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ALGEBRAIC SYSTEMS OF LINEARLY EQUIVALENT DIVISOR-LIKE SUBSCHEMES

Allen B. Altman and Steven L. Kleiman [1]

We develop a theory of algebraic systems of linearly equivalent divisor-like subschemes, which extends Grothendieck's theory (FGA 232 § 4) of the fiber functor of the canonical map from the scheme of divisors to the Picard scheme. Grothendieck used the theory in constructing the Picard scheme (FGA 232 § 5), and any light shed is valuable in its own right. We apply the extended theory and give a proof, in the geometric spirit of Grothendieck's work, that every closed subscheme of finite type of the scheme of divisors is complete when the ambient scheme is a geometrically normal, complete variety. This completeness theorem is equivalent to the analogous one for the Picard scheme (it is shown in (19) to imply its analogue; the proof of the converse is similar). Grothendieck suggested two proofs of the completeness theorem for the Picard scheme (FGA 236, Theorem 2.1); one involves the structure theorem for commutative algebraic groups (Chevalley-Borel), and the other involves the finiteness theorem for the Néron-Severi group and a Lefschetz theorem. The completeness theorem for the scheme of divisors, however, follows quickly [2], via the valuative criterion, from the theorem of Ramanujam-Samuel (EGA IV_4, 21.14.1), a purely local result.

Our proof of the completeness theorem for the scheme of divisors also uses the valuative criterion. Briefly, it runs as follows. Let X denote the ambient variety and k the ground field. Let R be a discrete valuation ring, K its quotient field, and k_0 its residue class field. Let D be an effective divisor on $X \otimes K$, and Y its scheme-theoretic closure in $X \otimes R$, which is flat over R. We have to show that the special fiber, $Y \otimes_R k_0$, is a divisor. Let H be a high multiple of an ample divisor. Let U be the scheme parametrizing the effective divisors on X that are linearly equivalent to the ones of the form $D + H'$, where H' runs through the divisors algebraically equivalent to H. Let Z be the scheme parametrizing the closed subschemes E of X that are linearly equivalent to the ones of the form,

[1] This author would like to thank the Mathematics Department of the California Institute of Technology for its generous hospitality during the preparation of this work and the National Science Foundation for its partial financial support.

[2] We wish to thank David Mumford for pointing this out.

$Y(k_0)+H'$ – that is, the ideal of E should be isomorphic to the tensor product of the ideals of $Y(k_0)$ and H' – where H' runs through the divisors algebraically equivalent to H; the normality of X is used in the construction of Z. There is a canonical monomorphism from U into the Hilbert scheme, $\mathrm{Hilb}_{(X\otimes K/K)}$, that specializes into a monomorphism from Z into $\mathrm{Hilb}_{(X\otimes k_0/k_0)}$, and both images in $\mathrm{Hilb}_{(X/k)}$ lie in the same irreducible component W; in fact, U is embedded as an open subscheme of $W\otimes K$, and the image of Z contains an open subscheme V of $W\otimes k_0$. Let k_1 be an algebraically closed field containing k-isomorphic copies of K and k_0. Since $W\otimes k_1$ is irreducible, its two open subsets $U\otimes k_1$ and $V\otimes k_1$ must intersect. Let E be a subscheme of $X\otimes k_1$ that corresponds to a point in the intersection. Then, on the one hand, E is a divisor, so its ideal is invertible, and, on the other hand, its ideal is isomorphic to the tensor product of the ideal of $(Y\otimes_R k_0)\otimes_{k_0} k_1$ and the ideal of a divisor. Therefore, the ideal of $(Y\otimes_R k_0)$ is invertible, and so $Y\otimes_R k_0$ is a divisor.

Most of the article is devoted to the study of algebraic systems of linearly equivalent divisor-like subschemes. More precisely, let S be a locally noetherian scheme, $f: X\to S$ a flat, proper morphism, P a locally noetherian S-scheme, and I a coherent $\mathcal{O}_{X_P}$-Module. We study the functor $\underline{\mathrm{Lin\ Syst}}_I$, whose value at a locally noetherian S-scheme T is the set of pairs (g, Y) consisting of S-morphism $g: T\to P$ and of a flat, closed subscheme Y of X_T whose ideal is, locally over T, isomorphic to $(g_X)^*I$. Notably, we represent $\underline{\mathrm{Lin\ Syst}}_I$ universally when the following technical conditions are satisfied: (i) $\mathcal{O}_S = f_*\mathcal{O}_X$ holds universally; (ii) there exists an open subset V of X_P containing every point x of X_P with $\mathrm{depth}(\mathcal{O}_{X_P(f_P(x)),x}) \leqq 1$ such that $I|V$ is invertible; and (iii) I is, locally over P, isomorphic to the cokernel of a homomorphism of locally free $\mathcal{O}_{X_P}$-Modules with finite rank. In fact, $\underline{\mathrm{Lin\ Syst}}_I$ is universally representable by an open subscheme of $\mathbb{P}(H)$, where H is the coherent $\mathcal{O}_P$-Module characterized by the condition that there be a functorial isomorphism,

$$\mathrm{Hom}\,(H, M) \xrightarrow{\sim} \mathrm{Hom}\,(I, (f_P)^*M),$$

where M is a quasi-coherent $\mathcal{O}_P$-Module; if the fibers of f are geometrically integral and if V contains each point x of X_P with $\mathrm{depth}\,(I(f_P(x))_x) = 0$, then $\underline{\mathrm{Lin\ Syst}}_I$ is universally representable by $\mathbb{P}(H)$ itself. In (EGA III_2, 7.7.9, (ii)), the remark is made that condition (iii) is fulfilled if f is projective. In (EGA III_2, 7.7.9, (iii)), it is stated that condition (iii) is superfluous to the existence of H, but no proof is given; for this reason alone, we have chosen to include condition (iii) as an assumption. The $\mathcal{O}_P$-Module H is closely related to a coherent $\mathcal{O}_P$-Module Q, which exists when $\underline{\mathrm{Hom}}\,(I, \mathcal{O}_{X_P})$ is flat over P; the Module Q is characterized by the condition that there exist a functorial isomorphism,

$$\mathrm{Hom}\,(Q, M) \simeq \Gamma(X_P, \mathrm{Hom}\,(I, \mathcal{O}_{X_P}) \otimes (f_P)^* M),$$

where M is a quasi-coherent $\mathcal{O}_P$-Module. The relationship between H and Q plays an important role in our applications. Grothendieck used Q along in his development of the theory.

The completeness theorem for the scheme of divisors may be restated, by virtue of the valuative criterion, in the following way: an effective divisor on a geometrically normal, projective variety remains a divisor under flat specialization. The result is optimal. A positive divisorial cycle – a closed subscheme that has pure codimension one and no embedded components – may acquire embedded components under flat specialization. Here, briefly, is an example; we plan to explain it in detail elsewhere. Let Y be a nonsingular plane cubic, and X the (projective) cone over Y. For each k-point y of Y, let $P(y)$ denote the (reduced) line determined by y and the vertex, and consider the divisorial cycle,

$$D_y = P(y_1) + P(y_2) + P(y),$$

where y_1 and y_2 are two fixed k-points of Y. Then D_y is a divisor if and only if y is equal to the third point y_3 in the intersection of Y and the secant determined by y_1 and y_2. Hence, the D_y are not isomorphic to the closed fibers of a flat family, for otherwise almost all of them would be divisors. There is, in fact, a flat family $\{Z(y)\}_{y \in Y}$ of subschemes of X such that $Z(y)$ is equal to D_y for each k-point y not equal to y_3 and $Z(y_3)$ is equal to the union of D_{y_3} and an embedded component located at the vertex of X.

1. Preliminary General Lemmas

1. Lemma: *Let X be a locally noetherian scheme, and I a coherent $\mathcal{O}_X$-Module. Assume there exists an open subset U of X containing every point x with $depth(\mathcal{O}_x) \leqq 1$ and every point x with $depth(I_x) = 0$, such that $I|U$ is invertible. Let F be a locally free $\mathcal{O}_X$-Module with finite rank, and G a coherent $\mathcal{O}_X$-Module. Then, the canonical map,*

$$s : \underline{\mathrm{Hom}}\,(G, F) \to \underline{\mathrm{Hom}}\,(G \otimes I, F \otimes I),$$

is an isomorphism.

Proof: The assertion is local on X. So, we may replace X by an arbitrary affine open subset and verify that the map $\Gamma(X, s)$ is bijective.

Construct a commutative diagram,

$$\begin{array}{ccc}
\Gamma(X, \underline{\mathrm{Hom}}\,(G, F)) & \xrightarrow{\Gamma(X,s)} & \Gamma(X, \underline{\mathrm{Hom}}\,(G \otimes I, F \otimes I)) \\
\downarrow r & \searrow u & \downarrow t \\
\Gamma(U, \underline{\mathrm{Hom}}\,(G, F)) & \xrightarrow{\Gamma(U,s)} & \Gamma(U, \underline{\mathrm{Hom}}\,(G \otimes I, F \otimes I)),
\end{array}$$

where r and t are the restrictions. The map s is an isomorphism on U, for this assertion is local on U and I is locally free with rank 1 on U; hence, $\Gamma(U, s)$ is bijective. Since U contains every point x with depth$(\mathcal{O}_x) \leqq 1$ and since F is locally free with finite rank, U contains every point x with depth$(\underline{\mathrm{Hom}}(G, F)_x) \leqq 1$ ([2], Lemma 2). Therefore, r is bijective ([2], Lemma 3). Hence, u is bijective, because it is the composition of r and $\Gamma(U, s)$.

Since U contains every point x with depth$(I_x) = 0$ and since F is locally free, U obviously also contains every point x with depth $(F_x \otimes I_x) = 0$; hence, the restriction t is injective ([2], Lemmas 2 and 3). Since u is bijective, $\Gamma(X, s)$ is therefore bijective.

2. Lemma: *Let $f : X \to S$ be a morphism of ringed spaces, and M and N two $\mathcal{O}_S$-Modules.*

(i) *Let $u : M \to N$ and $v : f^*M \to f^*N$ be homomorphisms, and consider the following diagram:*

(2.1)
$$\begin{array}{ccc}
M & \xrightarrow{u} & N \\
\downarrow \rho_M & & \downarrow \rho_N \\
f_* f^*M & \xrightarrow[f_*(v)]{} & f_* f^*N,
\end{array}$$

where ρ_M and ρ_N are the canonical maps. It is commutative if and only if $v = f^*(u)$ holds.

(ii) *If the canonical map $\rho_N : N \to f_* f^*N$ is an isomorphism, then the canonical maps,*

(2.2) $$\mathrm{Hom}\,(M, N) \to \mathrm{Hom}\,(f^*M, f^*N)$$

(2.3) $$\underline{\mathrm{Hom}}\,(M, N) \to f_*\, \underline{\mathrm{Hom}}\,(f^*M, f^*N),$$

are also isomorphisms.

Proof: (i) Applying the formulas for the adjoint of a composition (EGA 0_I, 3.5), we obtain the equalities,

(2.4) $$(\rho_N \circ u)^\# = id_N \circ f^*(u),$$

(2.5) $$(f_*(v) \circ \rho_M)^\# = v \circ id_M.$$

So, since the adjunction correspondence, $w \mapsto w^{\#}$, is bijective, $f^*(u) = v$ holds if and only if $\rho_N \circ u = f_*(v) \circ \rho_M$ holds.

(ii) Let $u_1, u_2 : M \to N$ be $\mathcal{O}_S$-homomorphisms satisfying $f^*(u_1) = f^*(u_2)$. Then, formula (2.4) yields $(\rho_N \circ u_1)^{\#} = (\rho_N \circ u_2)^{\#}$; hence, $\rho_N \circ \mathrm{u}_1 = \rho_N \circ u_2$ holds. So, since ρ_N is injective, $u_1 = u_2$ holds. Thus, (2.2) is injective.

Let $v : f^*M \to f^*N$ be an $\mathcal{O}_X$-homomorphism. Set $u = \rho_N^{-1} \circ f_*(v) \circ \rho_M$. Then, we obviously get a commutative diagram like (2.1). So, by (i), we have $f^*(u) = v$. Thus, (2.2) is surjective, so an isomorphism. It follows immediately that (2.3) is an isomorphism.

3. Lemma: *Let S be a locally noetherian scheme, $f : X \to S$ a flat morphism of finite type, and I a coherent $\mathcal{O}_X$-Module. Assume there exists an open subset U of X containing every point x with depth$(\mathcal{O}_{X(f(x)), x}) \leqq 1$ and every point x with depth$(I(f(x))_x) = 0$ such that $I|U$ is invertible, and assume I is flat over S.*

(i) *For each coherent $\mathcal{O}_X$-Module G and for each locally free $\mathcal{O}_X$-Module F with finite rank, the canonical map,*

$$(3.1) \qquad \underline{\mathrm{Hom}}\,(G, F) \to \underline{\mathrm{Hom}}\,(I \otimes G, I \otimes F),$$

is an isomorphism.

(ii) *Assume the comorphism, $\mathcal{O}_S \to f_*\mathcal{O}_X$, is an isomorphism. Then, for each coherent $\mathcal{O}_S$-Module M and each locally free $\mathcal{O}_S$-Module N with finite rank, the canonical map,*

$$(3.2) \qquad \underline{\mathrm{Hom}}\,(M, N) \to f_*\,\underline{\mathrm{Hom}}\,(I \otimes f^*M, I \otimes f^*N),$$

is an isomorphism.

Proof: (i) Let x be a point of $(X - U)$. Since $\mathcal{O}_X$ and I are flat over S and since depth$(\mathcal{O}_{X(f(x)), x}) \geqq 2$ and depth$(I(f(x))_x) \geqq 1$ hold, we have depth$(\mathcal{O}_{X, x}) \geqq 2$ and depth$(I_x) \geqq 1$ by (GD VII, 4.2). So, the map (3.1) is an isomorphism by (1).

(ii) The canonical map, $\rho_N : N \to f_* f^*N$, is an isomorphism, for the question is local and, by hypothesis, N is locally free with finite rank and the comorphism, $\rho_{\mathcal{O}_S} : \mathcal{O}_S \to f_*\mathcal{O}_X$, is an isomorphism. So, by (2, (ii)), the canonical map,

$$(3.3) \qquad \underline{\mathrm{Hom}}\,(M, N) \to f_*\,\underline{\mathrm{Hom}}\,(f^*M, f^*N),$$

is an isomorphism. Now, the canonical map,

$$(3.4) \qquad f_*\,\underline{\mathrm{Hom}}\,(f^*M, f^*N) \to f_*\,\underline{\mathrm{Hom}}\,(I \otimes f^*M, I \otimes f^*N),$$

is an isomorphism by (i). Composing the isomorphisms (3.3) and (3.4), we obtain (3.2); so, (3.2) is an isomorphism.

4. Lemma: *Let S be a locally noetherian scheme, $f: X \to S$ a morphism of finite type, and F, G two coherent $\mathcal{O}_X$-Modules. Assume there is an open subset U of X containing each point x with $depth(F(f(x))_x) = 0$ such that $G|U$ is locally free, and assume $\underline{\mathrm{Hom}}(G, F)$ is flat over S. For each quasi-coherent $\mathcal{O}_S$-Module M, consider the canonical map,*

$$b(M): \underline{\mathrm{Hom}}(G, F) \otimes f^*M \to \underline{\mathrm{Hom}}(G, F \otimes f^*M). \tag{4.1}$$

(*i*) *If U contains each point x of X with*

$$depth\,(\underline{\mathrm{Hom}}(G, F)(f(x))_x) = 0, \tag{4.2}$$

then $b(M)$ is injective for each quasi-coherent $\mathcal{O}_S$-Module M. Conversely, if the map $b(k(s))$ is injective for each point s of S, then U contains each point x of X where (4.2) *holds.*

(ii) *Assume that F is flat over S and that U contains each point x of X with $depth(F(f(x))_x) \leqq 1$. If U contains each point x of X with*

$$\mathrm{depth}\,(\underline{\mathrm{Hom}}(G, F)(f(x))_x) \leqq 1, \tag{4.3}$$

then $b(M)$ is bijective for each quasi-coherent $\mathcal{O}_S$-Module M. Conversely, if the map $b(k(s))$ is bijective for each point s of S, then U contains each point x of X where (4.3) *holds.*

Proof. All the hypotheses and assertions are clearly local on S and X; so, we may assume S and X are affine. We prove (i) only, because it is all we use; the proof of (ii) is similar.

First, assume M is coherent and U contains each point x of X where (4.2) holds. We shall show that the map $\Gamma(X, b(M))$ of global sections is injective. In the applications, we use only the case that M is coherent; on the other hand, the case that M is quasi-coherent follows formally because, over a ring, a module is the direct limit of its finitely generated submodules, because the functors, tensor product and $\mathrm{Hom}(F, -)$ with F finitely presented, commute with direct limits, and because a direct limit of injective maps is injective.

Consider the commutative diagram,

$$\begin{array}{ccc} \Gamma(X, \underline{\mathrm{Hom}}(G, F) \otimes f^*M) & \xrightarrow{\Gamma(X, b(M))} & \Gamma(X, \underline{\mathrm{Hom}}(G, F \otimes f^*M)) \\ \downarrow{\scriptstyle a} & & \downarrow \\ \Gamma(U, \underline{\mathrm{Hom}}(G, F) \otimes f^*M) & \xrightarrow{\Gamma(U, b(M))} & \Gamma(U, \underline{\mathrm{Hom}}(G, F \otimes f^*M)). \end{array}$$

Since $G|U$ is locally free with finite rank, the map $b(M)$ is an isomorphism on U; for, the assertion is local, and obvious when G is replaced by $\mathcal{O}_U$. Hence, the map $\Gamma(U, b(M))$ is bijective. Since U contains each point x

of X with depth $(\underline{\mathrm{Hom}}(G, F)(f(x))_x) = 0$, it contains each point x with depth $((\underline{\mathrm{Hom}}(G, F) \otimes f^*M)_x) = 0$ by (GD VII, 4.2); hence, a is injective by ([2], Lemma 3). Consequently, $\Gamma(X, b(M))$ is injective.

Conversely, assume $b(k(s))$ is injective for some point s of S. Since U contains each point x with depth $(F(f(x))_x) = 0$, it contains, in particular, each point x of $X(s)$ with depth $(F(s)_x) = 0$; hence, U contains each point x of $X(s)$ with depth $(\underline{\mathrm{Hom}}(G(s), F(s))_x) = 0$ ([2], Lemma 2). Consequently, since $b(k(s))$ is injective, U obviously contains each point x of $X(s)$ where (4.2) holds.

5. LEMMA: *Let $f: X \to S$ be a morphism of ringed spaces, and I an $\mathcal{O}_X$-Module. Assume the canonical map,*

$$m : \mathcal{O}_S \to f_* \underline{\mathrm{Hom}}(I, I),$$

is an isomorphism. Then, the functor taking an invertible $\mathcal{O}_S$-Module M to the $\mathcal{O}_X$-Module $I \otimes f^*M$ establishes an equivalence of categories between the category of invertible $\mathcal{O}_S$-Modules M and the category of $\mathcal{O}_X$-Modules G that are, locally over S, isomorphic to I.

PROOF: The map m and the functor, $M \mapsto I \otimes f^*M$, are related in the following way. Whether or not m is an isomorphism, it obviously induces a homomorphism of sheaves,

$$m^* : \mathcal{O}_S^* \to f_* \underline{\mathrm{Isom}}(I, I),$$

thence, a map of pointed sets,

$$\check{H}^1(S, m^*) : \check{H}^1(S, \mathcal{O}_S^*) \to \check{H}^1(S, f_* \underline{\mathrm{Isom}}(I, I)).$$

It is easy to see (EGA 0_{I}, 5.6.3) that the group $\check{H}^1(S, \mathcal{O}_S^*)$ classifies invertible sheaves on S; in the same way, it is easy to see that the set $\check{H}^1(S, f_* \underline{\mathrm{Isom}}(I, I))$ classifies the $\mathcal{O}_X$-Modules G that are isomorphic to I locally over S.

The map $\check{H}^1(S, m^*)$ may be explicitly described as follows. Let L be an invertible sheaf on S. Let (U_α) be an open covering of S such that there are isomorphisms, $v_\alpha : L|U_\alpha \xrightarrow{\sim} \mathcal{O}_S|U_\alpha$. Set $u_{\alpha\beta} = v_\beta \circ v_\alpha^{-1}$. Then, L corresponds to the class of $(u_{\alpha\beta})$ in $\check{H}^1((U_\alpha), \mathcal{O}_S^*)$, and $\check{H}^1(S, m^*)$ takes this class to the class of $(\mathrm{id}_I \otimes f^*(u_{\alpha\beta}))$ in $\check{H}^1(S, f_* \underline{\mathrm{Isom}}(I, I))$. Clearly, the class of $(\mathrm{id}_I \otimes f^*(u_{\alpha\beta}))$ corresponds to the $\mathcal{O}_X$-Module $I \otimes f^*L$. Thus, we obtain the formula,

$$\check{H}^1(S, m^*)(L) = I \otimes f^*L.$$

Since the map m is an isomorphism, the maps m^* and $\check{H}^1(S, m^*)$ are also isomorphisms. So, every $\mathcal{O}_X$-Module G that is, locally over S, isomorphic to I has the form $I \otimes f^*L$ for some invertible $\mathcal{O}_S$-Module L. Thus, the functor, $L \mapsto I \otimes f^*L$, is essentially surjective.

Let L and M be two invertible sheaves on S. Let (U_α) be an open covering of S such that $L|U_\alpha$ and $M|U_\alpha$ are isomorphic to $\mathcal{O}_S|U_\alpha$ for each α. Consider the following diagram:

$$\begin{array}{ccccc}
\operatorname{Hom}(L, M) & \longrightarrow & \prod_\alpha \operatorname{Hom}(L|U_\alpha, M|U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta} \operatorname{Hom}(L|U_{\alpha\beta}, M|U_{\alpha\beta}) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}(I \otimes f^*L, I \otimes f^*M) & \longrightarrow & \prod_\alpha \operatorname{Hom}((I \otimes f^*L)|V_\alpha, (I \otimes f^*M)|V_\alpha) & \rightrightarrows & \prod_{\alpha,\beta} \operatorname{Hom}((I \otimes f^*L)|V_{\alpha\beta}, (I \otimes f^*M)|V_{\alpha\beta})
\end{array},$$

where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $V_\alpha = f^{-1}(U_\alpha)$ and $V_{\alpha\beta} = f^{-1}(U_{\alpha\beta})$. Since a homomorphism of sheaves is determined locally, the two rows are exact. Since L and M are trivial on the covering, the middle and right-hand vertical maps are isomorphic to the products of the restrictions of m to U_α and to $U_{\alpha\beta}$; so, these two maps are bijective by hypothesis. Therefore, the left-hand vertical map is also bijective. Thus, the functor, $L \mapsto I \otimes f^*L$, is fully faithful, so an equivalence of categories.

6. Lemma: *Let k be a field, Y a geometrically normal, algebraic k-scheme, and y a point of Y. Assume depth$(\mathcal{O}_y) \leqq 1$ holds. Then, Y is smooth over k at y.*

Proof: Let k' be an algebraically closed field containing k, and y' a point of $Y \otimes k'$ that is a maximal point of the fiber over y. Then, by (GD VII, 4.2), depth$(\mathcal{O}_{y'})$ is equal to depth$(\mathcal{O}_y)$, for depth$(\mathcal{O}_{y'} \otimes_{\mathcal{O}_y} k(y))$ is zero since y' is maximal in the fiber over y; so, the inequality, depth$(\mathcal{O}_{y'}) \leqq 1$, holds. Since, by hypothesis, $Y \otimes k'$ is normal, $\mathcal{O}_{y'}$ is therefore regular by Serre's criterion (GD VII, 2.13). Hence, since k' is algebraically closed, $Y \otimes k'$ is smooth over k' at y' (GD VII, 6.3). Therefore, Y is smooth over k at y (GD VII, 5.11).

7. Lemma: *Let S be an irreducible, regular, noetherian scheme of dimension 1, and η the generic point of S. Let X be an S-scheme, and D a closed subscheme of $X(\eta)$. Then, the closure Y of D in X is the unique closed subscheme of X that is flat over S and satisfies the condition, $Y \cap X(\eta) = D$. Moreover, if X is smooth over S and D is a divisor in $X(\eta)$, then Y is a divisor in X.*

Proof: The first assertion is (EGA IV, 2.8.5). Assume X is smooth over S, and D is a divisor. Let y be a point of Y with depth$(\mathcal{O}_{Y,y}) = 0$, and let s be its image in S. Since Y is flat over S, we have the condition, depth$(\mathcal{O}_s) = 0$, by (GD VII, 4.2). So, since S is integral, s is equal to η.

Therefore, y is in D. So, since D has no embedded components, Y has no embedded components. Now, since S is regular, X is regular (GD VII, 4.9), hence, locally factorial (GD VII, 3.14). Therefore, Y will be a divisor if it has pure codimension 1.

Let z be the generic point of an irreducible component Z of Y. It is clear that, since Y is the closure of D, the point z is also the generic point of the irreducible component $Z \cap D$ of D and we have $\mathcal{O}_{X,z} = \mathcal{O}_{X(\eta),z}$. Now, $\operatorname{codim}(Z \cap D, X(\eta))$ is equal to $\dim(\mathcal{O}_{X(\eta),z})$, and $\operatorname{codim}(Z, X)$ is equal to $\dim(\mathcal{O}_{X,z})$. Hence, $\operatorname{codim}(Z, X)$ is equal to $\operatorname{codim}(Z \cap D, X(\eta))$, which is equal to 1 by hypothesis. Thus, we have $\operatorname{codim}(Z, X) = 1$. Hence, Y is regularly embedded of codimension 1.

8. Lemma: *Let S be the spectrum of a discrete valuation ring, $f: X \to S$ a flat, proper morphism with geometrically normal and geometrically integral fibers. Let U denote the open subset of X where f is smooth; let Y be a flat, closed subscheme of X/S; let P be a flat, locally noetherian S-scheme; let L be an invertible $\mathcal{O}_{X_P}$-Module; set $V = U_P$; set $I = I(Y)_P \otimes L$, where $I(Y)$ denotes the ideal of Y; and set $I^{\vee} = \underline{\mathrm{Hom}}(I, \mathcal{O}_{X_P})$. Assume the generic fiber of $I(Y)|U$ is invertible. Then*:

(i) $\mathcal{O}_S = f_* \mathcal{O}_X$ *holds universally.*
(ii) U *contains each point x of X with* $\operatorname{depth}(\mathcal{O}_{X(f(x)),x}) \leqq 1$.
(iii) V *contains each point x of X_P with* $\operatorname{depth}(I(f_P(x))_x) = 0$.
(iv) $I^{\vee}$ *is flat over P.*
(v) V *contains each point x of X_P with* $\operatorname{depth}((I^{\vee})(f_P(x))_x) = 0$.
(vi) $I|V$ *is invertible.*

Proof: Since f is flat, proper, and surjective and its fibers are geometrically integral, (i) holds (EGA III_2, 7.8). By (6) applied with $X(f(x))$ for Y, (ii) holds.

Since Y is flat over S, the sheaf $I(Y)_P(f_P(x))$ is clearly isomorphic to an ideal in $\mathcal{O}_{X_P(f_P(x))}$ for each x in X_P. So, $I(f_P(x))$ is locally isomorphic to an ideal of $\mathcal{O}_{X_P(f_P(x))}$. Hence, each x satisfying $\operatorname{depth}(I(f_P(x))_x) = 0$ clearly also satisfies $\operatorname{depth}(\mathcal{O}_{X_P(f_P(x)),x}) = 0$, and so x lies in V by (ii). Thus, (iii) holds.

Assertion (iv) is obviously local on X, so to prove it we may assume X is affine. Let t be a generator of the maximal ideal of $\Gamma(S, \mathcal{O}_S)$. Then, t is a non-zero-divisor of $\Gamma(X, \mathcal{O}_X)$ because X is flat over S. Hence, t is obviously a non-zero-divisor of $\mathrm{Hom}(I(Y), \mathcal{O}_X)$. Therefore, $\underline{\mathrm{Hom}}(I(Y), \mathcal{O}_X)$ is flat over S; so, $\underline{\mathrm{Hom}}(I(Y), \mathcal{O}_X)_P$ is flat over P. Since P is flat over S, we have a canonical isomorphism,

$$\underline{\mathrm{Hom}}(I(Y), \mathcal{O}_X)_P = \underline{\mathrm{Hom}}(I(Y)_P, \mathcal{O}_{X_P}), \tag{8.1}$$

(EGA 0_{I}, 5.7.6). Hence, $\underline{\mathrm{Hom}}(I(Y)_P, \mathcal{O}_{X_P})$ is flat over P. Therefore, $I^{\vee}$ is flat over P; that is, (iv) holds.

Let x be a point of $(X-U)$. By (ii) we have the inequality, $\operatorname{depth}(\mathcal{O}_{X(f(x)),x}) \geqq 2$. Since X is flat over S, we therefore have the inequality, $\operatorname{depth}(\mathcal{O}_x) \geqq 2$, (GD VII, 4.2). So, the inequality,

$$\operatorname{depth}(\mathrm{Hom}(I(Y)_x, \mathcal{O}_x)) \geqq 2,$$

holds ([2], Lemma 2). Therefore, (GD VII, 4.2), we have the inequality,

$$\operatorname{depth}(\underline{\mathrm{Hom}}(I(Y), \mathcal{O}_X)(f(x))_x) \geqq 1,$$

because $\underline{\mathrm{Hom}}(I(Y), \mathcal{O}_X)$ is flat over S (as was proved above) and the inequality, $\operatorname{depth}(\mathcal{O}_{f(x)}) \leqq 1$, holds. Consequently, for each x in $(X_P - V)$, we have the inequality,

$$\operatorname{depth}(\underline{\mathrm{Hom}}(I(Y)_P, \mathcal{O}_{X_P})(f_P(x))_x) \geqq 1,$$

because (8.1) holds and depth cannot decrease under a field extension (cf. proof of (6)). Hence, for each x in $(X_P - V)$, we have the inequality, $\operatorname{depth}((I^{\vee})(f_P(x))_x) \geqq 1$. Thus, (v) holds.

Obviously, $Y|U$ is a flat, closed subscheme of the smooth S-scheme U, and its generic fiber is a divisor. So, by (7), it is a divisor. Thus, $I(Y)|U$ is invertible. Therefore, $I(Y)_P|V$ is invertible, and so, $I|V$ is also. Thus, (vi) holds.

2. A Theory of $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_{\mathrm{I}}$

9. ($\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I$). Let S be a locally noetherian scheme, $f: X \to S$ a flat morphism of finite type, P a locally noetherian S-scheme, and I a coherent $\mathcal{O}_{X_P}$-Module. For each S-scheme T, let $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$ denote the subset of $(P \times_S \underline{\mathrm{Hilb}}_{(X/S)})(T)$ consisting of those pairs (g, Y) with Y in $\underline{\mathrm{Hilb}}_{(X/S)}(T)$ and g in $P(T)$ such that the ideal of Y is isomorphic, locally over T, to $(g_X)^*I$. Obviously, the sets, $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$, as T runs through all locally noetherian S-schemes, form a functor, $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I$, (obviously, a Zariski sheaf).

The functor, $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I$, comes equipped with maps to $\underline{\mathrm{Hilb}}_{(X/S)}$ and to P,

$$\begin{array}{ccc} \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I & \xrightarrow{p_2} & \underline{\mathrm{Hilb}}_{(X/S)} \\ {\scriptstyle p_1}\downarrow & & \\ P & & , \end{array}$$

namely, the restrictions to $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I$ of the projections from $P \times_S \underline{\mathrm{Hilb}}_{(X/S)}$.

Let $S' \to S$ be a morphism of locally noetherian schemes. Then, clearly, for each locally noetherian S'-scheme T, we have the formula,

$$(\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I \times_S S')(T) = \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_{I_{S'}}(T).$$

So, we also have the relations,

$$\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I \times_S S' = \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_{I_{S'}}$$

$$\underline{p}_i \times_S S' = (\underline{p}_i)_{S'} \qquad \text{for } i = 1, 2.$$

10. Lemma: *Let S be a locally noetherian scheme, $f : X \to S$ a flat morphism of finite type, P a locally noetherian S-scheme, and I a coherent $\mathcal{O}_{X_P}$-Module. Assume there is an open subset V of X_P containing every point x of X_P with* depth $(\mathcal{O}_{X_{P(f_P(x))},x}) \leqq 1$ *such that $I|V$ is invertible. For each locally noetherian S-scheme T and each morphism $g : T \to P$, let $\mathscr{N}_g$ denote the family of pairs (M, u) where M is an invertible $\mathcal{O}_T$-Module and u is an $\mathcal{O}_{X_T}$-homomorphism,*

$$u : (g_X)^* I \otimes (f_T)^* M \to \mathcal{O}_{X_T},$$

such that, for each point t of T, the induced map $u(t)$ of $\mathcal{O}_{X(t)}$-Modules is injective.

(i) *Assume that $\mathcal{O}_S = f_* \mathcal{O}_X$ holds universally. Let (g, Y) be an element of $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$ for some locally noetherian S-scheme T. Then, there exists a pair (M, u) in $\mathscr{N}_g$ such that the ideal $I(Y)$ of Y has the form,*

$$I(Y) = u((g_X)^* I \otimes (f_T)^* M).$$

Moreover, this pair is uniquely determined up to unique isomorphism in the sense that, if (M_1, u_1) is a second pair in $\mathscr{N}_g$ such that $u_1((g_X)^* I \otimes (f_T)^* M_1)$ is equal to $I(Y)$, then there exists a unique isomorphism $a : M \xrightarrow{\sim} M_1$ that makes the diagram,

(10.1)
$$\begin{tikzcd}
(g_X)^* I \otimes (f_T)^* M \arrow[d, "id \otimes (f_T)_S(a)"', "\wr"] \arrow[dr, "u"] & \\
(g_X)^* I \otimes (f_T)^* M_1 \arrow[r, "u_1"'] & \mathcal{O}_{X_T},
\end{tikzcd}$$

commutative.

(ii) *Assume f is proper. Then, for any locally noetherian S-scheme T, any S-morphism $g : T \to P$, and any pair (M, u) in $\mathscr{N}_g$, the image $u((g_X)^* I \otimes (f_T)^* M)$ in $\mathcal{O}_{X_T}$ is the ideal of a subscheme of $X \times_S T$ that is in $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$.*

Proof: (i) Since I_T is locally isomorphic to the ideal of a flat subscheme of X_T/T, it is obviously flat over T, and each point x of X_T with depth$(I_T(f_T(x))_x) = 0$ is in V_T because each point x of X_T with depth$(\mathcal{O}_{X_{T(f_T(x))},x}) = 0$ is in V_T in view of the hypothesis. By (3, (ii)) with T for S and with $\mathcal{O}_T$ for M and for N, the canonical map,

$$\mathcal{O}_T \to (f_T)_* \underline{\mathrm{Hom}}\,((g_X)^*I, (g_X)^*I),$$

is an isomorphism. So, by (5), there exist an invertible $\mathcal{O}_T$-Module M and an isomorphism,

$$u' : (g_X)^*I \otimes (f_T)^*M \xrightarrow{\sim} I(Y).$$

Let u denote the composition,

$$u : (g_X)^*I \otimes (f_T)^*M \xrightarrow{u'} I(Y) \xrightarrow{i} \mathcal{O}_{X_T},$$

where i is the inclusion. Then, we obviously have the relation,

$$u((g_X)^*I \otimes (f_T)^*M) = I(Y).$$

Since Y is flat over T, we obviously have, for each T-scheme T', an equality,

$$I(Y_{T'}) = I(Y)_{T'},$$

where $I(Y_{T'})$ denotes the ideal of $Y_{T'}$. In particular, for each point t of T, the map, $i(t) : I(Y)(t) \to \mathcal{O}_{X(t)}$, is injective. Since u' is an isomorphism, $u'(t)$ is an isomorphism. So, $u(t)$ is injective for each $t \in T$. Thus, the pair (M, u) is an element of $\varkappa_g$, and $I(Y)$ has the required form.

If (M_1, u_1) is a second pair of $\varkappa_g$ satisfying $u_1((g_X)^*I \otimes (f_T)^*M_1) = I(Y)$, then there obviously exists a commutative diagram,

$$\begin{tikzcd} (g_X)^*I \otimes (f_T)^*M \arrow[d, "(u_1')^{-1} \circ u'"', "\wr"] \arrow[dr, "u'"] & & \\ & I(Y) \arrow[r, "i"] & \mathcal{O}_{X_T}. \\ (g_X)^*I \otimes (f_T)^*M_1 \arrow[ur, "u_1'"'] & & \end{tikzcd}$$

Since, by (5), the functor $M \mapsto (g_X)^*I \otimes (f_T)^*M$ is fully faithful, there is a unique isomorphism $a : M \xrightarrow{\sim} M_1$ such that $\mathrm{id}_{(g_X)^*I} \otimes (f_T)^*(a)$ is equal to $(u_1')^{-1} \circ u'$. Thus, the uniqueness assertion holds.

(ii) Since X_T is flat over T and since $u(t)$ is injective for each point t of T, the quotient $\mathcal{O}_{X_T}/\mathrm{Im}\,(u)$ is flat over T and the map u is itself injective (GD VII, 4.1). So, $\mathrm{Im}\,(u)$ is the ideal of a closed subscheme Y of X_T that is flat over T, and $\mathrm{Im}\,(u)$ is isomorphic, locally over T, to $(g_X)^*I$ (which implies, in particular, that $(g_X)^*I$ is flat over T). Since f is proper, Y is in $\underline{\mathrm{Hilb}}_{(X/S)}(T)$. So, (g, Y) is in $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$.

11. **Proposition**: *Let S be a locally noetherian scheme, $f : X \to S$ a flat morphism of finite type such that $\mathcal{O}_S = f_*\mathcal{O}_X$ holds universally, and J a coherent $\mathcal{O}_X$-Module. Assume there is an open subset U of X containing*

each point x of X with $\operatorname{depth}(\mathcal{O}_{X(f(x)),x}) \leqq 1$ *such that $J|U$ is invertible. Let P be a locally noetherian S-scheme, L an invertible $\mathcal{O}_{X_P}$-Module, and $\underline{q}: P \to \underline{\mathrm{Pic}}_{(X/S)}$ the map of functors defined by L, where by definition $\underline{\mathrm{Pic}}_{(X/S)}(T)$ is equal to* $\mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T)$ *for each locally noetherian S-scheme T. Finally, set $I = J_P \otimes L^{-1}$.*

(i) *Suppose $\underline{q}: P \to \underline{\mathrm{Pic}}_{(X/S)}$ is a monomorphism. Then, so is the canonical map of functors* (9),

$$(11.1) \qquad \underline{p}_2 : \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I \to \underline{\mathrm{Hilb}}_{(X/S)}.$$

(ii) *Suppose J is invertible. Then, the canonical map $\underline{p}_2$ from $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I$ to $\underline{\mathrm{Hilb}}_{(X/S)}$ factors through $\underline{\mathrm{Div}}_{(X/S)}$ and yields a cartesian diagram,*

$$(11.2) \qquad \begin{array}{ccc} \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I & \longrightarrow & \underline{\mathrm{Div}}_{(X/S)} \\ {\scriptstyle \underline{p}_1}\downarrow & \square & \downarrow{\scriptstyle \underline{l}_J} \\ P & \xrightarrow{\ \underline{q}\ } & \underline{\mathrm{Pic}}_{(X/S)} \end{array},$$

where $\underline{l}_J$ is defined by sending a divisor E on $X \times_S T$ to the class of the invertible sheaf $\mathcal{O}_{X_T}(E) \otimes J_T$.

PROOF: (i) Let T be a locally noetherian S-scheme, (g, Y) and (g', Y') two elements of $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$ whose images in $\underline{\mathrm{Hilb}}_{(X/S)}(T)$ are equal, that is, for which $Y = Y'$ holds. Since Y is flat over T, for each point t of T, we obviously have $I(Y)(t) = I(Y(t))$, where $I(Y)$ denotes the ideal of Y and $I(Y(t))$ that of $Y(t)$. So, since U_T contains each point x with $\operatorname{depth}(\mathcal{O}_{X_T(f_T(x)),x}) = 0$, it contains each point x of X_T with

$$\operatorname{depth}(I(Y)(f_T(x))_x) = 0.$$

Since J_T is isomorphic, locally on X_T, to $I(Y)$, each point x of X_T with $\operatorname{depth}(J_T(f_T(x))_x) = 0$ is, therefore, in U_T. Again, since J_T is isomorphic locally on X_T to $I(Y)$, it is flat over T because Y is and so $I(Y)$ is.

By (10, (i)) there are invertible sheaves M and M' and isomorphisms,

$$u : (g_X)^*I \otimes (f_T)^*M \xrightarrow{\sim} I(Y) \quad \text{and} \quad u' : (g'_X)^*I \otimes (f_T)^*M' \xrightarrow{\sim} I(Y).$$

Therefore, we have an isomorphism,

$$v : (g_X)^*L^{-1} \otimes J_T \otimes (f_T)^*M \xrightarrow{\sim} (g'_X)^*L^{-1} \otimes J_T \otimes (f_T)^*M'.$$

By (3, (i)) with J for I and T for S, there is an isomorphism,

$$v' : (g_X)^*L^{-1} \otimes (f_T)^*M \xrightarrow{\sim} (g'_X)^*L^{-1} \otimes (f_T)^*M'.$$

Therefore, we have $\underline{q}(g) = \underline{q}(g')$. Hence, since $\underline{q}$ is a monomorphism, we have $g = g'$. Thus, the map $\underline{p}_2$ is a monomorphism.

(ii) Let T be a locally noetherian S-scheme, and (g, Y) an element of $\underline{\text{Lin}}\ \underline{\text{Syst}}_I(T)$. Since J and L are invertible, so is the ideal $I(Y)$ of Y. Thus, Y is in $\underline{\text{Div}}_{(X/S)}(T)$. Now, there are an invertible $\mathcal{O}_T$-Module M and an isomorphism, $(g_X)^*I \otimes (f_T)^*M \xrightarrow{\sim} I(Y)$, by (10, (i)); so, there is an isomorphism, $J_T \otimes (g_X)^*L^{-1} \otimes (f_T)^*M \xrightarrow{\sim} I(Y)$. Hence, $(g_X)^*L$ and $I(Y)^{-1} \otimes J_T$ represent the same element of $\underline{\text{Pic}}_{(X/S)}(T)$. However, $\mathcal{O}_{X_T}(Y) \otimes J_T = I(Y)^{-1} \otimes J_T$ represents the image of Y under $\underline{l}_J$, while $(g_X)^*L$ represents the image of g under $\underline{q}$. Thus, the diagram (11.2) is commutative.

Let T be a locally noetherian S-scheme, and (g, Y) a pair consisting of an S-morphism $g : T \to P$ and a relative effective divisor Y on $X \times_S T/T$ such that $(g_X)^*L$ and $I(Y)^{-1} \otimes J_T$ represent the same element of $\underline{\text{Pic}}_{(X/S)}(T)$ where $I(Y)$ denotes the ideal of Y. Then, clearly, $(g_X)^*L$ and $I(Y)^{-1} \otimes J_T$ are isomorphic locally over T. Hence, (g, Y) is an element of $\underline{\text{Lin}}\ \underline{\text{Syst}}_I(T)$. Thus, the commutative diagram (11.2) is cartesian.

12. (The sheaf $Q(F)$). Let $f : X \to S$ be a proper morphism of locally noetherian schemes, and F a coherent $\mathcal{O}_X$-Module that is flat over S. Then, by (EGA III_2, 7.7.6), there exist a coherent $\mathcal{O}_S$-Module $Q(F)$ and an element $q(F)$ in $\Gamma(X, F \otimes f^*Q(F))$ such that the Yoneda map,

$$y(q(F)) : \text{Hom}\,(Q(F), M) \to \Gamma(X, F \otimes f^*M),$$

is an isomorphism for each quasi-coherent $\mathcal{O}_S$-Module M (of course, $y(q(F))$ is defined for each $\mathcal{O}_S$-Module M, quasi-coherent or not, and it behaves functorially in M); in other words, the pair $(Q(F), q(F))$ represents the functor, $M \mapsto \Gamma(X, F \otimes f^*M)$, on the category of quasi-coherent $\mathcal{O}_S$-Modules.

By (EGA III_2, 7.7.9, (i)), the formation of the pair, $(Q(F), q(F))$, commutes with base change in the sense that, for each morphism $g : T \to S$ of schemes, the pair, $(g^*Q(F), g^*(q(F)))$, represents the functor,

$$N \mapsto \Gamma(X_T, (g_X)^*F \otimes (f_T)^*N),$$

on the category of quasi-coherent $\mathcal{O}_T$-Modules; if T is locally noetherian, then the commutativity of the pair, $(Q(F), q(F))$, with the base change g can be expressed by the formulas,

$$Q((g_X)^*F) = g^*Q(F) \quad \text{and} \quad q((g_X)^*F) = g^*(q(F)). \tag{12.1}$$

Moreover, for each morphism $g : T \to S$ and each $\mathcal{O}_S$-Module M, quasi-coherent or not, the diagram,

$$\begin{tikzcd} \text{Hom}\,(Q(F), M) \arrow[r, "y(q(F))"] \arrow[d] & \Gamma(X, F \otimes f^*M) \arrow[d, "\Gamma(X,\, \rho_{g_X}(F \otimes f^*M))"] \\ \text{Hom}\,(g^*Q(F), g^*M) \arrow[r, "y(g^*(q(F)))"] & \Gamma(X_T, (g_X)^*F \otimes (f_T)^*g^*M), \end{tikzcd} \tag{12.2}$$

is commutative by Yoneda's lemma since the identity map of $Q(F)$, considered as an element of $\operatorname{Hom}(Q(F), Q(F))$, is carried by both compositions to the element $g^*(q(F))$ of $\Gamma(X_T, (g_X)^*F \otimes (f_T)^*g^*Q(F))$.

Let G be another coherent $\mathcal{O}_X$-Module that is flat over S, and $u: F \to G$ an $\mathcal{O}_X$-homomorphism. Denote by

$$Q(u) : Q(G) \to Q(F)$$

the $\mathcal{O}_S$-homomorphism representing the map of functors in the quasi-coherent $\mathcal{O}_S$-Module M,

$$\Gamma(X, u \otimes f^*(M)) : \Gamma(X, F \otimes f^*(M)) \to \Gamma(X, G \otimes f^*(M)).$$

Then, by Yoneda's lemma, the diagram,

$$\begin{array}{ccc} \operatorname{Hom}(Q(F), M) & \xrightarrow{y(q(F))} & \Gamma(X, F \otimes f^*M) \\ {\scriptstyle \operatorname{Hom}(Q(u), M)}\downarrow & & \downarrow{\scriptstyle \Gamma(X, u\otimes f^*M)} \\ \operatorname{Hom}(Q(G), M) & \xrightarrow{y(q(G))} & \Gamma(X, G \otimes f^*M), \end{array} \tag{12.3}$$

is commutative for each $\mathcal{O}_S$-Module M, quasi-coherent or not.

A proof that the formation of the pair, $(Q(F), q(F))$, commutes with base change runs as follows. Let $g: T \to S$ be a morphism of schemes, and N a quasi-coherent $\mathcal{O}_T$-Module. We want to show that the map,

$$y(g^*(q(F))) : \operatorname{Hom}(g^*Q(F), N) \to \Gamma(X_T, (g_X)^* \otimes (f_T)^*N),$$

is an isomorphism. It is not hard to see that we may assume S and T are affine.

If we take g_*N for M in (12.2) and combine the resulting diagram with the commutative diagram expressing the functoriality of the map $y(g^*(q(F)))$ with respect to the canonical map, $\sigma_g(N): g^*g_*N \to N$, we obtain a commutative diagram,

$$\begin{array}{ccc} \operatorname{Hom}(Q(F), g_*N) & \xrightarrow{y(q(F))} & \Gamma(X, F \otimes f^*g_*N) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(g^*Q(F), N) & \xrightarrow{y(g^*(q(F)))} & \Gamma(X_T, (g_X)^*F \otimes (f_T)^*N). \end{array}$$

Obviously, the left-hand map is the adjunction isomorphism, and the right-hand map is induced by a canonical map,

$$F \otimes f^*g_*N \to (g_X)_*((g_X)^*F \otimes (f_T)^*N).$$

This map is easily seen to be an isomorphism because S and T are affine. So, the right-hand map is an isomorphism. Finally, the top map, $y(q(F))$, is an isomorphism because $g_* N$ is quasi-coherent. Hence, the bottom map, $y(g^*(q(F))$, is an isomorphism.

13. (The sheaf $H(G, F)$). Let $f : X \to S$ be a proper morphism of locally noetherian schemes, and F and G two coherent $\mathcal{O}_X$-Modules such that (i) F is flat over S and (ii) G is, locally over S, isomorphic to the cokernel of an $\mathcal{O}_X$-homomorphism of locally free $\mathcal{O}_X$-Modules with finite rank. Then, there exist a coherent $\mathcal{O}_S$-Module $H(G, F)$ and an element $h(G, F)$ in $\mathrm{Hom}(G, F \otimes f^*H(G, F))$ such that the Yoneda map,

$$y(h(G, F)) : \mathrm{Hom}\,(H(G, F), M) \to \mathrm{Hom}\,(G, F \otimes f^*M),$$

is an isomorphism for each quasi-coherent $\mathcal{O}_S$-Module M; in other words, the pair, $(H(G, F), h(G, F))$, represents the functor $M \mapsto \mathrm{Hom}(G, F \otimes f^*M)$ on the category of quasi-coherent $\mathcal{O}_S$-Modules. Indeed, the assertion results from (EGA III_2, 7.7.8).

By (EGA III_2, 7.7.9, (ii)), condition (ii) is always satisfied if f is projective; ((EGA III_2, 7.7.9, (iii)) states that it will be proved superfluous in Chapter V of EGA). By (EGA III_2; 7.7.9, (i)), the formation of the pair $(H(G, F), h(G, F))$, commutes with base change in the sense that, for each morphism $g : T \to S$ of schemes, the pair, $(g^*H(G, F), g^*h(G, F))$, represents the functor, $N \mapsto \mathrm{Hom}\,((g_X)^*G, (g_X)^*F \otimes (f_T)^*N)$, on the category of quasi-coherent $\mathcal{O}_T$-Modules; if T is locally noetherian, then the commutativity of the pair, $(H(G, F), h(G, F))$, with the base change g can be expressed by the formulas,

$$H((g_X)^*G, (g_X)^*F) = g^*H(G, F) \quad \text{and} \quad h((g_X)^*G, (g_X)^*F) = g^*(h(G, F)). \tag{13.1}$$

Moreover, for each morphism $g : T \to S$ and each $\mathcal{O}_S$-Module M, quasi-coherent or not, the diagram,

$$\begin{tikzcd}
\mathrm{Hom}\,(H(G, F), M) \arrow[r, "y(h(G, F))"] \arrow[d] & \mathrm{Hom}\,(G, F \otimes f^*M) \arrow[d] \\
\mathrm{Hom}\,(g^*H(G, F), g^*M) \arrow[r, "y(g^*(h(G, F)))"] & \mathrm{Hom}\,((g_X)^*G, (g_X)^*F \otimes (f_T)^*g^*M),
\end{tikzcd} \tag{13.2}$$

is commutative by Yoneda's lemma, where the vertical maps are the canonical ones.

14. (The natural map $c(G, F)$ from $H(G, F)$ to $Q(\mathrm{Hom}\,(G, F))$). Let $f : X \to S$ be a proper morphism of locally noetherian schemes, and F and G coherent $\mathcal{O}_X$-Modules such that (i) F is flat over S, (ii) G is, locally over S, isomorphic to the cokernel of an $\mathcal{O}_X$-homomorphism of locally free $\mathcal{O}_X$-Modules with finite rank, and (iii) $\underline{\mathrm{Hom}}(G, F)$ is flat over S. Denote by

$$c(G, F) : H(G, F) \to Q(\underline{\mathrm{Hom}}\,(G, F)),$$

the $\mathcal{O}_S$-homomorphism representing the map of functors in the quasi-coherent $\mathcal{O}_S$-Module M,

$$\Gamma(X, b(M)) : \Gamma(X, \underline{\mathrm{Hom}}\,(G, F) \otimes f^*M) \to \mathrm{Hom}\,(G, F \otimes f^*M).$$

Thus, by Yoneda's lemma, the diagram,

$$\begin{array}{ccc} \mathrm{Hom}\,(Q(\underline{\mathrm{Hom}}\,(G, F)), M) & \xrightarrow{\mathrm{Hom}\,(c(G,F),M)} & \mathrm{Hom}\,(H(G, F), M) \\ {\scriptstyle y(q(\underline{\mathrm{Hom}}\,(G,F)))}\downarrow & & \downarrow{\scriptstyle y(h(G,F))} \\ \Gamma(X, \underline{\mathrm{Hom}}\,(G, F) \otimes f^*M) & \xrightarrow{\Gamma(X, b(M))} & \mathrm{Hom}\,(G, F \otimes f^*M), \end{array} \tag{14.1}$$

is commutative for each $\mathcal{O}_S$-Module M, quasi-coherent or not.

If M is quasi-coherent and $b(M)$ is injective (resp. bijective), then $\mathrm{Hom}\,(c(G, F), M)$ is injective (resp. bijective) because then the vertical maps in (14.1) are bijective. Hence, if $b(M)$ is injective (resp. bijective) for every coherent $\mathcal{O}_S$-Module, then $c(G, F)$ is surjective (resp. bijective) (to prove surjectivity, take $\mathrm{coker}\,(c(G, F))$ for M; then, to prove injectivity, take $H(G, F)$ for M.)

Let E be a locally free $\mathcal{O}_X$-Module with finite rank. Then, with E for G, (ii) is obviously satisfied, and (iii) is also satisfied because $\underline{\mathrm{Hom}}\,(E, F)$ is locally isomorphic to a finite direct sum of copies of F, so flat over S. Moreover, in this case, $b(M)$ is an isomorphism for each $\mathcal{O}_S$-Module M because it obviously is for $E = \mathcal{O}_X$ and the question is local on X. Hence, $c(E, F)$ is an isomorphism,

$$c(E, F) : H(E, F) \xrightarrow{\sim} Q(\underline{\mathrm{Hom}}\,(E, F)). \tag{14.2}$$

Let $g : T \to S$ be a morphism of locally noetherian schemes. Then, the diagram,

$$\begin{array}{ccc} \Gamma(X, \underline{\mathrm{Hom}}\,(G, F) \otimes f^*M) & \longrightarrow & \Gamma(X_T, (g_X)^*\,\underline{\mathrm{Hom}}\,(G, F) \otimes (f_T)^*g^*M) \\ & & \downarrow{\scriptstyle \Gamma(X_T,\, b(T)\otimes id)} \\ {\scriptstyle \Gamma(X, b(M))}\downarrow & & \Gamma(X_T, \underline{\mathrm{Hom}}\,((g_X)^*G, (g_X)^*F) \otimes (f_T)^*g^*M) \\ & & \downarrow{\scriptstyle \Gamma(X_T,\, b(g^*M))} \\ \Gamma(X, \underline{\mathrm{Hom}}\,(G, F \otimes f^*M)) & \longrightarrow & \Gamma(X_T, \underline{\mathrm{Hom}}\,((g_X)^*G, (g_X)^*F \otimes (f_T)^*g^*M)), \end{array} \tag{14.3}$$

is clearly commutative, where

$$b(T) : (g_X)^*\,\underline{\mathrm{Hom}}\,(G, F) \to \underline{\mathrm{Hom}}\,((g_X)^*G, (g_X)^*F)$$

is the canonical map. Each map in (14.3) appears in a diagram like (12.2), (13.2), (12.3), or (14.1). Taking $Q(\underline{\mathrm{Hom}}\,(G, F))$ for M and following its identity map around these commutative diagrams, we see using Yoneda's lemma that the diagram,

$$\begin{array}{lccc} & H((g_X)^*G, (g_X)^*F) & \xrightarrow{\sim} & g^*H(G, F) \\ & \downarrow & & \Big| \\ (14.4) & Q(\underline{\mathrm{Hom}}\,((g_X)^*G, (g_X)^*F)) & & \Big| \\ & \downarrow & & \downarrow \\ & Q((g_X)^*\,\underline{\mathrm{Hom}}\,(G, F)) & \xrightarrow{\sim} & g^*Q(\underline{\mathrm{Hom}}\,(G, F)), \end{array}$$

is commutative; in short, we obtain the formula,

(14.5) $$g^*(c(G, F)) = Q(b(T)) \circ c((g_X)^*G, (g_X)^*F).$$

15. Theorem: *Let S be a locally noetherian scheme, $f : X \to S$ a flat, proper morphism, P a locally noetherian S-scheme, and I a coherent $\mathcal{O}_{X_P}$-Module. Assume I is isomorphic, locally over P, to the cokernel of an $\mathcal{O}_{X_P}$-homomorphism of locally free $\mathcal{O}_{X_P}$-Modules with finite rank (this condition is automatically satisfied if f is projective* (EGA III_2, 7.7.9, (ii)). *Assume there is an open set V of X_P containing each point x of X_P with* depth $(\mathcal{O}_{X_{P(f_P(x))}, x}) \leqq 1$ *such that $I|V$ is invertible, and assume $\mathcal{O}_S = f_* \mathcal{O}_X$ holds universally. Then, the functor* $\underline{\mathrm{Lin\ Syst}}_I$ *is universally representable by an open subscheme U of $\mathbb{P}(H(I, \mathcal{O}_{X_P}))$; that is, there is a canonical isomorphism of functors of locally noetherian S-schemes,*

(15.1) $$U \xrightarrow{\sim} \underline{\mathrm{Lin\ Syst}}_I,$$

whose formation commutes base change; if, also, the fibers of f are geometrically integral and V contains each point x of X_P with depth $(I(f_P(x))_x) = 0$, *then* $\underline{\mathrm{Lin\ Syst}}_I$ *is universally representable by $\mathbb{P}(H(I, \mathcal{O}_{X_P}))$ itself.*

Proof: Set $H = H(I, \mathcal{O}_{X_P})$. Let T be a locally noetherian S-scheme, $g : T \to P$ an S-morphism, and M an invertible $\mathcal{O}_T$-Module. By (13), there exists an isomorphism,

$$\mathrm{Hom}\,(g^*H, M) \xrightarrow{\sim} \mathrm{Hom}\,((g_X)^*I, (f_T)^*M),$$

which is functorial in M and g. For each quasi-coherent $\mathcal{O}_{X_T}$-Module F, there obviously exists a canonical isomorphism,

$$\mathrm{Hom}\,((g_X)^*I, \underline{\mathrm{Hom}}\,(F, \mathcal{O}_{X_T})) = \mathrm{Hom}\,((g_X)^*I \otimes F, \mathcal{O}_{X_T}).$$

Substituting $(f_T)^*M^{-1}$ for F and composing these isomorphisms we obtain a key isomorphism, (15.2) $\quad \kappa : \mathrm{Hom}\,(g^*H, M) \xrightarrow{\sim} \mathrm{Hom}\,((g_X)^*I \otimes (f_T)^*M^{-1}, \mathcal{O}_{X_T}),$

which is clearly functorial in M and g.

Fix a locally noetherian S-scheme T and an S-morphism $g : T \to P$. We first establish a canonical functorial bijection,

$$B_g : G_1(g^*H) \to \mathcal{M}_g,$$

from the set $G_1(g^*H)$ of 1-quotients of g^*H (that is, quotients of g^*H that are locally free with rank 1) to the set $\mathcal{M}_g$ of equivalence classes of pairs (M, u) where M is an invertible $\mathcal{O}_T$-Module and

$$u : (g_X)^*I \otimes (f_T)^*M^{-1} \to \mathcal{O}_{X_T}$$

is an $\mathcal{O}_{X_T}$-homomorphism with $u(t) \neq 0$ for each point t of T; pairs, (M, u) and (M_1, u_1), are considered equivalent if there exists an isomorphism, $a : M \xrightarrow{\sim} M_1$, that induces a commutative diagram,

(15.3)
$$\begin{tikzcd}
(g_X)^*I \otimes (f_T)^*M_1^{-1} \arrow[d, "id\otimes(f_T)^*(a^\vee)"'] \arrow[dr, "u_1"] & \\
& \mathcal{O}_{X_T}. \\
(g_X)^*I \otimes (f_T)^*M^{-1} \arrow[ur, "u"'] &
\end{tikzcd}$$

Let M be a 1-quotient of g^*H, and let $v : g^*H \to M$ denote the canonical surjection. Let t be a point of T. By the functoriality of κ in g, there is a relation,

$$(\kappa(v))(t) = \kappa(v(t)).$$

Obviously, $v(t)$ is nonzero. Hence, $(\kappa(v))(t)$ is nonzero since κ is injective. Thus, $(M, \kappa(v))$ represents an element of $\mathcal{M}_g$. Define B_g by the formula,

$$B_g(M) = \text{class}\,(M, \kappa(v)).$$

Let M_1 be a second 1-quotient of g^*H, let $v_1 : g^*H \to M_1$ denote the canonical surjection, and assume there is an isomorphism $a : M \xrightarrow{\sim} M_1$ inducing a commutative diagram like (15.3). Then, since κ is functorial in M, clearly $\kappa(a \circ v)$ is equal to $\kappa(v_1)$; so, since κ is injective, $a \circ v$ is equal to v_1. Hence, the 1-quotients, M and M_1, are equal. Thus, B_g is injective.

Let (M, u) represent an element of $\mathcal{M}_g$, and let $v : g^*H \to M$ denote $\kappa^{-1}(u)$. Let t be a point of T. Since $u(t)$ is nonzero, obviously $v(t) : (g^*H)(t) \to M(t)$ is nonzero. Since $M(t)$ is a 1-dimensional vector space, $v(t)$ is therefore surjective. Hence, v is surjective by Nakayama's lemma. Therefore, $g^*H/\mathrm{ker}\,(v)$ is a 1-quotient of g^*H, and there is an isomorphism $a : g^*H/\mathrm{Ker}\,(v) \xrightarrow{\sim} M$ such that $a^{-1} \circ v$ is equal to the canonical surjection. Since κ is functorial in M, there is a commutative diagram like (15.3).

So, $B_g(g^*H/\mathrm{Ker}\,(v))$ is equal to the element represented by (M, u). Thus, B_g is surjective, so bijective. Finally, B_g is clearly functorial in g because κ is.

Let $\alpha_1^\# : H_{\mathbb{P}(H)} \to \mathcal{O}_{\mathbb{P}(H)}(1)$ denote the canonical surjection, and set

$$\beta = \kappa(\alpha_1^\#).$$

Let $p : \mathbb{P}(H) \to S$ denote the structure morphism. Let $h : T \to \mathbb{P}(H)$ be an S-morphism satisfying the condition, $p \circ h = g$. Then, the functoriality of κ in M and of B_g in g yield the formula,

$$B_g(g^*H/\mathrm{Ker}\,(h^*(\alpha_1^\#))) = \mathrm{class}\,(h^*\mathcal{O}_p(1), h^*(\beta)).$$

There is a functorial bijection from the set of S-morphisms $h : T \to \mathbb{P}(H)$ satisfying the condition, $p \circ h = g$, to the set, $G_1(g^*H)$; it sends h to the 1-quotient of g^*H defined by $h^*(\alpha_1^\#)$, (EGA II, 4.2.3). Following this bijection with B_g, and letting g vary while keeping T fixed, we obtain a bijection,

$$A_S(T) : \mathbb{P}(H)(T) \xrightarrow{\sim} \mathcal{M}_S(T),$$

from the set $\mathbb{P}(H)(T)$ of S-morphisms, $h : T \to \mathbb{P}(H)$, to the set $\mathcal{M}_S(T)$ of classes of triples (g, M, u) consisting of an S-morphism, $g : T \to P$, an invertible $\mathcal{O}_T$-Module M, and an $\mathcal{O}_T$-homomorphism,

$$u : (g_X)^*I \otimes (f_T)^*M^{-1} \to \mathcal{O}_{X_T},$$

with $u(t) \neq 0$ for each point t of T; triples (g, M, u) and (g_1, M_1, u_1) are considered equivalent if g is equal to g_1 and there exists an isomorphism $a : M \xrightarrow{\sim} M_1$ inducing a commutative diagram exactly like (15.3). Obviously, $A_S(T)$ is given by the formula,

(15.4) $$A_S(T)(h) = \mathrm{class}\,(p \circ h, h^*(\mathcal{O}_{\mathbb{P}(H)}(1)), h^*(\beta)).$$

Clearly, the $A_S(T)$ form an isomorphism of functors, $A_S : \mathbb{P}(H) \to \mathcal{M}_S$. In short, the S-scheme $\mathbb{P}(H)$ represents the functor $\mathcal{M}_S$. It is evident from the construction that $\mathbb{P}(H)$ universally represents $\mathcal{M}_S$; that is, the formation of A_S from f, P, and I commutes with any base change $S' \to S$, with S' locally noetherian; for, the formation of H, $\mathbb{P}(H)$ and β do.

Next, we construct a canonical monomorphism of functors,

$$C_S : \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I \hookrightarrow \mathcal{M}_S,$$

whose formation commutes with any base change $S' \to S$, with S' locally noetherian. Let T be a locally noetherian S-scheme, and (g, Y) an element of $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I(T)$. By (10, (i)), the ideal of Y in $\mathcal{O}_{X_T}$ has the form $u((g_X)^*I \otimes (f_T)^*M^{-1})$, where M is an invertible $\mathcal{O}_T$-Module and

$$u : (g_X)^*I \otimes (f_T)^*M^{-1} \to \mathcal{O}_{X_T}$$

is an $\mathcal{O}_{X_T}$-homomorphism such that $u(t)$ is injective for each point t of T. Since $\mathcal{O}_S = f_* \mathcal{O}_X$ holds, f is surjective. Therefore, $X_T(t)$ is non-empty for each point t of T. Since $V_T(t)$ contains each point x of $X_T(t)$ with depth$(\mathcal{O}_{X_T(t),x}) = 0$, it is nonempty. Finally, since $I|V$ is invertible, $I_T(t)$ is nonzero. Therefore, $u(t)$ is nonzero for each point t of T. Thus, the triple (g, M, u) represents an element of $\mathcal{M}_S(T)$. Moreover, the uniqueness assertion of (10, (i)) implies that a different choice of such a pair (M, u) yields the same class in $\mathcal{M}_S(T)$. Define $C_S(T)$ by the formula,

$$C_S(T)(g, Y) = \text{class}\,(g, M, u).$$

The pair (M, u) determines the subscheme Y because the ideal of Y is equal to the image of u; so, $C_S(T)$ is injective. It is evident that the $C_S(T)$ form a natural transformation, C_S, and that the formation of C_S from f, P and I commutes with base change.

By construction, C_S carries $\underline{\text{Lin}}\ \underline{\text{Syst}}_I$ monomorphically into the subfunctor of $\mathcal{M}_S$ whose value at a locally noetherian S-scheme T is the set of equivalence classes of triples (g, M, u) such that $u(t)$ is injective for each point t of T. By (10, (ii)), every such triple (g, M, u) arises from some element (g, Y) of $\underline{\text{Lin}}\ \underline{\text{Syst}}_I(T)$. Thus, C_S carries $\underline{\text{Lin}}\ \underline{\text{Syst}}_I$ isomorphically onto this subfunctor.

Clearly, in view of (15.4), the map, $A_S^{-1} \circ C_S$, carries $\underline{\text{Lin}}\ \underline{\text{Syst}}_I$ isomorphically onto the subfunctor D_S of the functor of points of $\mathbb{P}(H)$ whose value at a locally noetherian S-scheme T is the set of S-morphisms, $h : T \to \mathbb{P}(H)$, such that $(h^*(\beta))(t)$ is injective for each point t of T. We shall now represent D_S by an open subscheme U of $\mathbb{P}(H)$; clearly, U then universally represents D_S, and so U also universally represents $\underline{\text{Lin}}\ \underline{\text{Syst}}_I$.

Let V_1 denote the set of points of $X_{\mathbb{P}(H)}$ where Coker (β) is flat over $\mathbb{P}(H)$; it is open by (GD V, 5.5). Set

$$U = \mathbb{P}(H) - (f_{\mathbb{P}(H)})[(X_{\mathbb{P}(H)} - V_1) \cup (\text{Supp}\,(\text{Ker}\,(\beta)))].$$

Since f is proper, U is an open subset of $\mathbb{P}(H)$. Moreover, clearly, a point t of $\mathbb{P}(H)$ lies in U if and only if β is injective and Coker(β) is flat over $\mathbb{P}(H)$ at each point x of $X_{\mathbb{P}(H)}$ lying over t. Therefore, since f is flat, a point t of $\mathbb{P}(H)$ lies in U if and only if $\beta(t)$ is injective (GD VII, 4.1). Consequently, an S-morphism, $h : T \to \mathbb{P}(H)$, factors through U if and only if $(h^*\beta)(t)$ is injective for each $t \in T$. Therefore, U represents D_S, and so U universally represents $\underline{\text{Lin}}\ \underline{\text{Syst}}_I$.

Finally, assume that the fibers of f are geometrically integral and that V contains each point x of X_P with depth$(I(f_P(x))_x) = 0$. We shall show that U is equal to $\mathbb{P}(H)$ or, equivalently, that $\beta(t)$ is injective for each point t of $\mathbb{P}(H)$. Since $\beta(t)$ is nonzero, it is nonzero at the generic point η of $X_{\mathbb{P}(H)}(t)$ because this scheme is integral. However, at η, the source of

$\beta(t)$ is a 1-dimensional vector space. So, $\beta(t)$ is injective at η. Since $V_{\mathbb{P}(H)}(t)$ contains each point x of $X_{\mathbb{P}(H)}(t)$ with $\operatorname{depth}(I_{\mathbb{P}(H)}(t)_x) = 0$ and since $I|V$ is invertible, η is the only point of $X_{\mathbb{P}(H)}(t)$ where $I_{\mathbb{P}(H)}(t)$ has depth 0. So, $\beta(t)$ is injective because its kernel has no point with depth 0. Thus, U is equal to $\mathbb{P}(H)$, and so $\mathbb{P}(H)$ universally represents $\underline{\text{Lin}}\ \underline{\text{Syst}}_I$.

16. Corollary: *Let S be a locally noetherian scheme, $f: X \to S$ a flat, proper morphism with geometrically integral fibers, P a locally noetherian S-scheme, and L an invertible $\mathcal{O}_{X_P}$-Module. Then, $\underline{\text{Lin}}\ \underline{\text{Syst}}_{L^{-1}}$ is universally representable by $\mathbb{P}(Q(L))$.*

Proof: Obviously, L^{-1} is isomorphic to the cokernel of an $\mathcal{O}_{X_P}$-homomorphism of locally free $\mathcal{O}_{X_P}$-Modules with finite rank, for example, the cokernel of a zero map into L^{-1}. Furthermore, since f is both open and closed, we may replace S by $f(X)$ and so assume f is surjective. Then, $\mathcal{O}_S = f_* \mathcal{O}_X$ holds universally (EGA III_2, 7.8). Hence, by (15) with X_P for V, the functor $\underline{\text{Lin}}\ \underline{\text{Syst}}_{L^{-1}}$ is universally representable by $\mathbb{P}(H(L^{-1}, \mathcal{O}_{X_P}))$. Finally, $H(L^{-1}, \mathcal{O}_{X_P})$ is universally isomorphic to $Q(L)$ by (14).

17. Remark: Let S be a locally noetherian scheme, $f: X \to S$ a flat, projective, surjective morphism with geometrically integral fibers, and P a locally noetherian S-scheme. Let V be an open subset of X_P containing each point x of X_P with $\operatorname{depth}(\mathcal{O}_{X_P(f_P(x)),x}) \leqq 1$. Let D be a closed subscheme of X_P; assume D is a divisor on V or, equivalently, its ideal $I(D)$ is invertible on V; and consider $\underline{\text{Lin}}\ \underline{\text{Syst}}_{I(D)}$. The hypotheses of (15) are satisfied; indeed, the relation $\mathcal{O}_S = f_* \mathcal{O}_X$ holds universally (EGA III_2, 7.8), and V contains each point x of X_P with $\operatorname{depth}(I(D)(f_P(x))_x) = 0$ because it contains each point x of X_P with $\operatorname{depth}(\mathcal{O}_{X_P(f_P(x)),x}) = 0$. So, $\underline{\text{Lin}}\ \underline{\text{Syst}}_{I(D)}$ is universally representable by $\mathbb{P}(H(I(D), \mathcal{O}_{X_P}))$.

An important case occurs when the fibers of f are geometrically normal. Then, the set V of smooth points of f_P contains each point x of X_P with $\operatorname{depth}(\mathcal{O}_{X_P(f_P(x)),x}) \leqq 1$ by (6), and a flat closed subscheme D of $X \times P/P$ is a divisor on V if and only if, for each point z of P, the restriction $D(z)|V(z)$ has pure codimension 1 and no embedded components (cf. proof of (7)).

It is interesting to note that, in this case, the flat, closed subschemes D of $X \times P/P$ that are divisors on V are parametrized by an open and closed subset of $\text{Hilb}_{(X \times P/P)}$; more precisely, this subset represents the functor whose value at a locally noetherian P-scheme T is the set of flat, closed subschemes of $X \times T$ that are divisors when restricted to $V \times T$ or, equivalently whose fibers are divisors on the fibers of $V \times T/T$. It clearly suffices to note that the set U of points t of $\text{Hilb}_{(X \times P/P)}$ such that the universal subscheme is a divisor at each point of $V \times \mathrm{Hilb}_{(X \times P/P)}$ lying over t is open and closed. The set U is open because f is proper and because U obviously has the form,

$$U = \mathrm{Hilb}_{(X \times P/P)} - (f_P)[(X \times \mathrm{Hilb}_{(X \times P/P)}) - (U_1 \cap (V \times \mathrm{Hilb}_{(X \times P/P)}))],$$

where U_1 is the open set on which the universal subscheme is a divisor. The set U is closed because it is closed under specialization by (7) and it is open (EGA I, 6.1.8).

3. Completeness Theorems

18. THEOREM: *Let k be a field, and X a geometrically normal, projective k-scheme. Let Z be a closed subscheme of* $\mathrm{Div}_{(X/k)}$. *If Z has finite type over k, then Z is complete.*

PROOF: We may assume k is algebraically closed, for the formation of $\mathrm{Div}_{(X/k)}$ commutes with base change and a scheme is complete if it becomes complete after an *fpqc* base change (EGA IV, 2.7.1). Moreover, since we clearly have a formula,

$$\mathrm{Div}_{((X_1 \coprod X_2)/k)} = \mathrm{Div}_{(X_1/k)} \times \mathrm{Div}_{(X_2/k)},$$

we may replace X by a connected component and so assume X is integral, hence geometrically integral because k is algebraically closed (EGA IV, 4.4.4).

We use the valuative criterion (EGA II, 7.3.8). Let R be a discrete valuation ring containing k. Let K denote the quotient field of R, and k_0 its residue class field. Let D be an effective divisor on $X \otimes K$ representing a K-point of Z. Let Y denote the closure of D in $X \otimes R$. Then, Y is flat over R by (7), so Y defines an R-point of $\mathrm{Hilb}_{(X/k)}$. We are going to prove that $Y \otimes_R k_0$ is a divisor. Then, clearly, this R-point of $\mathrm{Hilb}_{(X/k)}$ lies in the open subscheme $\mathrm{Div}_{(X/k)}$ and so also in Z. Thus, the hypotheses of the valuative criterion are fulfilled.

Set $P = \mathrm{Pic}^0_{(X/k)}$ and let L be a Poincaré sheaf on $X \times P$ (that is, a universal invertible sheaf; one exists because k is algebraically closed). Choose an ample invertible sheaf $\mathcal{O}_X(1)$, fix an integer n, and form the following coherent sheaves on $(X \times P) \otimes R$;

$$I = I(Y)_P \otimes L_R^{-1}(-n)$$
$$I^{\vee} = \underline{\mathrm{Hom}}\,(I, \mathcal{O}_{(X \times P) \otimes R}),$$

where $I(Y)$ denotes the ideal of Y.

All the hypotheses that appear in the various results we are about to apply hold by virtue of (8).

By (11, (i)), the canonical map of functors,

$$\underline{p}_2 : \underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I \to \underline{\mathrm{Hilb}}_{(X\otimes R/R)},$$

is a monomorphism. So, by (15), it is represented by a monomorphism of R-schemes,

$$p_2 : \mathbb{P}(H(I, \mathcal{O}_{X_{P\otimes R}})) \to \mathrm{Hilb}_{(X\otimes R/R)},$$

whose formation commutes with base change because the formation of p_2 does and because $\mathbb{P}(H(I, \mathcal{O}_{X_{P\otimes R}}))$ universally represents $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_I$.

By (4, (i)), the canonical map,

$$b(M) : \underline{\mathrm{Hom}}\,(I, \mathcal{O}_{X_{P\otimes R}}) \otimes (f_{P\otimes R})^* M \to \underline{\mathrm{Hom}}\,(I, (f_{P\otimes R})^* M),$$

is injective for each coherent $\mathcal{O}_{P\otimes R}$-Module M. So, by (14), there is a canonical surjection,

$$c : H(I, \mathcal{O}_{X_{P\otimes R}}) \to Q(I^{\check{}}).$$

The composition of the closed embedding $\mathbb{P}(c)$ and the monomorphism p_2 is a key monomorphism,

$$r : \mathbb{P}(Q(I^{\check{}})) \to \mathrm{Hilb}_{(X\otimes R/R)}.$$

Consider the generic fiber $r \otimes K$ of r. It factors through the composition,

$$\mathbb{P}(Q(I^{\check{}}) \otimes K) \xrightarrow{Q(b(K))} \mathbb{P}(Q(\underline{\mathrm{Hom}}\,(I_K, \mathcal{O}_{X_{P\otimes K}}))) \xrightarrow{\mathbb{P}(c_K)} \mathbb{P}(H(I_K, \mathcal{O}_{X_{P\otimes K}})),$$

by (14.5). Since K is flat over R, the canonical map,

$$b(K) : (I^{\check{}}) \otimes K \to \underline{\mathrm{Hom}}\,(I_K, \mathcal{O}_{X_{P\otimes K}}),$$

is an isomorphism (EGA 0_I, 5.7.6); hence $Q(b(K))$ is an isomorphism. Now, $I(Y) \otimes_R K$ is isomorphic to the ideal of D because Y is flat over R (or because K is). So, $I(Y) \otimes_R K$ is invertible because D is a divisor. Hence, I_K is invertible. Therefore, c_K is an isomorphism by (14), and so $\mathbb{P}(c_K)$ is an isomorphism. Consequently, by (11, (ii)), $r \otimes K$ factors through $\mathrm{Div}_{(X\otimes K/K)}$ and yields a cartesian diagram,

$$\begin{tikzcd} \mathbb{P}(Q(I^{\check{}}) \otimes K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Div}_{(X\otimes K/K)} \arrow[d] \\ P \otimes K \arrow[r, "q\otimes K"] & \mathrm{Pic}_{(X\otimes K/K)}, \end{tikzcd}$$

where q is the inclusion of P in $\mathrm{Pic}_{(X/k)}$. Since q is an open (and closed) embedding, the image of $\mathbb{P}(Q(I^{\check{}}) \otimes K)$ in $\mathrm{Div}_{(X\otimes K/K)}$ is equal to an open (and closed) subset U of $\mathrm{Div}_{(X \otimes K/K)}$, and we have the relation,

$$\text{(18.1)} \qquad \dim(U) = \dim(\mathbb{P}(Q(I^{\check{}}) \otimes K)).$$

Since $\mathrm{Div}_{(X \otimes K/K)}$ is open in $\mathrm{Hilb}_{(X \otimes K/K)}$, the set U is open in $\mathrm{Hilb}_{(X \otimes K/K)}$.

Consider the special fiber $r \otimes k_0$ of r. It is a monomorphism because r is. By Chevalley's Theorem (GD V, 4.6), the image, $(r \otimes k_0)(\mathbb{P}(Q(I\ ) \otimes k_0))$, contains an open set, V, of its closure in $\mathrm{Hilb}_{(X \otimes k_0/k_0)}$.

Since $\mathcal{O}_X(n)$ and L are each locally free with finite rank, we clearly have a canonical isomorphism,

$$I^{\check{}} = [\underline{\mathrm{Hom}}\,(I(Y)_P, \mathcal{O}_{X_{P \otimes R}}) \otimes L_R](n).$$

Since $\mathrm{Pic}_{(X/k)}$ is locally of finite type over k (SGA6 XIII, 3.1), P is of finite type over k (FGA, 236–02). Choose n so large that $R^q(f_{P \otimes R}(I^{\check{}})$ vanishes for each $q > 0$ (EGA III, 2.2.1). Then, $Q(I^{\check{}})$ is locally free (EGA III_2, 7). Since P is geometrically irreducible and R is irreducible, $P \otimes R$ is irreducible (EGA IV, 4.5.8, (i)). So, $\mathbb{P}(Q(I^{\check{}}))$ is also irreducible. Follow r by the projection from $\mathrm{Hilb}_{(X \otimes R/R)}$ to $\mathrm{Hilb}_{(X/k)}$, and let H denote the closure in $\mathrm{Hilb}_{(X/k)}$ of the image of this composition. Then, H is also irreducible. Since k is algebraically closed, H is geometrically irreducible (EGA IV, 4.4.4). Furthermore, $H \otimes k_0$ contains V because the projection from $\mathrm{Hilb}_{(X \otimes k_0/k_0)}$ to $\mathrm{Hilb}_{(X/k)}$ factors through $\mathrm{Hilb}_{(X \otimes R/R)}$. Similarly, $H \otimes K$ contains U.

Since U is open in $\mathrm{Hilb}_{(X \otimes K/K)}$, it is open in $H \otimes K$; since $H \otimes K$ is irreducible, U and $H \otimes K$ have the same dimension; so, by (18.1), $H \otimes K$ and $\mathbb{P}(Q(I^{\check{}}) \otimes K)$ have the same dimension. Since $\mathbb{P}(Q(I^{\check{}}) \otimes k_0)$ is irreducible, and since $(r \otimes k_0)^{-1}(V)$ is open in $\mathbb{P}(Q(I^{\check{}}) \otimes k_0)$, they have the same dimension; since $r \otimes k_0$ is a monomorphism, V and $(r \otimes k_0)^{-1}(V)$ have the same dimension; so V and $\mathbb{P}(Q(I^{\check{}}) \otimes k_0)$ have the same dimension. Since $Q(I^{\check{}})$ is locally free with finite rank, $\mathbb{P}(Q(I^{\check{}}) \otimes K)$ and $\mathbb{P}(Q(I^{\check{}}) \otimes k_0)$ have the same dimension (namely, $\dim(P) + \mathrm{rank}\,(Q(I^{\check{}})) - 1$). Therefore, $H \otimes K$ and V have the same dimension; hence, so do $H \otimes k_0$ and V, for $H \otimes K$ and $H \otimes k_0$ obviously do. Consequently, since $H \otimes k_0$ is irreducible and closed and contains V, the closure of V in $\mathrm{Hilb}_{(X \otimes k_0/k_0)}$ is equal to $H \otimes k_0$. Thus, since V is open in its closure in $\mathrm{Hilb}_{(X \otimes k_0/k_0)}$, it is open in $H \otimes k_0$.

Let k_1 be an algebraically closed field containing k-isomorphic copies of K and k_0. Then, since $H \otimes k_1$ is irreducible and since any two nonempty open subsets of an irreducible set intersect, we have the relation,

$$(V \otimes_{k_0} k_1) \cap (U \otimes_K k_1) \neq \phi.$$

Let E be a closed subscheme of $X \otimes k_1$ corresponding to a point in the intersection. Then, $I(E)$, the ideal of E, is isomorphic to $(I \otimes_R k_0) \otimes_{k_0} k_1$

since E corresponds to a k_1-point of V. So, we have an isomorphism,

$$I(E) \cong (((I(Y) \otimes_R k_0) \otimes_{k_0} k_1) \otimes L^{-1}(-n)).$$

On the other hand, $I(E)$ is invertible because E corresponds to a k_1-point of U. Hence, $(I(Y) \otimes_R k_0) \otimes_{k_0} k_1$ is invertible. However, $(I(Y) \otimes_R k_0) \otimes_{k_0} k_1$ is isomorphic to $I((Y \otimes_R k_0) \otimes_{k_0} k_1)$ because Y is flat over R. Therefore, $(Y \otimes_R k_0) \otimes_{k_0} k_1$ is a divisor; hence, so is $Y \otimes_R k_0$.

19. THEOREM: *Let k be a field, and X a geometrically normal, projective k-scheme. Let P be a closed subscheme of* $\mathrm{Pic}_{(X/k)}$. *If P has finite type over k, then P is complete.*

PROOF: We may assume k is algebraically closed, for the formation of $\mathrm{Pic}_{(X/k)}$ commutes with base change and a scheme is complete if it becomes complete after an *fpqc* base change (EGA IV, 2.7.1). Since we clearly have a formula,

$$\mathrm{Pic}_{((X_1 \coprod X_2)/k)} = \mathrm{Pic}_{(X_1/k)} \times \mathrm{Pic}_{(X_2/k)},$$

we may replace X by a connected component and so assume X is integral, hence geometrically integral because k is algebraically closed (EGA IV, 4.5.14).

Since k is algebraically closed, there is a Poincaré sheaf L' on $X \times \mathrm{Pic}_{(X/k)}$. Set $L = L'|X \times P$. Let $\mathcal{O}_X(1)$ be an ample invertible sheaf on X. By (EGA III, 2.2.1), there is an integer n such that the conditions,

(19.1)
$$R^q p_*(L(n)) = 0 \qquad \text{for } q > 0$$
$$p_*(L(n))_x \neq 0 \qquad \text{for each } x \in P,$$

hold, where $p: X \times P \to P$ denotes the projection, because P is of finite type over k.

By (16) with $S = \mathrm{Spec}(k)$, the scheme $\mathbb{P}(Q(L(n)))$ represents the functor $\underline{\mathrm{Lin}}\ \underline{\mathrm{Syst}}_{L^{-1}(-n)}$. By (11, (ii)) with $J = \mathcal{O}_X(-n)$, we obtain a cartesian diagram,

$$\begin{tikzcd} \mathbb{P}(Q(L(n))) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & \mathrm{Div}_{(X/k)} \arrow[d] \\ P \arrow[r] & \mathrm{Pic}_{(X/k)}. \end{tikzcd}$$

Hence, $\mathbb{P}(Q(L(n)))$ is embedded in $\mathrm{Div}_{(X/k)}$ as a closed subscheme of finite type. It is therefore complete by (18). Since conditions (19.1) hold, $Q(L(n))$ is locally free with finite nonvanishing rank on P; hence, the structure map $\mathbb{P}(Q(L(n))) \to P$ is surjective. Therefore, by (EGA I, 3.8.2, (iv)), P is complete.

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