(v)v; v \in S_1(M, g_0)|_p\}$ and define $C_{g_0}(p) := \exp_p(C(p))$, called the g_0-*cut locus* at p. Then the g_0-injectivity radius at p defined above satisfies

$$i_{g_0}(p) = \operatorname{dist}_{g_0}(p, C_{g_0}(p)) = \inf \{s(v); v \in S_1(M, g_0)|_p\}.$$

If M is compact, every radial geodesic from p has a cut point so $i_{g_0}(p)$ is finite for all $p \in M$. Klingenberg showed that $s : S_1(M, g_0) \to [0, \infty]$ is continuous and hence $p \mapsto i_{g_0}(p)$ from $M \to \mathbb{R}$ is continuous.

An important result in the local geometry of geodesics is the characterization of points q in $C_{g_0}(p)$ with $\operatorname{dist}_{g_0}(p, q) = i_{g_0}(p)$ for $i_{g_0}(p) < \infty$ and g_0 complete in terms of the behavior of the g_0-radial geodesic configuration at p.

BASIC LEMMA I: *Either one or both of the following holds.*

(A) *q is a first conjugate point to p along some radial geodesic from p, or*

(B) *there exist $v, w \in S_1(M, g_0)|_p$, $v \neq w$ such that if $t_0 := i_{g_0}(p)$ then $c_v(t_0) = c_w(t_0) = q$ and $\dot{c}_v(t_0) = -\dot{c}_w(t_0)$. Alternately, there is a geodesic loop at p through q.*

Let M be compact. Choose $p_0 \in M$ with

$$(*) \qquad i_{g_0}(p_0) = i_{g_0}(M) := \inf \{i_{g_0}(q); q \in M\}.$$

Choose $q_0 \in C_{g_0}(p)$ with $\mathrm{dist}_{g_0}(p_0, q_0) = i_{g_0}(p_0)$ and assume (B) of Basic Lemma I holds. Then by (*), p_0 must satisfy $i_{g_0}(q_0) = \mathrm{dist}_{g_0}(p_0, q_0)$ and the two loops given by (B) in fact form a smooth closed geodesic. This discussion together with the theory of conjugate points yields a minorization of Klingenberg, namely

BASIC LEMMA II: *Let M be compact, $g_0 \in R(M)$, and let $k(g_0) > 0$ be any upper bound for the g_0-sectional curvatures. Then*

$$i_{g_0}(M) \geqq \min \{\pi/\sqrt{k(g_0)}, \tfrac{1}{2} \cdot \mathit{Length}\,(g_0)\}$$

where

$$\mathit{Length}\,(g_0) = \inf \{L_g(c);\ c \textit{ is a smooth non-trivial closed } g_0\textit{-geodesic}\}.$$

It is then clear from Basic Lemma I that to see that $(g, p) \mapsto i_g(p)$ is continuous, we need to see why first conjugate points and geodesic loops cannot jump inward or outward for metrics close to a given metric. The analysis of the conjugate point behavior with the C^2 topology is fairly standard. After sketching below an argument of Dr. H. Karcher (personal communication) to indicate that (A) perturbs nicely, we will make no further mention of (A) in this paper, treating only (B) below in our proofs. We remark here that while C^2 closeness is clearly needed for the lower semicontinuity of $(g, p) \mapsto i_g(p)$ because of (A), C^1 closeness is all that is needed to prevent the geodesic loop of (B) from jumping inward.

Recall that conjugate points along the radial geodesics at p can be interpreted as singularities of the differential of the exponential map $\exp_p : M_p \to M$. Let $\varepsilon > 0$ be given. Suppose for $v \in S_1(M, g_0)|_p$, there is no conjugate point along $c_v : [0, d] \to M$. It is then standard that for metrics g sufficiently C^2 close to g_0 and tangent vectors w sufficiently close to v in TM, there will be no conjugate points along the g-radial geodesic with initial condition w up to at least time $d - \varepsilon$. In particular, this implies that $(g, p) \mapsto i_g(p)$ cannot fail to be lower semicontinuous because of conjugate point behavior.

To see that the first conjugate point cannot jump outward, we must consider the index form (formula (1), p. 142 of [6]). In the appendix to [5], Karcher shows that the index form for a given metric can be viewed as an operator of the form $I + k$ where k is a compact operator which changes continuously with a continuous perturbation of the curvature tensor. It is then standard that the spectrum of these operators is upper semicontinuous (but *not* necessarily lower semicontinuous) under continuous perturbations. This implies that the first conjugate point cannot 'jump outward' with C^2 perturbations of a given metric.

Define C contained in (M, g) to be *g-convex* iff for all p and q in C, there is precisely one minimal normal geodesic segment in C from p to q. The basic result on the existence of convex neighborhoods (from [6], p. 160) is

BASIC LEMMA III: *Let $B_{g,R}(p)$ satisfy the following two properties.*
(A′) *For all $q \in B_{g,R}(p)$, $\exp_q : U_{g,2R}(p) \to B_{g,2R}(p)$ is a diffeomorphism.*
(B′) *For all $v \in S_1(M, g_0)|_p$, the index form is positive definite for all Jacobi fields J along $c_v : [0, R] \to M$ with $g(J, \dot{c}_v) \equiv 0$ and $J(0) = 0$.*

Then $B_{g,R}(p)$ is g-convex.

By standard comparison theory in Riemannian geometry, it is clear that (B′) is locally minorized with the C^2 topology on $R(M)$. (See [4] for details.) Hence the key step in minorizing the convexity radius functional on $R(M)$ is to minorize the injectivity radius functional. We thus leave to the reader the formulation of the analogues of theorems 4 and 5 of section 4 for the convexity radius functional.

2. An estimate for systems of ordinary differential equations

For completeness, we prove an estimate for first order systems of O.D.E.'s similar to the estimate stated in [2] without proof which is not found in any standard text known to us. For $X = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$, let

$$\|X\|_2 := \Big(\sum_i (x_i)^2\Big)^{\frac{1}{2}}.$$

PROPOSITION 1: *Suppose $X(t) = (x_1(t), \cdots, x_m(t))$ is a solution of*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x_1, \cdots, x_m) = f_i(X, t)$$

for $t \in [0, R]$, $i = 1, \cdots, m$, where the f_i are continuous and satisfy a Lipschitz condition

$$(*) \qquad |f_i(X, t) - f_i(\bar{X}, t)| \leqq L\|X - \bar{X}\|_2$$

for $i = 1, \cdots, m$.
Suppose $Y(t) = (y_1(t), \cdots, y_m(t))$ is a solution of

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = g_i(y_1, \cdots, y_m, t) = g_i(Y, t)$$

for $t \in [0, R]$, $i = 1, \cdots, m$ where $X(0) = Y(0)$, the g_i are continuous, and

$$|f_i(X, t) - g_i(X, t)| \leqq \delta$$

for all (X, t) and $i = 1, \cdots, m$.

Then for $t \in [0, R]$ *and* $i = 1, \cdots, m$

$$|x_i(t) - y_i(t)| \leqq m\delta t e^{mLt}.$$

PROOF: Recall the following elementary facts. First, if $X(t) \neq 0$ then

$$(\|X(t)\|_2)' \leqq \|X'(t)\|_2$$

and second, if $F(X, t) := (f_1(X, t), \cdots, f_m(X, t))$ then from (*)

$$\|F(X, t)\|_2 \leqq mL\|X\|_2.$$

It is enough to show that $\|X(t) - Y(t)\|_2 e^{-mLt} \leqq m\delta t$. Now

$$\begin{aligned} |x_i'(t) - y_i'(t)| &= |f_i(X, t) - g_i(Y, t)| \\ &\leqq |f_i(X, t) - f_i(Y, t)| + |f_i(Y, t) - g_i(Y, t)| \leqq L\|X(t) - Y(t)\|_2 + \delta. \end{aligned}$$

Hence

$$\|X'(t) - Y'(t)\|_2 \leqq mL\|X(t) - Y(t)\|_2 + m\delta.$$

Thus if $X(t) \neq Y(t)$, we have

$$\begin{aligned} (\|X(t) - Y(t)\|_2 e^{-mLt})' &\leqq (\|X'(t) - Y'(t)\|_2 - mL\|X(t) - Y(t)\|_2)e^{-mLt} \\ &\leqq m\delta e^{-mLt} \leqq m\delta \end{aligned}$$

since $t \geqq 0$, $L \geqq 0$. If $X(t) = Y(t)$, the desired estimate clearly holds. Thus suppose $X(t) \neq Y(t)$. Choose $t_0 \in [0, t)$ such that $X(t_0) = Y(t_0)$ and $X(s) \neq Y(s)$ for all $s \in (t_0, t]$. Then

$$\begin{aligned} \|X(t) - Y(t)\|_2 e^{-mLt} &= \int_{s=t_0}^{t} (\|X(s) - Y(s)\|_2 e^{-mLs})' \mathrm{d}s \\ &\leqq m\delta \int_{s=t_0}^{t} \mathrm{d}s = m\delta(t - t_0) \leqq m\delta t. \qquad \text{Q.E.D.} \end{aligned}$$

3. The local behavior of the configuration of radial geodesics

In this section, let M^n be a fixed smooth manifold not necessarily compact with $n \geqq 2$. Fix a complete $g_0 \in R(M)$ and $p \in M$. Choose R_0 with $0 < R_0 < i_{g_0}(p)$. Fix for this section a g_0-orthonormal basis $\{e_1, \cdots, e_n\} \subset M_p$. Let $x = (x_1, \cdots, x_n)$ be g_0-Riemann normal coordinates centered at p for $\overline{B_{g_0, R_0}(p)}$ defined by $\{e_1, \cdots, e_n\}$. Explicitly if $q \in \overline{B_{g_0, R_0}(p)}$ we may choose a unique $t > 0$ and $v \in S_1(M, g_0)|_p$ such that $q = \exp_p tv$ where $\exp_p : M_p \to M$ is the exponential map determined by g_0. If $v = \sum_i a_i e_i$, then $x_i(q) = ta_i$. Define for $i = 1, \cdots, n$

$$x_{i+n} : \overline{B_{g_0, R_0}(p)} - \{p\} \to \mathbb{R}$$

as follows. Given $q \in \overline{B_{g_0, R_0}(p)} - \{p\}$, choose $t, v = \sum_i a_i e_i$ with $t > 0$ uniquely so that $q = \exp_p tv$ and put $x_{i+n}(q) := a_i$. Thus $|x_{i+n}| \leqq 1$ on $\overline{B_{g_0, R_0}(p)} - \{p\}$ for $i = 1, \cdots, n$.

Let $g_1 \in R(M)$ be complete. We want to study the difference in the configuration of radial geodesics at p determined by g_0 and g_1. We fix the following notation. For $v \in S_1(M, g_0)|_p$ let

$$c_{0,v} : [0, R_0] \to M$$

be the unique g_0-geodesic with $c_{0,v}(0) = p$ and $\dot{c}_{0,v}(0) = v$. Let

$$c_{1,v} : [0, R_0] \to M$$

be the unique g_1-geodesic with $c_{1,v}(0) = p$ and $\dot{c}_{1,v}(0) = v$. Fix $v = \sum_i a_i e_i$ in $S_1(M, g_0)|_p$. Identifying as usual x_i and $x_i \circ c_{0,v}$, the differential equation for $c_{0,v}$ written in terms of the g_0-Riemann normal coordinates is

$$(*) \quad \begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = x_{i+n} \\[2ex] \dfrac{\mathrm{d}x_{i+n}}{\mathrm{d}t} = -\displaystyle\sum_{j,k=1}^{n} \Gamma^i_{jk}(g_0, x_1, \cdots, x_n) x_{j+n} x_{k+n} \end{cases}$$

with initial conditions $x_i(0) = 0$ and $x_{i+n}(0) = a_i$ for $i = 1, \cdots, n$. Let $X = (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n})$ and $Y = (y_1, \cdots, y_n, y_{n+1}, \cdots, y_{2n})$ be arbitrary points in the domain of definition of (*) which is of course the diagonal of $\overline{B_{g_0, R_0}(p)} \times \overline{B_{g_0, R_0}(p)}$ modulo the identification of M_p and $\mathbb{R}^n$ given by the g_0-frame $\{e_1, \cdots, e_n\}$ in M_p. Then we may write (*) as

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(X, t)$$

for $i = 1, \cdots, 2n$ where

$$f_i(X, t) = x_{i+n}$$

and

$$f_{i+n}(X, t) = -\sum_{j,k=1}^{n} \Gamma^i_{jk}(g_0, x_1, \cdots, x_n) x_{j+n} x_{k+n}$$

for $i = 1, \cdots, n$. Let

$$\|X - Y\|_1 := \sum_{i=1}^{2n} |x_i - y_i|.$$

Then for $i = 1, \cdots, n$

$$|f_i(X, t) - f_i(Y, t)| = |x_{i+n} - y_{i+n} \leqq \|X - Y\|_1$$

and

$$\begin{aligned}
&|f_{i+n}(X, t) - f_{i+n}(Y, t)| \\
&\quad \leqq \sum_{j,k} |\Gamma^i_{jk}(g_0, x_1, \cdots, x_n)x_{j+n}x_{k+n} - \Gamma^i_{jk}(g_0, y_1, \cdots, y_n)y_{j+n}y_{k+n}| \\
&\quad \leqq \sum_{j,k} |\Gamma^i_{jk}(g_0, x_1, \cdots, x_n)x_{j+n}x_{k+n} - \Gamma^i_{jk}(g_0, y_1, \cdots, y_n)x_{j+n}x_{k+n}| \\
&\quad + \sum_{j,k} |\Gamma^i_{jk}(g_0, y_1, \cdots, y_n)| \cdot |x_{j+n}x_{k+n} - y_{j+n}y_{k+n}| \\
&\quad \leqq n^2\|\partial\Gamma\|(g_0, B_{g_0, R_0}(p))\|X - Y\|_1 + 2n^2\|\Gamma\|(g_0, B_{g_0, R_0}(p))\|X - Y\|_1 \\
&\quad \leqq (n^3\|\partial\Gamma\|(g_0, B_{g_0, R_0}(p)) + 2n^3\|\Gamma\|(g_0, B_{g_0, R_0}(p)))\|X - Y\|_2
\end{aligned}$$

since $|x_{j+n}|$, $|y_{j+n}| \leqq 1$ by construction. Thus if we put

$$L_0 := \min\{1, n^3\|\partial\Gamma\|(g_0, B_{g_0, R_0}(p)) + 2n^3\|\Gamma\|(g_0, B_{g_0, R_0}(p))\}$$

we have

$$|f_i(X, t) - f_i(Y, t)| \leqq L_0\|X - Y\|_2$$

for all X, Y in the domain of definition of (*) and $i = 1, \cdots, 2n$.

In terms of the Riemann normal coordinates $x = (x_1, \cdots, x_n)$ for $B_{g_0, R_0}(p)$, the system of differential equations for the g_1-radial geodesic $c_{1,v}$ has the form

$$\frac{dy_i}{dt} = g_i(y_1, \cdots, y_n, t) = g_i(Y, t)$$

for $i = 1, \cdots, 2n$ with the initial condition $Y(0) = X(0)$ where

$$g_i(Y, t) = y_{i+n}$$

and

$$g_{i+n}(Y, t) = -\sum_{j,k=1}^{2n} \Gamma^i_{jk}(g_1, y_1, \cdots, y_n)y_{j+n}y_{k+n}$$

for $i = 1, \cdots, n$.

Suppose

$$|\Gamma^i_{jk}(g_1, x, q) - \Gamma^i_{jk}(g_0, x, q)| \leqq \delta$$

for all $q \in \overline{B_{g_0, R_0}(p)}$ and for all $i, j, k = 1, \cdots, n$. Then

$$|g_i(X, t) - f_i(X, t)| = |x_{i+n} - x_{i+n}| = 0$$

and

$$|g_{i+n}(X, t) - f_{i+n}(X, t)|$$

$$\leqq \sum_{j,k=1}^{n} |\Gamma^i_{jk}(g_1, x_1, \cdots, x_n) - \Gamma^i_{jk}(g_0, x_1, \cdots, x_n)||x_{j+n}x_{k+n}| \leqq n^2\delta.$$

Hence Proposition 1 of Section 1 with $m := 2n$ implies

PROPOSITION 2: *Let $v \in S_1(M, g_0)|_p$ and define L_0 as above. Suppose*

$$|\Gamma^i_{jk}(g_1, x, q) - \Gamma^i_{jk}(g_0, x, q)| \leqq \delta$$

for all $q \in \overline{B_{g_0, R_0}(p)}$ and all $i, j, k = 1, \cdots, n$. Then for all $t \in [0, R_0]$ and $i = 1, \cdots, n$,

$$|(x_i \circ c_{0,v})(t) - (x_i \circ c_{1,v})(t)| \leqq 2n^3\delta t e^{2nL_0 t}$$

and

$$|(x_i \circ c_{0,v})'(t) - (x_i \circ c_{1,v})'(t)| \leqq 2n^3\delta t e^{2nL_0 t}.$$

Since $e^{2nL_0 t} \leqq e^{2nL_0 R_0}$ for $t \in [0, R_0]$ this estimate quantitatively measures the fact that for g_1 $\delta - C^1$ close to g_0 (as in Proposition 2) and δ small, the g_1-radial geodesic configuration at p is close to the g_0-radial configuration at p near p. We can interpret the first estimate geometrically as follows: for all $v \in S_1(M, g_0)|_p$ and for all metrics g_1 in a C^1 δ-ball about g_0, the g_1 radial geodesic $c_{1,v}$ lies in a δ 'cone neighborhood' of the g_0 radial geodesic $c_{0,v}$.

In order to make this more precise, define a distance function

$$\text{dist} : \overline{B_{g_0, R_0}(p)} \times \overline{B_{g_0, R_0}(p)} \to [0, \infty)$$

by

$$\text{dist}(q, r) := \Big(\sum_{i=1}^{n} (x_i(q) - x_i(r))^2\Big)^{\frac{1}{2}}$$

where $x = (x_1, \cdots, x_n)$ are the fixed Riemann normal coordinates for $\overline{B_{g_0, R_0}(p)}$. It is elementary that $[\overline{B_{g_0, R_0}(p)}, \text{dist}]$ is a metric space.

We say $g_1 \in R(M)$ is *δ-close to g_0 on $B_{g_0, R_0}(p)$ in the C^1 topology*, written $|g_1 - g_0|_{C^1, x, B_{g_0, R_0}(p)} < \delta$, with coordinates $x = (x_1, \cdots, x_n)$ iff

$$(1-\delta)^2 g_0(v, v) \leqq g_1(v, v) \leqq (1+\delta)^2 g_0(v, v)$$

for all $v \in TM|_{\overline{B_{g_0, R_0}(p)}}$ and the Christoffel symbols $\Gamma^i_{jk}(g_0, x, \cdot)$ and $\Gamma^i_{jk}(g_1, x, \cdot)$ satisfy the condition of Proposition 2.

From the transformation formulas for the Christoffel symbols under a change of coordinates, it is clear that although for $\delta > 0$ fixed the inequality $|g_1 - g_0|_{C^1, x, \bar{B}_{g_0, R_0}(p)} \leqq \delta$ is not invariant under coordinate change, the notion of a sequence of metrics $\{g_n\} \subset R(M)$ with

$$|g_n - g_0|_{C^1, x, B_{g_0, R_0}(p)} \to 0$$

is well defined. This will become quite explicit in the construction of Section 4.

By Proposition 2, for $g_1 \in R(M)$ that is δ-close to g_0 on $B_{g_0, R_0}(p)$ in the C^1 topology, $t \in [0, R_0]$, and $v \in S_1(M, g_0)|_p$ we have

$$\operatorname{dist}(c_{0,v}(t), c_{1,v}(t)) \leqq 2n^4 \delta t e^{2nL_0 t}.$$

Let $g_1 \in R(M)$ be δ-close to g_0 on $B_{g_0, R_0}(p)$ in the C^1 topology. Suppose there is a smooth closed non-trivial g_1-geodesic c through p contained in $B_{g_0, R_0}(p)$. We may choose a smallest $t_0 > 0$ and $v \in S_1(M, g_0)|_p$ such that $s := c_{1,v}(t_0) = c_{1,v}(-t_0) = c_{1,-v}(t_0)$ (that is, c is the union of the two g_1 radial geodesics $c_{1,v} : [0, t_0] \to M$ and $c_{1,-v} : [0, t_0] \to M$). Assume c is sufficiently short that $t_0 \leqq R_0$. Let $q := c_{0,v}(t_0)$ and $r := c_{0,-v}(t_0)$. Then since $R_0 < i_{g_0}(p)$, by basic Riemannian geometry, $\operatorname{dist}(q, r) = \operatorname{dist}(p, q) + \operatorname{dist}(p, r) = 2t_0$. But the triangle inequality for the metric 'dist' implies

$$\begin{aligned}
\operatorname{dist}(q, r) &= \operatorname{dist}(c_{0,v}(t_0), c_{0,-v}(t_0)) \\
&\leqq \operatorname{dist}(c_{0,v}(t_0), c_{1,v}(t_0)) + \operatorname{dist}(c_{1,v}(t_0), c_{1,-v}(t_0)) \\
&\qquad\qquad + \operatorname{dist}(c_{1,-v}(t_0), c_{0,-v}(t_0)) \\
&\leqq 2n^4 \delta t_0 e^{2nL_0 t_0} + \operatorname{dist}(s, s) + 2n^4 \delta t_0 e^{2nL_0 t_0} \leqq 4n^4 \delta t_0 e^{2nL_0 R_0}.
\end{aligned}$$

Thus we have the inequality $1 \leqq 2\delta n^4 e^{2nL_0 R_0}$ which is false as $\delta \to 0$. Hence

THEOREM 3: *Given $g_0 \in R(M)$ complete, $R_0 < i_{g_0}(p)$, and a fixed g_0-Riemann normal coordinate system $x = (x_1, \cdots, x_n)$ on $B_{g_0, R_0}(p)$ as above. There exists a constant $\delta(g_0, x, p) \in (0, 1)$ such that $g \in R(M)$ and*

$$|g - g_0|_{C^1, x, B_{g_0, R_0}(p)} < \delta(g_0, x, p)$$

implies there is no smooth closed non-trivial g-geodesic c through p of g-length $\leqq R_0$. Hence there is no sequence of metrics $\{g_n\} \subset R(M)$ with $|g_n - g_0|_{C, x, B_{g_0, R_0}(p)} \to 0$ and such that g_n has a smooth closed non-trivial geodesic c_n through p with $L_{g_0}(c_n) \to 0$.

PROOF: If $(1-\delta)^2 g_0 \leqq g \leqq (1+\delta)^2 g_0$, then for any sectionally smooth curve c, we have

$$(1-\delta) L_{g_0}(c) \leqq L_g(c) \leqq (1+\delta) L_{g_0}(c).$$

Thus if c is a smooth closed g-geodesic through p of g-length $\leqq R_0$, it follows that the 't_0' of the paragraph preceding Theorem 3 satisfies

$$t_0 \leqq (R_0/2)(1+\delta(g_0, x, p))$$

so

$$t_0 < R_0 \quad \text{if} \quad \delta(g_0, x, p) < 1. \qquad \text{Q.E.D.}$$

From the proof, it is clear that making the upper bound on $\delta(g_0, x, p)$ smaller, the upper bound on $L_g(c)$ can be improved.

4. The local minorization of the length of the shortest smooth closed non-trivial geodesic on $R(M)$ for M compact

Fix $g_0 \in R(M)$ and choose $R_0 > 0$ with $4R_0 < i_{g_0}(M)$. Since M is compact, fix $p_1, \cdots, p_{m_0}$ in M so that

$$M = \bigcup_{i=1}^{m_0} B_{g_0, R_0/2}(p_i).$$

For each i, fix a g_0-orthonormal basis $\{e_{i,1}, \cdots, e_{i,n}\}$ for M_{p_i} thus defining once and for all g_0-Riemann normal coordinates $x^i = (x^i_1, \cdots, x^i_n)$ on $B_{g_0, 4R_0}(p_i)$ for $i = 1, \ldots, m_0$.
For each p_i, parallel translate (using g_0) the basis $\{e_{i,1}, \cdots, e_{i,n}\}$ for M_{p_i} along radial geodesics getting a g_0-orthonormal frame $\{E_{i,1}, \cdots, E_{i,n}\}$ on $B_{g_0, 4R_0}(p_i)$ for each $i = 1, \cdots, m_0$. Hence for each point q in M we obtain at most m_0 orthonormal bases for M_q by this procedure which we will call *distinguished bases* for M_q.

DEFINITION: $g \in R(M)$ is *δ-close to g_0 in the C^1 topology* iff

$$(1-\delta)^2 g_0 \leqq g \leqq (1+\delta)^2 g_0$$

and for each $i = 1, \cdots, m_0$, using the fixed Riemann normal coordinates $x^i = (x^i_1, \cdots, x^i_n)$ on $\overline{B_{g_0, 2R_0}(p_i)}$,

$$|\Gamma^k_{ij}(g, x^i, q) - \Gamma^k_{ij}(g_0, x^i, q)| \leqq \delta$$

for all $q \in \overline{B_{g_0, 2R_0}(p_i)}$.

We define smooth maps

$$G^i_{kl}, F^i_{kl} : \overline{B_{g_0, R_0/2}(p_i)} \times \overline{B_{g_0, R_0/2}(p_i)} \to \mathbb{R}$$

for each i and $1 \leqq k, l \leqq n$ as follows. Fix i for the moment and write $(x_1, \cdots, x_n)$ for $(x^i_1, \cdots, x^i_n)$. Given (q, s) in $\overline{B_{g_0, R_0/2}(p_i)} \times \overline{B_{g_0, 2R_0}(p_i)}$ parallel translate $\{e_{i,1}, \cdots, e_{i,n}\}$ from M_{p_i} along the unique unit speed g_0-radial geodesic from p_i to q getting a distinguished basis $\{E_{1|_q}, \cdots, E_{n|_q}\}$ for M_q. Let $y = (y_1, \cdots, y_n)$ be g_0-Riemann normal coordinates defined on $B_{g_0, 2R_0}(q)$ by $\{E_{1|_q}, \cdots, E_{n|_q}\}$. Then for $s \in \overline{B_{g_0, 2R_0}(p_i)}$

$$\mathrm{dist}_{g_0}(s, q) \leqq \mathrm{dist}_{g_0}(s, p_i) + \mathrm{dist}_{g_0}(p_i, q) \leqq \frac{5R_0}{2} < 4R_0$$

so $(y_1, \cdots, y_n)$ are smooth at g. Thus we may define

$$F^i_{kl}(q, s) := \left.\frac{\partial x^i_k}{\partial y_l}\right|_s \quad \text{and} \quad G^i_{kl}(q, s) := \left.\frac{\partial y_k}{\partial x^i_l}\right|_s .$$

Since $\overline{B_{g_0, R_0/2}(p_i)} \times \overline{B_{g_0, 2R_0}(p_i)}$ is compact, we may choose a constant C_i such that

$$|F^i_{kl}(q, s)|, |G^i_{kl}(q, s)| \leqq C_i$$

for all k and l and all $(q, s) \in \overline{B_{g_0,R_0/2}(p_i)} \times \overline{B_{g_0, 2R_0}(p_i)}$. In particular, for a fixed $q \in B_{g_0, R_0/2}(p_i)$, the maps from $\overline{B^{g}{}_{0, R_0}(q)} \to \mathbb{R}$ given by $s \mapsto |F^i_{kl}(q, s)|$ and $s \to |G^i_{kl}(q, s)|$ are bounded by C_i. Doing this construction for all p_i, $i = 1, \cdots, m_0$, we get constants C_i for $i = 1, \cdots, m_0$. Put $C := \max\{C_1, \cdots, C_{m_0}\}$.

Recall that if (U, x) and (V, y) are two local coordinate systems with $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ then for $s \in U \cap V$,

$$\Gamma^k_{ij}(g_0, x, s) = \sum_{l=1}^{n} \frac{\partial^2 y_l}{\partial x_i \partial x_j} \frac{\partial x_k}{\partial y_l} + \sum_{p,q,r=1}^{n} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} \frac{\partial x_k}{\partial y_r} \Gamma^r_{pq}(g_0, y, s).$$

Thus if $q \in B_{g_0, R_0/2}(p_i)$ and $y = (y_1, \cdots, y_n)$ are the g_0-Riemann normal coordinates on $B_{g_0, R_0}(q)$ obtained by g_0-parallel translation of $\{e_{i,1}, \cdots, e_{i,n}\}$ to M_q, we have for $g \in R(M)$ δ-close to g_0 in the C^1 topology

$$|\Gamma^k_{ij}(g, y, s) - \Gamma^k_{ij}(g_0, y, s)| \leqq n^3 C^3 \delta$$

Hence for any $q \in M$, if $g \in R(M)$ is δ-close to g_0 in the C^1 topology, then using g_0-Riemann normal coordinates $x = (x_1, \cdots, x_n)$ on $B_{g_0, R_0}(q)$ defined by any distinguished basis for M_q we have

$$|\Gamma^k_{ij}(g_0, x, s) - \Gamma^k_{ij}(g, x, s)| \leqq n^3 C^3 \delta.$$

for all $s \in B_{g_0, R_0}(q)$. In particular, the C^1 neighborhoods of g_0 defined as above are independent of the choice of distinguished bases.

To apply the results of section 3, it only remains to see that for any q in M and any Riemann normal coordinate system defined by any distinguished basis for M_q that we have a uniform Lipschitz condition on the O.D.E.'s for the g_0 radial geodesics. Fix i with $1 \leqq i \leqq m_0$. Let $S := S_1(M, g_0)_{\overline{B_{g_0, R_0/2}(p_i)}} \times [0, R_0]$. We define maps $S^i_{jkl} : S \to \mathbb{R}$ and $T^i_{pqrs} : S \to \mathbb{R}$ as follows. Let $(v, t) \in S$. Let $\bar{q} := \pi(v)$ and let $y = (y_1, \cdots, y_n)$

be g_0 Riemann normal coordinates defined on $B_{g_0, 2R_0}(\bar{q})$ by g_0 parallel translation of $\{e_{i,1}, \cdots, e_{i,n}\} \hookrightarrow M_{p_i}$ along the g_0-unit speed radial geodesic from p_i to $\bar{q}$ in $B_{g_0, R_0}(p_i)$. Then for all $1 \leqq j, k, l, p, q, r, s \leqq n$ put

$$S^i_{jkl}(v, t) := \Gamma^l_{jk}(g_0, y, c_{0,v}(t))$$

and

$$T^i_{pqrs}(v, t) := \frac{\partial}{\partial y_p}(\Gamma^s_{qr}(g_0, y, c_{0,v}(t)))$$

where $c_{0,v}$ is the unique g_0 geodesic with $c_{0,v}(0) = q$ and $c'_{0,v}(0) = v$ as before. From basic Riemannian geometry these maps are continuous. Thus we can choose constants $\Gamma_i, \partial\Gamma_i > 0$ such that $|S^i_{jkl}(v, t)| \leqq \Gamma_i$ and $|T^i_{pqrs}(v, t)| \leqq \partial\Gamma_i$ for all $(v, t) \in S$ and $1 \leqq j, k, l, p, q, r, s \leqq n$. Doing this construction for all $i = 1, \cdots, m_0$, put

$$\|\Gamma(g_0)\| := \max\{\Gamma_1, \cdots, \Gamma_{m_0}\}$$

and

$$\|\partial\Gamma(g_0)\| := \max\{\partial\Gamma_1, \cdots, \partial\Gamma_{m_0}\}.$$

Put

$$\mathrm{Lip}\,(g_0) := \max\{1, n^3\|\partial\Gamma(g_0)\| + 2n^3\|\Gamma(g_0)\|\}.$$

For any $p \in M$, using g_0-Riemann normal coordinates $x = (x_1, \cdots, x_n)$ on $\overline{B_{g_0, R_0}(p)}$ from any distinguished basis at p to define the system of differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x_1, \cdots, x_{2n}, t) = f_i(X, t)$$

for any radial geodesic $c_{0,v}$ at p as in section 2, we have

$$|f_i(X, t) - f_i(Y, t)| \leqq \mathrm{Lip}\,(g_0) \cdot \|X - Y\|_2$$

on $x(\overline{B_{g_0, R_0}(p)})$. Now determine $\delta(g_0, x^i, p_i)$ for $B_{g_0, 2R_0}(p_i)$ for $i = 1, \cdots, m_0$ as in Theorem 3, Section 3. Let

$$\delta(g_0) = \max\left\{\frac{\delta(g_0, x^1, p_1)}{n^3 C^3}, \cdots, \frac{\delta(g_0, x^{m_0}, p_{m_0})}{n^3 C^3}\right\}.$$

In particular,

$$(**) \qquad 2\delta(g_0) n^4 e^{2n\,\mathrm{Lip}\,(g_0) R_0} < 1.$$

THEOREM 1: *Let M be compact, $g_0 \in R(M)$ and $4R_0 < i_{g_0}(M)$. With the C^1 neighborhoods of g_0 defined as above, there exists a constant $\delta(g_0)$*

with $0 < \delta(g_0) < 1$ *such that* $g_1 \in R(M)$ *and*

$$|g_1 - g_0|_{C^1} < \delta(g_0)$$

implies the g_1*-length of the shortest smooth non-trivial* g_1 *geodesic is greater than or equal to* R_0.

PROOF: Suppose there exists a smooth closed non-trivial geodesic c with $L_{g_1}(c) < R_0$ for $g_1 \in R(M)$ with $|g_1 - g_0|_{C^1} < \delta(g_0)$. As in Theorem 3, Section 3, $L_{g_0}(c) < 2R_0$. Let $p := c(0)$ and choose v and $t_0 < R_0$ as in the proof of Theorem 3, Section 3. Put $s := c_{1,v}(t_0) = c_{1,-v}(t_0)$. Choosing any distinguished basis at p, put g_0-Riemann normal coordinates on $\overline{B_{g_0, R_0}(p)}$. Write the O.D.E.'s for $c_{0,v}$ and $c_{0,-v}$ in the form

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_{2n}, t) = f_i(X, t)$$

and for $c_{1,v}$ and $c_{1,-v}$ as

$$\frac{dy_i}{dt} = g_i(y_1, \cdots, y_{2n}, t) = g_i(Y, t)$$

as before. By the construction of $\delta(g_0)$, we have

$$|f_i(X, t) - f_i(\bar{X}, t)| \leqq \operatorname{Lip}(g_0) \cdot \|X - \bar{X}\|_2$$

and

$$|f_i(X, t) - g_i(X, t)| \leqq \delta(g_0).$$

Hence as in the proof of theorem 3, we obtain

$$2\delta(g_0) n^4 e^{2n \operatorname{Lip}(g_0) R_0} \geqq 1$$

contradicting (**). Q.E.D.

REMARK: L. Berard Bergery has shown us an example of a perturbation of a surface of revolution shaped like a bowling pin to show that the map from $R(M) \to \mathbb{R}$ given by $g \mapsto \text{Length}(g)$ (defined as in Basic Lemma II, Section 1) is not upper or lower semicontinuous with the C^1 topology on $R(M)$. Thus the local minorization of $g \mapsto \text{Length}(g)$ given by Theorem 1 is the best possible result in general.

The following result is clear but seems not to be present in the standard literature so we state it. A proof can be found in [4].

LEMMA: *Let* M *be non-compact. Let* g_0 *be a complete metric for* M. *If* g *is any other metric agreeing with* g_0 *off a compact subset of* M, *then* g *is complete.*

Hence for M non-compact, given a complete metric g_0 for M and a compact subset C contained in M, we may define a family of complete metrics $F_{C,g_0}(M)$ by

$$F_{C,g_0}(M) := \{g \in R(M); g = g_0 \text{ on } TM|_{M-\mathrm{Int}(C)}\}$$

The following result is a consequence of the lower semicontinuity of $g \mapsto i_g(p)$ proven in Section 5 together with the type of uniformity argument given in proving Theorem 4 of this section from Theorem 3 of Section 3. Let $i_g(C) := \inf\{i_g(q); q \in C\}$.

THEOREM 5: *Let $g_1 \in F_{C,g_0}(M)$. Then there exists constants $\delta(g_1, C) > 0$ and $I(g_1, C) > 0$ such that $g_2 \in F_{C,g_0}(M)$ and*

$$|g_1 - g_2|_{C^2} < \delta(g_1, C) \textit{ implies } i_{g_2}(C) > I(g_1, C).$$

5. The lower semicontinuity of $g \mapsto i_g(p)$ from $R(M) \to \mathbb{R}$ for M compact

In this section we show using a modified version of Proposition 3.2.

THEOREM 1: *Let M be compact and fix any point $p \in M$. With the C^2 topology on $R(M)$, the map $R(M) \to [0, \infty]$ given by*

$$g \mapsto i_g(p)$$

is lower semicontinuous.

Fix $g_0 \in R(M)$. By compactness, $i_{g_0}(p) < \infty$. Given $\varepsilon > 0$ we must show that there exists a $\delta > 0$ such that $g \in R(M)$ and

$$|g - g_0|_{C^2} < \delta$$

implies $i_g(p) \geqq i_{g_0}(p) - \varepsilon$. Put $R_0 := i_{g_0}(p) - \varepsilon/100$ and $R_1 := i_{g_0}(p) - \varepsilon$. By Basic Lemma II of section 1 and our subsequent remarks, it suffices to show that given $\varepsilon > 0$, there exists $\delta > 0$ with the following property. For $g_1 \in R(M)$ with

$$|g_1 - g_0|_{C^1} < \delta,$$

there does not exist a $t_0 \in (0, R_1)$ and two g_1-radial geodesics from p

$$c_{1,v} : [0, t_0] \to M$$

and

$$c_{1,w} : [0, t_0] \to M$$

with $v \neq w$, v, $w \in S_1(M, g_1)|_p$, $s := c_{1,v}(t_0) = c_{1,w}(t_0)$, and $\dot{c}_{1,v}(t_0) = -\dot{c}_{1,w}(t_0)$.

Given $\delta > 0$, we will suppose we have such a $t_0 \in (0, R_0)$ and two such g_1-radial geodesics forming a loop and see what inequality this forces δ to satisfy.

Choose $\delta_0 > 0$ such that $\delta \in [0, \delta_0]$ and $g_1 \in R(M)$ with

$$|g_1 - g_0|_{C^2} \leqq \delta$$

implies that $R_1\sqrt{g_0(v, v)} \leqq R_0$ and $g_0(v, v) \leqq 2$ for all $v \in S_1(M, g_1)|_p$. (This is possible because $R_1 = R_0 + (99/100) \cdot \varepsilon$ and $g_0(v, v) \leqq (1+\delta)g(v, v)$ from C^0 closeness for all $v \in TM$.)

Let $\{e_1, \cdots, e_n\} \subset M_p$ be a g_0-orthonormal basis. Let $x = (x_1, \cdots, x_n)$ be g_0-Riemann normal coordinates defined on $\overline{B_{g_0, R_0}(p)}$ by $\{e_1, \cdots, e_n\}$. Define $x_{i+n} : \overline{B_{g_0, R_0}(p)} \to \mathbb{R}$ as in Section 3. Then if $g_1(v, v) = 1$ and $|g_0 - g_1|_{C^0} < \delta_0$ we have

$$|(x_{i+n} \circ c_{0,v})(t)| = |(x_i \circ c_{0,v})'(t)| \leqq 2$$

for all $i = 1, \cdots, n$. Thus substituting $|x_{i+n}| \leqq 2$ for $|x_{i+n}| \leqq 1$ in the proof of Proposition 3.2 we obtain

PROPOSITION 3.2′: *Let*

$$L_0 := \min\{1, 4n^3\|\partial\Gamma\|(g_0, B_{g_0, R_0}(p)) + 4n^3\|\Gamma\|(g_0, B_{g_0, R_0}(p))\}.$$

Suppose $g_1 \in R(M)$ satisfies $|g_1 - g_0|_{C^0} < \delta_0$ and

$$|\Gamma_{ij}^k(g_1, x, s) - \Gamma_{ij}^k(g_0, x, s)| < \delta$$

for all $s \in \overline{B_{g_0, R_0}(p)}$. Then for any $v \in S_1(M, g)|_p$ and $t \in [0, R_1]$ we have

$$|(x_i \circ c_{0,v})(t) - (x_i \circ c_{1,v})(t)| \leqq 2n^3\delta t e^{2nL_0 t}$$

and

$$|(x_i \circ c_{0,v})'(t) - (x_i \circ c_{1,v})'(t)| \leqq 2n^3\delta t e^{2nL_0 t}.$$

Let $c : [0, A] \to B_{g_0, R_0}(p)$ be a smooth curve. Then for $t_0 \in (0, A)$,

$$\dot{c}(t_0) = \sum_{i=1}^n \dot{c}(t_0)(x_i) \left.\frac{\partial}{\partial x_i}\right|_{c(t_0)}$$
$$= \sum_{i=1}^n (x_i \circ c)'(t_0) \left.\frac{\partial}{\partial x_i}\right|_{c(t_0)}.$$

Thus $\dot{c}_{1,v}(t_0) = -\dot{c}_{1,w}(t_0)$ iff $(x_i \circ c_{1,v})'(t_0) = -(x_i \circ c_{1,w})'(t_0)$ for all $i = 1, \cdots, n$.

Write $v = \sum_{i=1}^n a_i e_i$ and $w = \sum_{i=1}^n b_i e_i$ in terms of the fixed g_0-orthonormal frame. Let $\theta_0(v, w)$ be the g_0-angle between v and w.

We have three cases.

Case I: $\cos\theta_0(v, w) \geqq 1-1/100$.
We have

$$\begin{aligned}|a_i+b_i| &= |(x_i \circ c_{0,v})'(t_0)+(x_i \circ c_{0,w})'(t_0)| \\ &= |(x_i \circ c_{0,v})'(t_0)-(x_i \circ c_{1,v})'(t_0)+(x_i \circ c_{1,v})'(t_0)+(x_i \circ c_{0,w})'(t_0)| \\ &\leqq |(x_i \circ c_{0,v})'(t_0)-(x_i \circ c_{1,v})'(t_0)|+|(x_i \circ c_{0,w})'(t_0)-(x_i \circ c_{1,w})'(t_0)| \\ &\leqq 8n^3\delta t_0 e^{2nL_0t_0}.\end{aligned}$$

Thus

$$\sum_{i=1}^{n}(a_i+b_i)^2 \leqq 64n^7\delta^2R_0^2e^{4nL_0R_0}.$$

But

$$\begin{aligned}\sum_{i=1}^{n}(a_i+b_i)^2 &= \sum_{i=1}^{n}a_i^2+2\sum_{i=1}^{n}a_ib_i+\sum_{i=1}^{n}b_i^2 \\ &= g_0(v, v)+2g_0(v, w)+g_0(w, w) \\ &\geqq 2(1-\delta)+2\cos(\theta_0(v, w))\cdot\sqrt{g_0(v, v)g_0(w, w)} \\ &\geqq 2(1-\delta)+2(1-1/100)(1-\delta) \geqq 2-1/100.\end{aligned}$$

Thus

$$2-1/100 \leqq \sum_{i=1}^{n}(a_i+b_i)^2 \leqq 64n^7\delta^2R_0^2e^{4nL_0R_0}$$

Choose $\delta_1 \in (0, \frac{1}{2}]$ such that for any $\delta \in [0, \delta_1]$

$$2-1/100 > 64n^7\delta^2R_0^2e^{4nL_0R_0}.$$

Case II: $0 \leqq \cos\theta_0(v, w) \leqq 1-1/100$.
Define $\mathrm{dist}: \overline{B_{g_0,R_0}(p)} \times \overline{B_{g_0,R_0}(p)} \to \mathbb{R}$ by

$$\mathrm{dist}\,(q, r) := \Big(\sum_{i=1}^{n}(x_i(q)-x_i(r))^2\Big)^{\frac{1}{2}}$$

as before. We have

$$\mathrm{dist}\,(q, r) \leqq \mathrm{dist}\,(q, s)+\mathrm{dist}\,(s, r) \leqq 4n^3\delta t_0 e^{2nL_0t_0}$$

and

$$\begin{aligned}\mathrm{dist}\,(q, r) &:= \Big(\sum_{i=1}^{n}(t_0a_i-t_0b_i)^2\Big)^{\frac{1}{2}} \\ &= t_0(g_0(v, v)-2\cos(\theta_0(v, w))\sqrt{g_0(v, v)g_0(w, w)}+g_0(w, w))^{\frac{1}{2}} \\ &\geqq t_0(2(1-\delta)-2(1-1/100)(1+\delta))^{\frac{1}{2}} \geqq t_0\sqrt{2}(1/100-(199/100)\delta)^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\sqrt{2}(1/100-(199/100)\delta)^{\frac{1}{2}} \leqq \operatorname{dist}(q, r)/t_0 \leqq 4n^3\delta e^{2nL_0R_0}.$$

Choose $\delta_2 > 0$ such that for all $\delta \in [0, \delta_2]$

$$\sqrt{2}(1/100-(199/100)\delta)^{\frac{1}{2}} > 4n^3\delta e^{2nL_0R_0}.$$

Case III: $-1 \leqq \cos\theta_0(v, w) \leqq 0$.
The idea is the same as in Case II but the arithmetic is different. Again, $\operatorname{dist}(q, r) \leqq 4n^3\delta t_0 e^{2nL_0t_0}$. But

$$\operatorname{dist}(q, r) = t_0(g_0(v, v)-2\cos(\theta_0(v, w)\sqrt{g_0(v, v)g_0(w, w)}+g_0(w, w))^{\frac{1}{2}}$$
$$\geqq t_0(g_0(v, v)+g_0(w, w))^{\frac{1}{2}}.$$

Thus

$$\sqrt{2}\sqrt{1-\delta} \leqq 4n^3\delta e^{2nL_0R_0}.$$

Choose $\delta_3 > 0$ such that $\delta \in [0, \delta_3]$ implies

$$\sqrt{2}\sqrt{1-\delta} > 4n^3\delta e^{2nL_0R_0}.$$

Let $\delta(g_0, x, p, \varepsilon) := \min\{\delta_0, \delta_1, \delta_2, \delta_3\}$. We have shown

PROPOSITION 2: *Let* $\varepsilon > 0$ *be given. There exists a constant* $\delta(g_0, x, p, \varepsilon) > 0$ *such that* $g \in R(M)$ *and*

$$|g-g_0|_{C^1, x, B_{g_0, i_{g_0}(p)-\varepsilon/100}(p)} < \delta(g_0, x, p, \varepsilon)$$

implies there do not exist minimal g-normal radial g-geodesics c_1, c_2: $[0, t_0] \to M$ *with* $c_1(0) = c_2(0) = p$, $t_0 < i_{g_0}(p)$,

$$c_1(t_0) = c_2(t_0) \in B_{g_0, i_{g_0}(p)-\varepsilon}(p),$$

and $\dot{c}_1(t_0) = -\dot{c}_2(t_0)$.

With Proposition 2 and the compactness of M insuring that the C^2 topology on $R(M)$ is well defined, the proof of Theorem 1 is now clear.

REMARK: The added difficulty in proving Theorem 1 of this section over Theorem 4 of section 4 is that the following situation may occur. Fix p in M and $\varepsilon > 0$. Let $B = B_{g_0, i_{g_0}(p)-\varepsilon}(p)$. Suppose $g \in R(M)$ and the closest points on the cut locus $C_g(p)$ to p lie in B. Let q be such a point and suppose there is a loop at p through q with initial vectors v and w as in (B) of Basic Lemma I of Section 1. Let $\theta_0(g)$ be the g_0-angle between v and w. The method of proof of Theorem 4, Section 4, fails precisely when there exist $\{g_n\}_{n=1}^{\infty} \subset R(M)$ with $g_n \to g_0$ in the C^1 topology but $\theta_0(g_n) \to 0$. H. Karcher noticed that in the C^2 topology on $R(M)$, Toponogoff's triangle comparison theorem ([6], p. 183) implies no such sequence exists. However, to apply this result, a lower bound on the sectional curvatures of the metrics g_n is needed which we do not have with the C^1 topology on $R(M)$.

Now let M be non-compact. Suppose for $p \in M$ and $g_0 \in R(M)$ complete we have $i_{g_0}(p) < \infty$. Fix a g_0-orthonormal frame at p to define Riemann normal coordinates x on $B_{g_0, i_{g_0}(p)}(p)$. Since $K := \overline{B_{g_0, i_{g_0}(p)}(p)}$ is compact, we can define C^2 closeness of $g \in R(M)$ to g_0 on K independent of a choice of Riemann normal coordinates. Hence, the proof of Proposition 2 carries through and we have lower semicontinuity at $(g_0, p) \in R(M) \times M$ in the sense that given $\varepsilon > 0$, there exists a $\delta > 0$ such that $g \in R(M)$ and $|g - g_0|_{C^2, K} < \delta$ implies $i_g(p) \geqq i_{g_0}(p) - \varepsilon$.

If M is non-compact and $i_{g_0}(p) = \infty$, then M is diffeomorphic to $\mathbb{R}^n$. In this case, we cannot necessarily define a C^2 neighborhood of g_0 independent of the choice of the g_0-orthonormal basis at p used to define Riemann normal coordinates. However, the following analogue of Proposition 2 holds. Fix $N > 0$ and a g_0-orthonormal basis for M_p thus defining Riemann normal coordinates x on any ball $B_{g_0, R}(p)$ for any $R > 0$. Then given N, let $R_1 := N$ and $R_0 := 2N$. Then the same proof (using Lipschitz estimates on $B_{g_0, R_0}(p)$) shows that there exists a constant $\delta(g_0, x, p, N) > 0$ such that $g \in R(M)$ and

$$|g - g_1|_{C^1, x, B_{g_0, 2N}(p)} < \delta(g_0, x, p, N)$$

implies that no g-geodesic loop through p lies in $B_{g_0, N}(p)$. Hence given $N > 0$, there exists a constant $\bar{\delta}(g_0, x, p, N) > 0$ such that

$$|g - g_0|_{C^2, x, B_{g_0, 2N}(p)} < \bar{\delta}(g_0, x, p, N)$$

implies $i_g(p) \geqq N$.

6. The lower semicontinuity of $(g, p) \mapsto i_g(p)$ from $R(M) \times M \to \mathbb{R}$ for M compact

We prove

THEOREM 1: *For M compact, $(g, p) \mapsto i_g(p)$ from $R(M) \times M \to \mathbb{R}$ is lower semicontinuous with the C^2 topology on $R(M)$.*

PROOF: Fix $(g_0, p_0) \in R(M) \times M$. By compactness, $i_{g_0}(p_0)$ is finite. Let $\varepsilon > 0$ be given. Fix a g_0-orthonormal basis $\{e_1, \cdots, e_n\} \hookrightarrow M_{p_0}$.

Step 1: By continuity of $p \mapsto i_{g_0}(p)$ from $M \to \mathbb{R}$, choose $R_0 > 0$ with $2R_0 < i_{g_0}(p_0)$ such that $\mathrm{dist}_{g_0}(p_0, q) \leqq R_0$ implies $|i_{g_0}(p_0) - i_{g_0}(q)| \leqq \varepsilon/2$.

Then we will show there exists $\delta > 0$ such that $q \in \overline{B_{g_0, R_0}(p_0)}$ and $|g - g_0|_{C^2} < \delta$ implies $i_g(q) \geqq i_{g_0}(q) - \varepsilon/2$. This completes the proof for then

$$i_g(q) \geqq i_{g_0}(q) - \varepsilon/2 = (i_{g_0}(q) - i_{g_0}(p_0)) + i_{g_0}(p) - \varepsilon/2 \geqq i_{g_0}(p_0) - \varepsilon.$$

Step 2: Let

$$S_\varepsilon := \{(q, s) \in \overline{B_{g_0, R_0}(p_0)} \times M;\ \mathrm{dist}_{g_0}(q, s) \leqq i_{g_0}(p_0) - \varepsilon/100\}$$
$$= \bigcup_{q \in \overline{B_{g_0, R_0}(p_0)}} \{q\} \times \overline{B_{g_0, i_{g_0}(q) - \varepsilon/100}(q)}.$$

Since $(q, s) \mapsto \mathrm{dist}_{g_0}(q, s) - i_{g_0}(q) - \varepsilon/100$ is continuous from $M \times M \to \mathbb{R}$, S_ε is closed in $M \times M$ and hence compact.

Step 3: Parallel translate with the g_0 metric the g_0-orthonormal basis $\{e_1, \cdots, e_n\}$ for M_{p_0} along the g_0-unit speed radial geodesics from p_0 getting a g_0-orthonormal frame field $\{E_1, \cdots, E_n\}$ for $\overline{B_{g_0, R_0}(p)}$. For each $q \in \overline{B_{g_0, R_0}(p)}$ we define g_0-Riemann normal coordinates $x(q)$ in $\overline{B_{g_0, i_{g_0}(q) - \varepsilon/100}(q)}$ from the g_0-orthonormal basis $\{E_1|_q, \cdots, E_n|_q\}$. Fix closed balls $B_1, \cdots, B_m$ covering M to define the C^2 topology on $R(M)$. Given any $\delta > 0$, there exists a $\tilde{\delta} > 0$ such that $g_1 \in R(M)$ and $|g_1 - g_0|_{C^1} < \tilde{\delta}$ implies for all $q \in \overline{B_{g_0, R_0}(p_0)}$ that

$$|\Gamma_{ij}^k(g_0, x(q), s) - \Gamma_{ij}^k(g_1, x(q), s)| < \delta$$

for all $s \in \overline{B_{g_0, i_{g_0}(q) - \varepsilon/100}(q)}$ and $1 \leqq i, j, k \leqq n$.

Step 4: Define continuous maps $F_{ij}^k : S_\varepsilon \to \mathbb{R}$ and $G_{ijk}^l : S_\varepsilon \to \mathbb{R}$ for $1 \leqq i, j, k, l \leqq n$ by

$$F_{ij}^k(q, s) := \Gamma_{ij}^k(g_0, x(q), s)$$

and

$$G_{ijk}^l(q, s) := \frac{\partial}{\partial x_l(q)} (\Gamma_{ij}^k(g_0, x(q), s))$$

where $x(q) = (x_1(q), \cdots, x_n(q))$ are the g_0-Riemann normal coordinates on $\overline{B_{g_0, i_{g_0}(q) - \varepsilon/100}(q)}$. Choose by compactness a constant $B < \infty$ with $|F_{ij}^k|$, $|G_{ijk}^l| \leqq B$ on S_ε. (Note that these maps can be defined on a slightly larger open set containing S_ε since $\varepsilon > 0$ and $2R_0 < i_{g_0}(p_0)$.)

Step 5: Using

$$|x_{i+n}(q)| \leqq 2,$$

calculate a Lipschitz constant $\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)$ using Step 4 such that for all $q \in \overline{B_{g_0, R_0}(p_0)}$, on $\overline{B_{g_0, i_{g_0}(q)-\varepsilon/100}(q)}$ the g_0-O.D.E. system for the g_0 radial geodesics written as in Section 2 in terms of the g_0-Riemann normal coordinates $x(q)$ satisfies

$$|f_i(X, t) - f_i(\bar{X}, t)| \leqq \mathrm{Lip}(g_0, p_0, R_0, \varepsilon)\|X - \bar{X}\|_2.$$

Let $k_0 := \max\{i_{g_0}(q);\, q \in \overline{B_{g_0, R_0}(p_0)}\} > 0$ and $k_1 := k_0 - \varepsilon/100$.

Then as in Section, if

$$|\Gamma^k_{ij}(g_1, x(q), \cdot) - \Gamma^k_{ij}(g_0, x(q), \cdot)| \leqq \delta$$

on $B_{g_0, i_{g_0}(q)-\varepsilon/100}(q)$ and some pair of g_1-radial geodesics from q meets at an angle of 180 degrees in $\overline{B_{g_0, i_{g_0}(q)-\varepsilon}(q)}$ we have the three estimates

(i) $2 - 1/100 \leqq 64n^7\delta^2 k_1^2\, e^{4n\,\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)k_1}$

(ii) $\sqrt{2}\,(1/100 - (199/100)\delta)^{\frac{1}{2}} \leqq 4n^3\delta e^{2n\,\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)k_1}$

(iii) $\sqrt{2}\sqrt{1-\delta} \leqq 4n^3\delta e^{2n\,\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)k_1}$.

Step 6: Choose $\delta_0 > 0$ such that for all $q \in \overline{B_{g_0, R_0}(p_0)}$, $|g - g_0|_{C^0} \leqq \delta_0$ on $B_{g_0, R_0}(p_0)$ implies $(i_{g_0}(q) - \varepsilon/100)\sqrt{g_0(v, v)} \leqq i_{g_0}(q) - \varepsilon$ and $g_0(v, v) \leqq 2$ for all $v \in S_1(M, g)|_q$. (This is possible by the continuity of $p \mapsto i_{g_0}(p)$.) Make δ_0 smaller if necessary so that $0 \leqq \delta \leqq \delta_0$ implies

(i) $2 - 1/100 > 64n^7\delta^2 k_1^2\, e^{4n\,\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)k_1}$

(ii) $\sqrt{2}(1/100 - (199/100)\delta)^{\frac{1}{2}} > 4n^3\delta e^{4n\,\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)k_1}$

(iii) $\sqrt{2}\sqrt{1-\delta} > 4n^3\delta e^{2n\,\mathrm{Lip}(g_0, p_0, R_0, \varepsilon)k_1}$.

Step 7: By step 3, choose $\delta > 0$ such that $|g - g_0|_{C^1} < \delta$ implies $|\Gamma^k_{ij}(g_0, x(q), \cdot) - \Gamma^k_{ij}(g, x(q), \cdot)| < \delta_0$ on $\overline{B_{g_0, i_{g_0}(q)-\varepsilon/100}(q)}$ for all $q \in \overline{B_{g_0, R_0}(p_0)}$. The proof of Theorem 1 is now clear. Q.E.D.

Suppose M is non-compact. Let g_0 be a complete metric for M. Suppose $p_0 \in M$ and $i_{g_0}(p_0) < \infty$. Then C^2 neighborhoods of g_0 restricted to $\overline{B_{g_0, i_{g_0}(p_0)}(p_0)}$ are well defined by the compactness of this set. From the proof above, it is clear that given $\varepsilon > 0$, we can find a $\delta > 0$ such that $g \in R(M)$,

$$|g - g_0|_{C\ ,B_{g_0, i_{g_0}(p_0)}(p_0)} < \delta$$

and $\mathrm{dist}_{g_0}(p \cdot q) < \delta$ implies that $i_g(q) \geqq i_{g_0}(p_0) - \varepsilon$. However, if $i_{g_0}(p_0) = \infty$, difficulties similar to those mentioned at the end of Section 5 occur.

7. The upper semicontinuity of $g \mapsto i_g(p)$ from $R(M) \to \mathbb{R}$ for M compact

Fix $p \in M$ and let M be compact.

THEOREM 1: *With the C^2 topology on $R(M)$, the map $g \mapsto i_g(p)$ from $R(M) \to \mathbb{R}$ is upper semicontinuous and hence continuous.*

REMARK: Upper semicontinuity is more delicate than lower semicontinuity in that to control the 'closing up' of the radial geodesics to form a loop (alternative (B) of Basic Lemma I, Section 1) in our proof of Theorem 1 we need the C^2 topology whereas the C^1 topology on $R(M)$ sufficed for alternative (B) in the proof of Theorem 5.1.

PROOF: Fix $g_0 \in R(M)$. If $g \mapsto i_g(p)$ is not upper semicontinuous at g_0, then there exists an $\varepsilon > 0$ and $\{g_m\}_{m=1}^{\infty} \subset R(M)$ with $|g_m - g_0|_{C^2} < 1/m$ and $i_{g_m}(p) > i_{g_0}(p) + \varepsilon$. As a matter of notation, for $z \in S_1(M, g_0)|_p$ let

$$c_{m,z} : [0, \infty) \to M$$

be the g_m-radial geodesic from p with $\dot{c}_{m,z}(0) = z$.

For $g \in R(M)$ let $\operatorname{diam}(M, g, p) = \sup\{\operatorname{dist}_{g_0}(p, q); q \in M\}$. Suppose first that $i_{g_0}(p) = \operatorname{diam}(M, g_0, p)$. Recall that

$$|g - g_0|_{C^0} < \delta$$

implies that $\sqrt{1-\delta}\,\operatorname{dist}_{g_0} \leqq \operatorname{dist}_g \leqq \sqrt{1+\delta}\,\operatorname{dist}_{g_0}$ (see [4], section 2). Thus $|g - g_0|_{C^2} < \delta$ implies that

$$i_g(p) \leqq \operatorname{diam}(M, g, p) \leqq \sqrt{1+\delta}\,\operatorname{diam}(M, g_0, p) \leqq i_{g_0}(p) \cdot \sqrt{1+\delta}$$

It is then clear that $g_m \to g_0$ and $i_{g_m}(p) \geqq i_{g_0}(p) + \varepsilon$ is impossible.

Now we may suppose $i_{g_0}(p) < \operatorname{diam}(M, g_0, p)$ so choosing a new $\varepsilon > 0$ if necessary we may as well assume $i_{g_0}(p) < \operatorname{diam}(M, g_0, p) - \varepsilon$.

Choose $q \in C_{g_0}(p)$ with $\operatorname{dist}_{g_0}(p, q) = i_{g_0}(p)$. It is clear from our remarks following Basic Lemma I of Section 1 that $i_{g_m}(p) > i_{g_0}(p) + \varepsilon$ and $g_m \to g_0$ in the C^2 topology on $R(M)$ implies that q cannot be a conjugate point to p. Thus alternative (B) of Basic Lemma I must hold. That is, there exist distinct $v, w \in S_1(M, g_0)|_p$ such that putting $t_0 := i_{g_0}(p)$ we have g_0-radial geodesics

$$c_{0,v}, c_{0,w} : [0, t_0] \to \overline{B_{g_0, i_{g_0}(p)}(p)}$$

with $s := c_{0,v}(t_0) = c_{0,w}(t_0)$ and $\dot{c}_{0,v}(t_0) = -\dot{c}_{0,w}(t_0)$. We will show that this is impossible hence deriving the required contradiction and showing that $g \mapsto i_g(p)$ is upper semicontinuous at g_0. The idea is first to fix a metric g_{m_0} and thus minorize $|x_i \circ c_{m,z}|$ for all $m \geqq m_0$ and all $z \in S_1(M, g_0)|_p$

and second to use this minorization to find a uniform Lipschitz constant for all g_m with $m \geqq m_0$ so as to be able to apply the proof of the lower semicontinuity of the map $g \mapsto i_g(p)$ to the sequence $\{g_m\}$.

We may choose $m_0 > 0$ with the following properties. First just using C^0 closeness of metrics we may suppose that for all $w \in S_1(M, g_0)|_p$ and $m \geqq m_0$ that

$$c_{m,w}([0, t_0]) \hookrightarrow \overline{B_{g_{m_0}, i_{g_0}(p)+\varepsilon}(p)} \cap \overline{B_{g_m, i_{g_0}(p)+\varepsilon}(p)}.$$

Second we may suppose that $|g_0 - g_m|_{C^2} < 1/100$ and $|g_m - g_{m_0}|_{C^2} < 1/100$ for all $m \geqq m_0$.

Let $B := \overline{B_{g_{m_0}, i_{g_0}(p)+\varepsilon}(p)}$. Fixing a g_{m_0}-orthonormal basis at p, define fixed g_{m_0}-Riemann normal coordinates $x = (x_1, \cdots, x_n)$ that are smooth on an open set containing B. We will use these coordinates to make all our estimates.

We may assume $|g_m - g_{m_0}|_{C^2, x, B} < 1/100$ and $|g_m - g_0|_{C^2, x, B} < 1/100$ for all $m \geqq m_0$. Explicitly, for all $m \geqq m_0$ and all $1 \leqq i, j, k, p \leqq n$ we may assume

$$|\Gamma^k_{ij}(g_m, x, \cdot) - \Gamma^k_{ij}(g_0, x, \cdot)| < 1/m \qquad \text{on } B,$$

$$\left| \frac{\partial}{\partial x_p} (\Gamma^k_{ij}(g_m, x, \cdot) - \Gamma^k_{ij}(g_0, x, \cdot))\right| < 1/m \qquad \text{on } B,$$

$$|\Gamma^k_{ij}(g_0, x, \cdot) - \Gamma^k_{ij}(g_{m_0}, x, \cdot)| < 1/100 \qquad \text{on } B,$$

and

$$\left| \frac{\partial}{\partial x_p} (\Gamma^k_{ij}(g_m, x, \cdot) - \Gamma^k_{ij}(g_{m_0}, x, \cdot)) \right| < 1/100 \qquad \text{on } B, \text{ etc.}$$

Let $L(g_{m_0})$ be the appropriate Lipschitz constant calculated on B for the system of g_{m_0}-radial geodesics (with $|x_i| \leqq 2$). Then by Proposition 3.2, for all $m \geqq m_0$

$$|(x_i \circ c_{m,v})(t) - (x_i \circ c_{m_0})(t)| \leqq \frac{n^3}{50} t e^{2nL(g_{m_0})t}$$

and

$$|(x_i \circ c_{m,v})'(t) - (x_i \circ c_{m_0,v})'(t)| \leqq \frac{n^3}{50} t e^{2nL(g_{m_0})t}$$

for all $v \in S_1(M, g_0)|_p$. Write for $m \geqq m_0$

$$|(x_i \circ c_{m,v})(t)| \leqq |(x_i \circ c_{m_0,v})(t)| + |(x_i \circ c_{m,v})(t) - (x_i \circ c_{m_0,v})(t)|$$

and

$$|\Gamma_{ij}^k(g_m, x, \cdot)| \leqq |\Gamma_{ij}^k(g_{m_0}, x, \cdot)| + |\Gamma_{ij}^k(g_m, x, \cdot) - \Gamma_{ij}^k(g_{m_0}, x, \cdot)|,$$

etc. Clearly we can find a constant $\operatorname{Lip}(g_{m_0})$ such that for any $m \geqq m_0$ the system of O.D.E.'s for the g_m-radial geodesics $c_{m,v}$ written on B as in Section 3 in terms of the fixed g_{m_0}-Riemann normal coordinates $x = (x_1, \cdots, x_n)$ has the Lipschitz constant $\operatorname{Lip}(g_{m_0})$.

Set $R_0 := i_{g_0}(p) + \varepsilon$. Let $\theta_0(v, w)$ be the g_0-angle between the g_0-unit vectors v and w which are the initial directions of the g_0-loop through p contained in B assumed to exist above. By the arguments of Section 5 applied to g_m and g_0 we derive the inequalities

(i) if $\cos\theta_0(v, w) \geqq 1 - 1/100$, then for all $m \geqq m_0$

$$2 - 1/100 \leqq \frac{64}{m^2} n^7 e^{4n \operatorname{Lip}(g_{m_0}) R_0}$$

(ii) if $0 \leqq \cos\theta_0(v, w) \leqq 1 - 1/100$, then for all $m \geqq m_0$

$$\sqrt{2}(1/100 - 199/100\, m)^{\frac{1}{2}} \leqq \frac{4n^3}{m} e^{2n \operatorname{Lip}(g_{m_0}) R_0}$$

(iii) if $\cos\theta_0(v, w) \leqq 0$, then for all $m \geqq m_0$

$$\sqrt{2}\sqrt{1 - 1/m} \leqq \frac{4n^3}{m} e^{2n \operatorname{Lip}(g_{m_0}) R_0}.$$

Evidently these inequalities fail to hold as $m \to \infty$ so that the g_0-geodesics $c_{0,v}$ and $c_{0,w}$ cannot meet at s to form a loop giving the required contradiction. Q.E.D.

We now consider the map $(g, p) \mapsto i_g(p)$ from $R(M) \times M \to \mathbb{R}$. We claim this map is also upper semicontinuous. Fix $(g_0, p_0) \in R(M) \times M$. If the map is not upper semicontinuous at (g_0, p_0), then there exists a sequence $\{g_m\}_{m=1}^{\infty} \subset R(M)$ and $\{p_m\} \subset M$ with $g_m \to g_0$ in the C^2 topology on $R(M)$, $p_m \to p_0$ on M, and $i_{g_m}(p_m) \geqq i_{g_0}(p_0) + \varepsilon$ for some $\varepsilon > 0$ and all m. Choose $R_0 > 0$ such that $q \in B_{g_0, R_0}(p_0)$ implies

$$|i_{g_0}(q) - i_{g_0}(p_0)| \leqq \varepsilon/100.$$

Then there exists $m_0 > 0$ such that $m \geqq m_0$ implies

$$\left.\begin{aligned} & i_{g_0}(p_m) \leqq i_{g_0}(p_0) + \varepsilon/100 \\ \text{and} \quad & i_{g_m}(p_m) \geqq i_{g_0}(p_0) + \varepsilon. \end{aligned}\right\} \qquad (*)$$

Modulo uniformizing the estimates used in proving $g \mapsto i_g(p)$ is upper semicontinuous, it is clear that essentially the same argument given for the upper semicontinuity of $g \mapsto i_g(p)$ yields a contradiction in equations (*) thus proving the upper semicontinuity of $(g, p) \mapsto i_g(p)$ at the point (g_0, p_0). But in light of the proofs of Theorem 4.4 and 6.1, taking R_0 sufficiently small, the uniformity follows just as before.

THEOREM 2: *Let M be compact. Let $R(M)$ be given the C^2 topology defined as in Section 4. Then in the product topology on $R(M) \times M$, the map $(g, p) \mapsto i_g(p)$ from $R(M) \times M \to \mathbb{R}$ is continuous.*

8. The continuity of $g \mapsto i_g(M)$ from $R(M) \to \mathbb{R}$ for M compact

We prove

THEOREM: *Let M be compact. Then the map $g \mapsto i_g(M)$ is continuous with the C^2 topology on $R(M)$.*

Step 1: The upper semicontinuity of $g \mapsto i_g(M)$.

Fix g_0 in $R(M)$. If the map is not upper semicontinuous at g_0, then there exists $\varepsilon > 0$ and $\{g_n\}_{n=1} \subset R(M)$ with $i_{g_n}(M) \geqq i_{g_0}(M) + \varepsilon$ and $g_n \to g_0$ in the C^2 topology on $R(M)$. Choose p_0 with $i_{g_0}(p_0) = i_{g_0}(M)$. Then

$$i_{g_n}(p_0) \geqq i_{g_n}(M) \geqq i_{g_0}(p_0) + \varepsilon$$

which is impossible by the upper semicontinuity of $g \mapsto i_g(p_0)$.

Step 2: The lower semicontinuity of $g \mapsto i_g(M)$.

Fix g_0 in $R(M)$. Suppose $g \mapsto i_g(M)$ is not lower semicontinuous at g_0. Then there exists $\varepsilon > 0$ and $\{g_n\}_{n=1}^{\infty} \subset R(M)$ with $|g_0 - g_n|_{C^2} < 1/n$ and $i_{g_n}(M) \leqq i_{g_0}(M) - \varepsilon$. Since M is compact, choose p_n with $i_{g_n}(p_n) = i_{g_n}(M)$ for all n. By compactness, $\{p_n\}$ has a convergent subsequence which we will relabel as $\{p_n\}$ with $p_n \to p_0$. We have

$$i_{g_n}(p_n) = i_{g_n}(M) \leqq i_{g_0}(M) - \varepsilon \leqq i_{g_0}(p_0) - \varepsilon.$$

By the continuity of $p \mapsto i_{g_0}(p)$, choose $\delta > 0$ such that $\mathrm{dist}_{g_0}(p, p_0) < \delta$ implies

$$|i_{g_0}(p) - i_{g_0}(p_0)| < \varepsilon/2.$$

Choose n_0 so that $n \geqq n_0$ implies $\mathrm{dist}_{g_0}(p_n, p_0) < \delta$. Then $n \geqq n_0$ implies

$$\begin{aligned} i_{g_n}(p_n) &\leqq i_{g_0}(p_0) - \varepsilon = (i_{g_0}(p_0) - i_{g_0}(p_n)) + (i_{g_0}(p_n) - \varepsilon) \\ &\leqq \varepsilon/2 + i_{g_0}(p_n) - \varepsilon = i_{g_0}(p_n) - \varepsilon/2. \end{aligned}$$

Thus we have $\{g_n\}_{n \geqq n_0}$ contained in $R(M)$ and $\{p_n\}_{n \geqq n_0}$ contained in $B_{g_0, \delta}(p_0)$ with $i_{g_n}(p_n) \leqq i_{g_0}(p_n) - \varepsilon/2$ and $g_n \to g_0$ in the C^2 topology on $R(M)$. But this is impossible by the proof of the lower semicontinuity of $(g, p) \mapsto i_g(p)$. Q.E.D.

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