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COMPUTATIONS OF IWASAWA INVARIANTS AND K_2

Alan Candiotti

Introduction

Let l be an odd prime, and let k be a number field which contains μ_l, the group of l-th roots of unity. Let W be the group of l^n-th roots of unity, $1 \leqq n < \infty$. Then if $K = k(W)$, we know that K/k is Galois and that $\Gamma = \mathrm{Gal}(K/k) \simeq \boldsymbol{Z}_l$. Corresponding to the subgroups $\Gamma^n \simeq l^n\boldsymbol{Z}_l$, there is a unique tower of fields

$$k = k_0 \subset k_1 \subset \cdots \subset K \tag{1}$$

such that $\bigcup_{n=1}^{\infty} k_n = K$ and $G(k_n/k_{n-1}) \simeq \boldsymbol{Z}/l\boldsymbol{Z}$.

Let A_n be the l-primary subgroup of the ideal class group of k_n and define e_n to be the integer such that the order of A_n is l^{e_n}. Then Iwasawa [6] has proven:

THEOREM: *There exist integers μ_c, λ_c, v_c depending only on k and l such that $e_n = \mu_c l^n + \lambda_c n + v_c$ for sufficiently large n.*

By Steinitz's theorem, the ideal class group of a finite extension of $\boldsymbol{Q}$ may be identified with the reduced K_0 of its ring of algebraic integers [8]. We are then led to consider the behavior of the K_2 of the ring of integers in the tower (1). Garland [3] has shown that these groups are finite. Let $B_n = K_2 \mathcal{O}_n$ where $\mathcal{O}_n$ is the ring of integers of k_n and define d_n to be the integer so that the order of B_n is l^{d_n}. Coates [1] has shown:

THEOREM: *There exist another set of integers μ_R, λ_R, v_R, depending only on k and l so that $d_n = \mu_R l^n + \lambda_R n + v_R$ for sufficiently large n.*

This theorem is proven in [1] for the tame kernel, which we denote $R_2 k_n$ rather than for $K_2 \mathcal{O}_n$, but, by a recent theorem of Quillen [9], these two groups are canonically isomorphic for finite extensions of $\boldsymbol{Q}$. We will also deal directly with the tame kernel.

In this paper we will study the invariants λ_c, μ_c, λ_R, μ_R. It can be shown that $\mu_c = \mu_R$ (a result of little interest since both are conjectured to be 0). It is also known that $\lambda_R \geqq \lambda_c$, and we give in this paper a class of fields for which this inequality is strict. In the special case in which there is only one prime of K above l, it turns out that $\lambda_c = \lambda_R$, a result which has been known for some time.

It is natural to make a further restriction on k, namely (J), to assume that k is a totally imaginary quadratic extension of a totally real field, which we call k^+. This implies directly that the same holds for k_n for all n. This additional assumption implies that there is an action of complex conjugation, which we denote by σ, on the groups A_n and $R_2 k_n$, and that the action is independent of the embedding of k in $\boldsymbol{C}$. If we set $A_n^+ = (1+\sigma)A_n$ and $A_n^- = (1-\sigma)A_n$, noting that l is odd, we get $A_n = A_n^+ \oplus A_n^-$. Similarly we have $R_2 k_n = (R_2 k_n)^+ \oplus (R_2 k_n)^-$. These decompositions give rise to invariants μ_c^+, λ_c^+, μ_R^-, λ_R^-, etc. It is shown in [1] that we always have $\lambda_c^- = \lambda_R^+$ and $\lambda_c^+ \leqq \lambda_R^-$.

Much of this paper is devoted to the study of λ_c^+ and λ_R^-. (Entirely different techniques have been used for λ_c^- and λ_R^+, cf. [2]). We obtain a lower bound for λ_R^- which yields many examples of fields for which $\lambda_R^- > 0$. By contrast, it has been conjectured that λ_C^+ is always 0. [4]. We have been able to verify this conjecture for a number of fields. For example, let $l = 3$ and $k = \boldsymbol{Q}(\sqrt{d}, \sqrt{-3})$ for $d > 0$ and $d \equiv 2 \pmod 3$. Let ε be the fundamental unit of $k^+ = \boldsymbol{Q}(\sqrt{d})$. Let A^+ and A^- be the 3-primary subgroups of the ideal class groups of $\boldsymbol{Q}(\sqrt{d})$ and $\boldsymbol{Q}(\sqrt{-3d})$ respectively. Then we obtain

THEOREM: *Assume that* (1) A^- *is cyclic,* (2) A^+ *has exponent* 3, *and* (3) $k(\sqrt[3]{\varepsilon})$ *is not embeddable in a* Z_3*-extension of* k. *Then* $\lambda_c^+ = \mu_c^+ = \lambda_R^- = \mu_R^- = 0$.

A similar result has been obtained independently by R. Greenberg [4]. This theorem is useful in practice because we show how (3) can be decided by computations with norm residue symbols. This fact, although long known in principle, has not been referred to explicitly in the literature (Greenberg [4] uses a more ad hoc procedure). In this connection, it is interesting to note that we have found, apparently for the first time, examples of fields (e.g. $k = \boldsymbol{Q}(\sqrt{254}, \sqrt{-3})$ for which (1) and (2) are satisfied with both A^+ and A^- non-trivial, but for which (3) fails. For these fields it remains an open problem to decide whether $\lambda_c^+ = \mu_c^+ = 0$.

Finally, we use our general methods to obtain some information concerning the size of the 3-primary subgroup of the tame kernel of a quadratic field and, in fact, compute the order of the 3-primary part of the tame kernel of all imaginary quadratic fields with descriminant d satisfying $|d| < 200$, except $d = -107$.

The index of the wild kernel

Let F be a number field, $[F:\boldsymbol{Q}] < \infty$, and let $X' \subset K_2 F$ be the intersection of the kernels of the Hilbert symbols at all primes of F. Let X be the intersection of the kernels of all tame symbols. We then have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d] & K_2F \arrow[r, "\lambda_{\text{wild}}"] \arrow[d, equal] & \Sigma \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & X \arrow[r] & K_2F \arrow[r, "\lambda_{\text{tame}}"] & T \arrow[r] & 0 \end{tikzcd} \tag{1}$$

where $\Sigma = \operatorname{Im}(\lambda_{\text{wild}})$ and $T = \operatorname{Im}(\lambda_{\text{tame}})$. In all cases $T = \sum_v \dot{k}_v$ where k_v is the residue class field of F at v and v ranges over all non-archimedean primes of F. For each non-archimedean prime v of F, let μ_v be the group of roots of unity in the completion F_v. Let $\mu'_v = \mu_v(p)$ where p is the unique rational prime such that $v|p$. For v real, let $\mu'_v = (\pm 1)$.

Using Moore's theorem, we have

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] & \bigoplus\limits_{\substack{v \text{ non-}\\ \text{archimedean}}} \dot{k}_v \arrow[r] & 0 & \\ 0 \arrow[r] & \Sigma \arrow[r] \arrow[u, "\alpha"] & \bigoplus\limits_{\text{all } v} \mu_v \arrow[r] \arrow[u] & \mu_F \arrow[r] & 0 \end{tikzcd} \tag{2}$$

By (1) we have $X/X' \simeq \operatorname{Ker}\alpha$. From (2) we get the exact sequence

$$0 \to \operatorname{Ker}\alpha \to \bigoplus_v \mu'_v \to \mu_F \to 0$$

Thus

$$|X/X'| = \frac{1}{|\mu(F)|}\Big(\prod_v |\mu'_v|\Big).$$

Let n_0 be the smallest integer such that K/k_{n_0} is totally ramified at all primes dividing l. Let s be the number of divisors of l in k_{n_0} (hence in K) and let s^+ be the number of divisors of l in $k^+_{n_0}$.

LEMMA 1: *Let $\mathscr{P}$ be a prime of k^+_0 dividing l. Let $\mathfrak{P}$ be a prime of k^+_n lying above $\mathscr{P}$ (for any n). Then $\mathscr{P}$ splits in the extension k/k^+ if and only if $\mathfrak{P}$ splits in k_n/k^+_n.*

PROOF: We have, for all n, $k_n = k^+_n(\mu_l)$. Then $\mathscr{P}$ splits in k/k^+ if and only if $\mu_l \subset (k^+)_{\mathscr{P}}$, and $\mathfrak{P}$ splits in k_n/k^+_n if and only if $\mu_l \subset (k^+_n)_{\mathfrak{P}}$. Clearly, if $\mu_l \subset (k^+)_{\mathscr{P}}$, then $\mu_l \subset (k^+_n)_{\mathfrak{P}}$, so one implication is obvious. On the other hand, since $\mu_l \subset k$ and $[k:k^+] = 2$, we have $[(k^+)_{\mathscr{P}}(\mu_l):(k^+)_{\mathscr{P}}]$ divides 2. Suppose $\mu_l \subset (k^+_n)_{\mathfrak{P}}$. Since $[k^+_n : k^+] = l^n$, we have $[(k^+_n)_{\mathfrak{P}}:(k^+)_{\mathscr{P}}]$ divides l^n. Then, since $(k^+)_{\mathscr{P}} \subset (k^+)_{\mathscr{P}}(\mu_l) \subset (k^+_n)_{\mathfrak{P}}$ we have $(k^+)_{\mathscr{P}}(\mu_l) = (k^+)_{\mathscr{P}}$ and $\mu_l \subset (k^+)_{\mathscr{P}}$.

Let $\mathscr{R}_2 k_n$ be the l-primary part of the wild kernel for k_n. Then the maps $i_{n,m}: R_2 k_n \to R_2 k_m$ induce maps $\overline{i_{n,m}}: R_2 k_n/\mathscr{R}_2 k_n \to R_2 k_m/\mathscr{R}_2 k_m$.

PROPOSITION 2: *For $m \geqq n \geqq n_0$, the map $\overline{i_{m,n}}$ is injective.*

PROOF: Let $\mathscr{P}$ be a prime dividing l in k_n and let $\mathfrak{P}$ be the unique prime lying above $\mathscr{P}$ in k_m. Since $n > n_0$ we have $|\mu'_{\mathscr{P}}| = l^{q+n}$ and $|\mu'_{\mathfrak{P}}| = l^{q+m}$. Let $\lambda_{\mathfrak{P}}: K_2 k_m \to \mu_{\mathfrak{P}}$ and $\lambda_{\mathscr{P}}: K_2 k_n \to \mu_{\mathscr{P}}$ be the Hilbert symbols. Let $j: \mu_{\mathscr{P}} \to \mu_{\mathfrak{P}}$ be the inclusion and $r: \mu_{\mathfrak{P}} \to \mu_{\mathscr{P}}$ be the homomorphism given by raising elements to the power l^{m-n}. By a lemma proved by Tate, the following diagram commutes:

$$\begin{array}{ccc} K_2 k_m & \xrightarrow{\lambda_{\mathfrak{P}}} & \mu_{\mathfrak{P}} \\ \uparrow{\scriptstyle i_{n,m}} & & \uparrow{\scriptstyle \phi_{n,m}} \\ K_2 k_n & \xrightarrow{\lambda_{\mathscr{P}}} & \mu_{\mathscr{P}} \end{array}$$

where $\varphi_{n,m} \circ r = j \circ N$ and N is the norm map from $\mu_{\mathfrak{P}}$ to $\mu_{\mathscr{P}}$. We notice immediately that on $\mu_{\mathfrak{P}}$ we have $N = r$. Let $\xi \in R_2 k_n$, $\xi \notin \mathscr{R}_2 k_n$. Then for some prime $\mathscr{P}$ of k_n, dividing l, we have $\lambda_{\mathscr{P}}(\xi) \neq 1$. Suppose $\lambda_{\mathfrak{P}}(i_{n,m}(\xi)) = 1$. Then $\varphi_{n,m}(\lambda_{\mathscr{P}}(\xi)) = 1$. Let $\theta \in \mu_{\mathfrak{P}}$ so that $r(\theta) = \lambda_{\mathscr{P}}(\xi)$. Then $\varphi_{n,m} \circ r(\theta) = 1$, hence $j \circ N(\theta) = 1$ and $N(\theta) = r(\theta) = \lambda_{\mathscr{P}}(\xi) = 1$. This is a contradiction.

PROPOSITION 3: *For $n \geqq n_0$, $|R_2 k_n/\mathscr{R}_2 k_n| = (l^{q+n})^{s-1}$ and $|(R_2 k_n)^-/(\mathscr{R}_2 k_n)^-| = (l^{q+n})^{s^+-1}$.*

PROOF: We have $|R_2 k_n/\mathscr{R}_2 k_n| = (1/|\mu(k_n)(l)|)\prod_{i=1}^{s} |\mu_{\mathscr{P}_i}|$. Since $n \geqq n_0$, for each i we have $|\mu'_{\mathscr{P}_i}| = l^{q+n}$. Thus, since $|\mu(k_n)(l)| = l^{q+n}$, we have $|R_2 k_n/\mathscr{R}_2 k_n| = (l^{q+n})^{s-1}$.

Now, in k_n^+ we first observe that $k_n^+/k_{n_0}^+$ is totally ramified at all primes dividing l. Thus the number of primes dividing l in k_n^+ is also s^+. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_t$ be the primes above l in k_n^+ which split in k_n and let $\mathfrak{P}_{t+1}, \cdots, \mathfrak{P}_{s^+}$ be those which do not. Then $\mu_{q+n} \subset (k_n^+)_{\mathfrak{P}_i}$ for $i = 1, \cdots, t$ and $\mu_l \not\subset (k_n^+)_{\mathfrak{P}_i}$ for $i = t+1, \cdots, s^+$. The number of primes of k_n lying above l is then $(s^+)+t$. Then we have $|R_2 k_n/\mathscr{R}_2 k_n| = (l^{q+n})^{(s^+)+t-1}$. $(R_2 k_n)^+/(\mathscr{R}_2 k_n)^+$ is isomorphic to $R_2(k_n^+)/\mathscr{R}_2(k_n^+)$ which is isomorphic to

$$\prod_{i=1}^{s^+} (\mu'_{\mathfrak{P}_i}) = \prod_{i=1}^{t} (\mu'_{\mathfrak{P}_i}),$$

so its order is $(l^{q+n})^t$ and the conclusion follows.

We may define the group $R_2 K/\mathscr{R}_2 K = \varinjlim R_2 k_n/\mathscr{R}_2 k_n$ using the maps $\overline{i_{n,m}}$.

PROPOSITION 4: *$(R_2 K)/\mathscr{R}_2 K) \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^{s-1}$ and $(R_2 K)^-/(\mathscr{R}_2 K)^- \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^{s^+-1}$.*

PROOF: From the exact sequence

$$0 \to R_2 k/\mathscr{R}_2 k_n \to \prod_{\mathscr{P}/l} \mu'_{\mathscr{P}} \xrightarrow{\alpha} \mu(k_n)(l) \to 0$$

since the map $\alpha: \prod_{\mathscr{P}|l} \mu'_{\mathscr{P}} \to \mu(k_n)(l)$ is given by $\alpha(x_1, \cdots, x_s) = x_1 \cdots x_s$, we see that $R_2 k_n/\mathscr{R}_2 k_n$ is isomorphic to $(\mathbf{Z}/l^{n+q}\mathbf{Z})^{s-1}$ and, from similar considerations, that $(R_2 k_n)^-/(\mathscr{R}_2 k_n)^-$ is isomorphic to $(\mathbf{Z}/l^{n+q}\mathbf{Z})^{s^+-1}$. The fact that the maps $\overline{i_{n,m}}$ are injective yields the desired result.

If k is also Galois over $\mathbf{Q}$, our assumption (J) is equivalent to the fact that k^+ is also Galois. Then the same holds for k_n and k_n^+, so that either $s = s^+$ or $s = 2s^+$, the former if the primes of k^+ do not split in k, the latter if they do.

COROLLARY 1: *If $k/\mathbf{Q}$ is Galois and the primes of k^+ dividing l split in k, then for $n \geqq n_0$*

$$(R_2 k_n)^-/(\mathscr{R}_2 k_n)^- \simeq (\mathbf{Z}/l^{q+n}\mathbf{Z})^{s/2-1}.$$

COROLLARY 2: *If $k/\mathbf{Q}$ is Galois and the primes of k^+ dividing l do not split in k, then for $n \geqq n_0$*

$$(R_2 k_n)^-/(\mathscr{R}_2 k_n)^- \simeq (\mathbf{Z}/l^{q+n}\mathbf{Z})^{s-1}.$$

If we consider the Iwasawa invariants of the extension K/k, we get the following:

THEOREM 1: $\lambda_R^- \geqq s^+ - 1$.

PROOF: This is clear from the fact that $|(R_2 k_n)^-/(\mathscr{R}_2 k_n)^-| = (l^{n+q})^{s^+-1}$ for $n \geqq n_0$.

COROLLARY 1: *Let F be a Galois field satisfying (J), $[F:\mathbf{Q}] < \infty$, such that $F/\mathbf{Q}$ is unramified at l. Let d^+ be the number of primes of F^+ which divide l. Let $k = F(\mu_l)$ and let K/k be the cyclotomic Z_l-extension. Then $(R_2 K)^-/(\mathscr{R}_2 K)^- \simeq (\mathbf{Q}_l/\mathbf{Z}_l)^{d^+-1}$ and $\lambda_R^- \geqq d^+ - 1$.*

PROOF: Since l is unramified in F, we have k/F completely ramified at all primes over l, and $[k:F] = l-1$. Let $\mathscr{P}$ be a prime of F^+ which divides l. Then if $\mathscr{P}$ splits in F we find that σ does not fix either of the primes above $\mathscr{P}$ in F. Then certainly $\mathscr{P}$ cannot split in k^+/F^+. Thus k^+/F^+ must be totally ramified at $\mathscr{P}$ and k/k^+ split at the prime above $\mathscr{P}$. On the other hand, if $\mathscr{P}$ fails to split in F/F^+, since $\mathscr{P}$ is totally ramified in k/F, we find that $\mathscr{P}$ fails to split in k/F^+. In either case, there is a unique prime above $\mathscr{P}$ in k^+. Since $k/\mathbf{Q}$ has ramification $l-1$ at primes dividing l, we have $\mu_{l^2} \not\subset k_{\mathscr{P}}$ for any prime $\mathscr{P}$ dividing l, so the extension K/k is totally ramified at all primes above l (i.e. $n_0 = 0$). Thus $d^+ = s^+$ and the result follows.

COROLLARY 2: *If F is a real quadratic field in which l splits, then* $(R_2K)^-/(\mathscr{R}_2K)^- \simeq \boldsymbol{Q}_l/\boldsymbol{Z}_l$ *and* $\lambda_R^- \geqq 1$.

The invariants μ_c^+ and λ_c^+

In this section we will describe some classes of fields for which $\lambda_c^+ = \mu_c^+ = 0$, including some examples for which $\lambda_R^- > 0$. Let F be a totally real field, $[F:\boldsymbol{Q}] < \infty$, and let S be the set of primes of F dividing l. Let K be the cyclotomic $\boldsymbol{Z}_l$-extension of F and M be the maximal abelian l-ramified l-extension of F. Let J be the group of ideles of F and A and A_S the l-primary subgroups of the ideal class group and l-ideal class group of F respectively.

We define $J^S = \{I \in J : (I)_v = 1 \text{ for } v \in S, (I)_v \in U_v \text{ for } v \notin S\}$ (there is no requirement on $(I)_v$ for v archimedean). Then, by class field theory, we have $G(M/k) \simeq J/\overline{\dot{F}J^S}(l)$ and

$$0 \to \dot{F}J_S/\overline{\dot{F}J^S}(l) \to J/\overline{\dot{F}J^S}(l) \to A_S \to 0$$

The term on the left is isomorphic to $\prod_{v \in S} \dot{F}_v/\bar{U}_S(l)$ where U_S is the group of S-units of F, embedded diagonally in $\prod_{v\in S} \dot{F}_v$. Let $U_l = \prod_{v \in S} U_v$. Then we have

PROPOSITION 5: *Assume that l does not divide the class number of F, that $\mu_l \not\subset F_v$ for any $v \in S$, and that the fundamental units of F are linearly independent in $U_l/(U_l)^l$. Then $M = K$.*

PROOF: If the class number of F is prime to l, we have $A_S = 0$, hence $G(M/F) \simeq \prod_{v\in S} \dot{F}_v/\bar{U}_S(l)$, and also that $\prod_{v \in S} \dot{F}_v/\bar{U}_{S1} \simeq U_l/\bar{U}_1$ where U is the group of units of F, embedded in U_l. Let $n = [F:\boldsymbol{Q}]$. Since F is totally real, we have $U = (\pm 1) \times U'$ where U' is free on $n-1$ generators $\varepsilon_1, \cdots, \varepsilon_{n-1}$. Since $\mu_l \not\subset F_v$ for all $v \in S$, we know that U_l has no l-torsion. From the exact sequence

$$0 \to \bar{U} \to U_l \to U_l/\bar{U} \to 0$$

we deduce that if $\varepsilon_1, \cdots, \varepsilon_{n-1}$ are linearly independent in $U_l/(U_l)^l$, then $U_l/\bar{U}$ has no l-torsion. Since $U_l(l)$ is a free $\boldsymbol{Z}_l$-module of rank n, and since rank $(\bar{U}) = n-1$, we have $U_l/\bar{U} \simeq \boldsymbol{Z}_l$. Hence $G(M/F) \simeq \boldsymbol{Z}_l$ and, since $K \subset M$, we have $K = M$.

We note that if the class number of $F(\mu_l)$ is also prime to l, then we must have $\{\varepsilon_1, \cdots, \varepsilon_{n-1}\}$ linearly independent in $U_l/(U_l)^l$. Otherwise, suppose

$$\beta = \prod_{i=1}^{n-1} \varepsilon_i^{m_i} \in (U_l)^l.$$

Then $F(\mu_l)$ $(\sqrt[l]{\beta})$ gives an unramified extension of $F(\mu_l)$ which cannot exist.

Let $F = k_0 \subset k_1 \subset \cdots \subset K$ be the intermediate fields in the $\boldsymbol{Z}_l$-extension K/F. Let A_n be the l-primary part of the ideal class group of k_n. Then

THEOREM 2: *Under the hypothesis of Proposition* 5, $A_j = (0)$ *for all j.*

PROOF: Since the class number of F is prime to l and $[k_1 : F] = l$, k_1/F must ramify at some prime above l(K/F is l-ramified). Then K/F is totally ramified at that prime $\mathscr{P}_1$. Let L_j be the maximal abelian unramified l-extension of k_j, and let L'_j be the maximal subfield of L_j which is abelian over F. Then $k_j \subset L'_j \subset L_j$. Since L'_j must be l-ramified over F, we have $L'_j \subset M$. But, by Proposition 1, $M = K$, so $L'_j \subset K$. Then L'_j/k_j must be both unramified and totally ramified at the prime of k_j lying over $\mathscr{P}_1$, so $L'_j = k_j$. Let γ be a generator of $G(k_j/F)$. We then have $G(L_j/k_j) \simeq A_j$ and $G(L'_j/k_j) \simeq A_j/(\gamma-1)A_j$. Since $L'_j = k_j$, we have $A_j = (\gamma-1)A_j$, so $A_j = (0)$.

COROLLARY 1: *If* $F = \boldsymbol{Q}$, *the class number of* k_n *is prime to l when l is odd.* (This result was first proven by Iwasawa) [5].

PROOF: Since l is odd, we have $\mu_l \not\subset \boldsymbol{Q}_l$.The other hypotheses of proposition 5 are verified trivially for $\boldsymbol{Q}$.

COROLLARY 2: *Let F be a real quadratic field. Let* ε *be the fundamental unit of F and assume that* $\varepsilon \notin (F_v)^l$ *for some v dividing l, and that if* $l = 3$, *the discriminant* $d \not\equiv -3 \pmod 9$. *Assume that the class number of F is prime to l. Then* $A_n = (0)$ *for all n.*

PROOF: For $l > 3$, we have $\mu_l \not\subset F_v$ for $v|l$. For $l = 3$, the condition $d \not\equiv -3 \pmod 9$ guarantees that $\mu_3 \not\subset F_v$ for $v|3$. The rest of the hypotheses of Theorem 2 are assumed, hence the result follows.

We remark that if $l = 3$ and $k = F(\mu_3)$, then $k^+ = F$, so we have shown that $\mu_c^+ = \lambda_c^+ = 0$.

We note that if $F = \boldsymbol{Q}(\sqrt{d})$, then $F(\mu_3) = \boldsymbol{Q}(\sqrt{d}, \sqrt{-3})$. Then

$$A_{F(\mu_3)} \simeq A_F \oplus A_{\boldsymbol{Q}(\sqrt{-3d})},$$

so if both $\boldsymbol{Q}(\sqrt{d})$ and $\boldsymbol{Q}(\sqrt{-3d})$ have class numbers prime to 3, then we have $\varepsilon \notin (F_v)^3$ for some v dividing 3. We note that if $F = \boldsymbol{Q}(\sqrt{7})$ $(d = 28)$, $\varepsilon = 8 + 3\sqrt{7}$. $\boldsymbol{Q}(\sqrt{7})$ has class number 1 and we verify either directly or by noting that 3 does not divide the class number of $\boldsymbol{Q}(\sqrt{-21})$, that $\varepsilon \notin (F_v)^3$ for $v|3$. We conclude that $\mu_c^+ = \lambda_c^+ = 0$ for this field. On the other hand, we have shown in Theorem 1 that $\lambda_R^- \geqq 1$.

We will now investigate a special situation, when $l = 3$ and $k = \boldsymbol{Q}(\sqrt{d}, \sqrt{-3})$ where d is a positive square-free integer > 1, and there is only one prime of k dividing 3. It is easy to see that this happens exactly when $d \equiv 2 \pmod 3$ or $d = 3(3m+1)$ for some positive integer m. We have $k^+ = \boldsymbol{Q}(\sqrt{d})$ and we let $k^- = \boldsymbol{Q}(\sqrt{-3d})$. Let A_0^+, A_0^-, and A_0 be the 3-primary parts of the ideal class groups of k^+, k^- and k respectively. Then let σ and τ be the non-trivial elements of $G(k/k^+)$ and $G(k/k^-)$ respectively. Let K be the cyclotomic $\boldsymbol{Z}_3$-extension of k and F be the extension obtained from K by adjoining all 3^n-th roots of 3-units of k $(1 \leqq n < \infty)$. Let F_0 be the maximal abelian extension of k contained in F, P_0 the compositium of all $\boldsymbol{Z}_3$-extensions of k, and M_0 the maximal abelian 3-ramified 3-extension of k.

REMARK: Since l is odd, we see that the natural maps

$$R_2 k^+ \to (R_2 k)^\sigma, \qquad A_0^+ \to (A_0)^\sigma$$

$$R_2 k^- \to (R_2 k)^\tau, \qquad A_0^- \to (A_0)^\tau$$

are all isomorphisms and, since the class number of $\boldsymbol{Q}(\sqrt{-3})$ is 1 and $R_2 \boldsymbol{Q}(\sqrt{-3}) = 0$ for all l [11], we see that, in fact

$$R_2 k^+ \simeq (R_2 k)_{(\tau+I)}, \qquad A_0^+ \simeq (A_0)_{(\tau+I)}$$

$$R_2 k^- \simeq (R_2 k)_{(\sigma+I)}, \qquad A_0^- \simeq (A_0)_{(\sigma+I)}$$

Since $G(k/\boldsymbol{Q})$ acts on $G(F_0/k)$, $G(P_0/k)$ and $G(M_0/k)$ in a natural way and $\tau^2 = 1$, we can decompose these groups into eigenspaces. We will write $X = X^+ \oplus X^-$ where X^+ will denote the eigenspace $X_{(\tau+I)}$ and X^- the space $X_{(\tau-I)}$. Let F_0^-, P_0^-, M_0^- be the fixed fields of $G(F_0/k)^+$, $G(P_0/k)^+$ and $G(M_0/k)^+$ respectively. Then we have $G(P_0^-/k) = G(P_0/k)^-$ and $G(M_0^-/k) = G(M_0/k)^-$. Clearly we have $P_0^- \subseteq M_0^-$. It follows from Iwasawa's work [7] that we also have $P_0^- \subseteq F_0^-$.

PROPOSITION 6: *Let $\mathfrak{P}$ be a representative of a class of order 3 in A_0^+. Let $\mathfrak{P}^3 = (\alpha)$, $\alpha \in k^+$. Then $\mathfrak{P}$ becomes principal in k_n for some n if and only if $\alpha \in (F_0)^3$.*

PROOF: First let $\beta \in F_0$ be such that $\beta^3 = \alpha$. Then $K(\beta) = K(\sqrt[3]{\kappa})$ for some 3-unit κ of K. Then $\beta^3 = \kappa\gamma^3$ for some $\gamma \in K$. Choose n large enough so that $\kappa, \gamma \in k_n$. Then, in k_n, we have $\alpha = \beta = \kappa\gamma^3$, so in A_n', we have $i_{0,n}(\mathfrak{P}')$ is trivial. Since $A_n = A_n'$, $\mathfrak{P}$ becomes principal in k_n.

Conversely, suppose $\mathfrak{P}$ becomes principal in k_n. Then, in k_n we have $i_{0,n}(\mathfrak{P})^3 = (\alpha) = (\gamma)^3$. Then $\alpha\kappa = \gamma^3$ for some unit κ of k_n. But $\kappa \in (F)^3$ hence $\alpha \in (F)^3$. Since $k(\sqrt[3]{\alpha})/k$ is an abelian extension, we have $\alpha \in (F_0)^3$.

We remark that Greenberg [3] has proved that under our assumptions, a necessary and sufficient condition so that $\mu_c^+ = \lambda_c^+ = 0$ is that the map $i: A_0^+ \to A$ be the zero map. Using this fact we get:

COROLLARY: *Suppose A_0^+ has exponent 3. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_s$ be representative for a set of independent generators of A_0^+ and let $\alpha_1, \cdots, \alpha_s$ be elements of k^+ so that $(\mathfrak{P}_i)^3 = (\alpha_i)$ for $i = 1, \cdots, s$. Then $\mu_c^+ = \lambda_c^+ = 0$ if and only if $\alpha_i \in (F_0)^3$ for all i.*

Let J and J_1 be the idele groups of k and k^- respectively. In any subfield of K, there is only one prime dividing 3 and we will use S to denote the one point set containing that prime in a given field. We then have the exact sequence

$$0 \to \dot{k}^-(J_1)_S/\overline{\dot{k}^- J_1^S}(3) \to G(M_0^-/k) \to (A_0')^- \to 0 \qquad (*)$$

and the isomorphism $\dot{k}^-(J_1)_S/\overline{\dot{k}^- J_1^S} \simeq k_{\mathscr{P}'}^-/\overline{U_S}$ where U_S is the group of S-units of k^- and $\mathscr{P}'$ is the prime above 3 in k^-. When $d \equiv 2 \pmod 3$ we have $k_{\mathscr{P}'}^- = \boldsymbol{Q}_3(\sqrt{3})$ and when $d = 3(3m+1)$ we have $k_{\mathscr{P}'}^- = \boldsymbol{Q}_3(\sqrt{2})$. In either case $U_S = (\pm 1)\langle 3 \rangle$ and $k_{\mathscr{P}'}^-/\overline{U_S}(3)$ is torsion free, hence isomorphic to $\boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$.

We remark that since the unique prime $\mathscr{P}$ dividing 3 in k is principal, we have $A_0' = A_0$, hence $(A_0')^- = (A_0)^-$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_s$ be a set of representatives for independent generators of $(A_0^+)_3$, and $\mathfrak{U}_1, \cdots, \mathfrak{U}_t$ a set of representatives for independent generators of $(A_0^-)_3$. Then there are elements $\alpha_1, \cdots, \alpha_s \in k^+$, $\beta_1, \cdots, \beta_t \in k^-$ so that $\mathfrak{P}_i^3 = (\alpha_i)$ and $\mathfrak{U}_j^3 = (\beta_j)$ for all i and j. Any cubic extension of k is of the form $k(\sqrt[3]{x})$ for some $x \in k$. Clearly the extension depends only on the residue class of x in k/k^3. Let

$$N = \{y \in k/k^3 \colon k(\sqrt[3]{y'}) \subset M_0 \text{ when } y = y'k^3\}$$

$$N^- = \{y \in k/k^3 \colon k(\sqrt[3]{y'}) \subset M_0^- \text{ when } y = y'k^3\}$$

Let ε be the fundamental unit of k^+ and ζ be a primitive cube root of unity. Then the classes of the elements $\{\zeta, 3, \varepsilon, \alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_t\}$ form a $\boldsymbol{Z}/3\boldsymbol{Z}$ basis of N. By observing the action of τ on the Kummer pairing $N \times G(M_0/k) \to \mu_3$, we see that the classes of $\{\zeta, \varepsilon, \alpha_1, \cdots, \alpha_s\}$ form a basis for N^-.

PROPOSITION 7: $s \leqq t \leqq s+2$.

PROOF: Let $X = G(M_0^-/k)$, $Y = k_{\mathscr{P}'}^-/\bar{U}_S(3)$. Then we have $X \simeq \boldsymbol{Z}_3^2 \oplus T$, where T is a finite 3-group. Using the snake lemma on the commutative diagram obtained from the cubing of (*), we obtain the exact sequence of $\boldsymbol{Z}/3\boldsymbol{Z}$ vector spaces

$$0 \to T_3 \xrightarrow{f} (A_0^-)_3 \to Y/Y^3 \to X/X^3 \xrightarrow{g} A_0^-/(A_0^-)^3 \to 0$$

because Y is torsion free. X/X^3 is the Galois group of the maximal subfield of M_0^- of type $(3, 3, \cdots, 3)$. Then we have

$$\dim_{\mathbf{Z}/3\mathbf{Z}}(X/X^3) = \dim_{\mathbf{Z}/3\mathbf{Z}}(N^-) = 2+s.$$

Since $X/X^3 \simeq (\mathbf{Z}/3\mathbf{Z})^2 \oplus T/T^3$, we have $\dim_{\mathbf{Z}/3\mathbf{Z}}(T/T^3) = s$. The injectivity of f shows that $s \leqq t$ and the surjectivity of g shows that $2+s \geqq t$. (This result was originally obtained by Scholz [10] in 1932, using a slightly different method.)

COROLLARY: *If A_0^- is cyclic, then A_0^+ is cyclic.*

PROOF: If A_0^- is cyclic then $t = 1$. Hence $s \leqq 1$ and A_0^+ is cyclic too.

THEOREM 3: *Suppose A_0^- is cyclic and that A_0^+ has exponent* 3. *If $k(\sqrt[3]{\varepsilon})$ is not embeddable in a $\mathbf{Z}_3$-extension of k, then $\mu_c^+ = \lambda_c^+ = 0$.*

PROOF: We note that, by the above corollary, A_0^+ is cyclic of order 3. Since $k(\sqrt[3]{\varepsilon})$ is not embeddable in a $\mathbf{Z}_3$-extension of k, we know that ε is not orthogonal to T in the Kummer pairing. Since $k(\sqrt[3]{\zeta}) \subset K$, we know that ζ is orthogonal to T. The only possibility is that there is an element $\alpha \in k^+$ so that $(\alpha) = \mathfrak{P}^3$, the class of $\mathfrak{P}$ generates A_0^+, and α is orthogonal to T. Hence $k(\sqrt[3]{\alpha}) \subset P_0^- \subseteq F_0^-$. Thus $\alpha \in (F_0)^3$ and $\mu_c^+ = \lambda_c^+ = 0$.

Let L be the maximal unramified abelian 3-extension of k. Then $G(L/k) \simeq A_0 = A_0'$. Hence every unramified cubic extension of k splits completely at $\mathscr{P}$. When A_0^- is cyclic, there is a unique unramified cubic extension of k lying in M_0^-. Then there is a subgroup $N^* \subset N^-$ such that $|N^*| = 3$ and satisfying $xk^3 \in N^*$ if and only if $xk^3 \in N^-$ and $x \in (k_{\mathscr{P}})^3$, that is, that the extension $k(\sqrt[3]{x}) \subset M_0^-$ is completely split at $\mathscr{P}$.

PROPOSITION 8: *Assume A_0^- is cyclic. Let $xk^3 \in N^*$. Then $k(\sqrt[3]{x})$ is embeddable in a Z_3-extension of k if and only if $|T| < |A_0^-|$.*

PROOF: Since Y is torsion free, we know that T injects into A_0^-. Since $k(\sqrt[3]{x})$ is the unique unramified cubic extension of k in M_0^-, we have $G(k\sqrt[3]{x})/k) \simeq A_0^-/(A_0^-)^3$, and this isomorphism is compatible with the exact sequence (*). Thus $\mathrm{Im}\,(T) \subset (A_0^-)^3$ if and only if $|T| < |A_0^-|$, hence a generator of T fixes $\sqrt[3]{x}$ if and only if $|T| < |A_0^-|$. Thus x is orthogonal to T if and only if $|T| < |A_0^-|$ and the result follows.

COROLLARY: *If $\varepsilon \in (k_{\mathscr{P}})^3$, then $k(\sqrt[3]{\varepsilon})$ is embeddable in a Z_3-extension of k if and only if $|T| < |A_0^-|$.*

Before giving some examples of how one computes whether $k(\sqrt[3]{\varepsilon})$ is embeddable in a $\mathbf{Z}_3$-extension of k, we make the following remark. When, as in this case, there is only one prime above l in a cyclotomic Z_l-extension K/k, we have the isomorphism $R_2 K \cong \mathscr{T} \otimes A[1]$ and, observing the action of complex conjugation on $\mathscr{T}$, we have $(\mu_c^+, \lambda_c^+) = (\mu_R^-, \lambda_R^-)$, so in the preceding cases we have also shown that $\mu_R^- = \lambda_R^- = 0$.

We will now describe how to determine whether $k(\sqrt[3]{\varepsilon})$ is embeddable in a $\boldsymbol{Z}_3$-extension of k and will give the result of this determination for some fields. We are considering fields $\boldsymbol{Q}(\sqrt{a}, \sqrt{-3})$ in which there is only one prime above 3, and such that the class number of $k^+ = \boldsymbol{Q}(\sqrt{a})$ is divisible by 3 (otherwise $\mu_c^+ = \lambda_c^+ = 0$ by Greenberg). There are six such fields for $a \leqq 500$ and they are $a = 254, 257, 326, 359, 443$, and 473. We also consider the values $a = 761$, 1223, and 1367 since they fall easily within the scope of these calculations. In five of these fields, when $a = 257, 326, 359, 1223$, and 1367, we have $|A_0^+| = |A_0^-| = 3$ and $\varepsilon \in (k_{\mathscr{P}})^3$ where $\mathscr{P}$ is the prime above 3. Since T is non-trivial (c.f. Prop. 7) and injects in A_0^-, we have $|T| = 3$, and, by Proposition 8, $k(^3\sqrt{\varepsilon})$ is not embeddable in a Z_3-extension of k. Then for these fields $\mu_c^+ = \lambda_c^+ = 0$. The values of ε are tabulated below

a	257	326	359	1223	1367
ε	$16+\sqrt{257}$	$325+18\sqrt{326}$	$360+19\sqrt{359}$	$1224+35\sqrt{1223}$	$1368+37\sqrt{1367}$

In the rest of the cases we proceed as follows: Suppose $|T| = 3^r$, and $|A_0^-| = 3^u$.Then $r \leqq u$ because T injects into A_0^-, and since $T \simeq (A_0^-)_{3^r}$, we find that the map $Y/Y^{3^r} \to X/X^{3^r}$ is injective. Let $\mathfrak{P}$ be an ideal which represents a generator of the cyclic group A_0^-, and for convenience, choose $\mathfrak{P}$ so that only one prime divides it. Let $\beta \in k^-$ be an element so that $(\beta) = \mathfrak{P}^{3^u}$, and let $N_{k^-/\boldsymbol{Q}}(\mathfrak{P}) = q^b$ for some rational prime q. Clearly we may assume that 3 does not divide b. Consider the ideles I_0 and I_1 of k^- given by

$$(I_0)_v = q^b \text{ when } v|\mathfrak{P}, \quad (I_0)_v = 1 \text{ when } v \nmid \mathfrak{P}$$
$$(I_1)_v = \beta \text{ when } v|3, \quad (I_1)_v = 1 \text{ when } v \nmid 3.$$

By class field theory, we have the isomorphism $X \simeq J_1/\overline{k^- J_1^S}$ (3). Making this identification, and, identifying T with its image in $J_1/\overline{k^- J_1^S}$ we see the following:

I_0 maps to the class of $\mathfrak{P}$ in A_0^- and I_1 represents a non-trivial class in Y. Also, $I_1 I_0{}^{3^u} \in k^- J_1^S$, so the class of $I_1 I_0{}^{3^u}$ is trivial in X. Since the class of I_1 is then in $Y \cap X^{3^u}$, hence in $Y \cap X^{3^r}$, it must be in Y^{3^r}; hence $\mathfrak{P}$ must be in $(k^-)^{3^r}_{\mathscr{P}'}$ and r is necessarily the largest integer $\leqq u$ for which this is true. Let $\gamma = \beta^{1/3^r}$ in $k^-_{\mathscr{P}'}$ and let I_2 be given by $(I_2)_{\mathscr{P}'} = \gamma$, $(I_2)_v = q^{3^{u-r}b}$ for $v|\mathfrak{P}$ and $(I_2)_v = 1$ for all other v. Then $(I_2)^{3^r} = I_1(I_0)^{3^u}$ in X and it is clear that the class of I_2 generates T.

To decide whether $\sqrt[3]{\varepsilon}$ is embeddable in a $\boldsymbol{Z}_3$-extension of k, we must decide whether T (considered as a subgroup of X) fixes $\sqrt[3]{\varepsilon}$, since the fixed field of T is the compositum of the $\mathbf{Z}_3$-extensions of k contained in M_0^-. To make matters simpler, we will perform our calculations in $G(M_0/k) \simeq J/kJ^S$ (3), identifying J_1 with its image under the canonical map $J_1 \to J$. Then using class field theory the Kummer pairing

$$N \times G(M_0/k) \to \mu_3$$

translates into the 3-symbol $N \times J/kJ^S J^3 \to \mu_3$ given by

$$(\alpha, I_v) \to \prod_v (\alpha, I_v)_{v,3}.$$

Explicitly we must evaluate

$$(\varepsilon, I_2)_3 = (\varepsilon, \gamma)_{\mathscr{P}}^{m_{\mathscr{P}}/3} \prod_{v'|v} (\varepsilon, q^{3^u - rb})_{v'}^{m_{v'}3}$$

where v is the unique prime of k^- dividing $\mathfrak{P}$. Then $\sqrt[3]{\varepsilon}$ is embeddable in a $\mathbf{Z}_3$-extension of k if and only if it is orthogonal to T in the sense that $(\varepsilon, I_2)_3$ given above is 1.

To compute $(\varepsilon, \gamma)_{\mathscr{P}}^{m_{\mathscr{P}}/3}$ we note that it certainly suffices to approximate by ε and γ modulo U^3 where U is the group of units in $k_{\mathscr{P}}$. We have $k_{\mathscr{P}} = \mathbf{Q}_3(\sqrt{-3}, \sqrt{2})$ and a basis for the $\mathbf{Z}/3\mathbf{Z}$-vector space U/U^3 is given by $\{1+\sqrt{-6}, 1+\sqrt{-3}, 1+3\sqrt{2}, 4, 1+3\sqrt{-3}\}$. We note that ε and γ will always be units taken from the subfields $\mathbf{Q}_3(\sqrt{2})$ and $\mathbf{Q}_3(\sqrt{-6})$ and that they will not both be in the same subfield.

LEMMA 9: *Let x be a unit in $\mathbf{Q}_3(\sqrt{2})$ and y be a unit in $\mathbf{Q}_3(\sqrt{-6})$. Then $(x, y)_{\mathscr{P}}^{m_{\mathscr{P}}/3} = \zeta^{\varphi(x)\psi(y)}$ where $\varphi(x)$ and $\psi(y)$ are the coefficients of $1+3\sqrt{2}$ and $1+\sqrt{-6}$ respectively in the expansions of x and y with respect to the above basis of U/U^3.*

PROOF: Since $x \in \mathbf{Q}_3(\sqrt{2})$, we may write x(modulo U^3) as $(1+3\sqrt{2})^{x_1}$ $(4)^{x_2}$, and since $y \in \mathbf{Q}_3(\sqrt{-6})$, we may write y(modulo U^3) as $(1+\sqrt{-6})^{y_1}$ $(4)^{y_2}$. Then

$$(x, y)_{\mathscr{P}} = (1+3\sqrt{2}), \; 1+\sqrt{-6})_{\mathscr{P}}^{x_1 y_1}(1+3\sqrt{2}, 4)_{\mathscr{P}}^{x_1 y_2}(4, 1+\sqrt{-6})_{\mathscr{P}}^{x_2 y_1}(4, 4)_{\mathscr{P}}^{x_2 y_2}.$$

We then raise to the power $m_{\mathscr{P}}/3$ and see immediately that $(4, 4)_{\mathscr{P}}^{m_{\mathscr{P}}/3} = 1$. Furthermore, since μ_3 is not contained in either $\mathbf{Q}_3(\sqrt{2})$ or $\mathbf{Q}_3(\sqrt{-6})$, we have

$$(Tr_{k_{\mathscr{P}}/\mathbf{Q}_3(\sqrt{2})}\{1+3\sqrt{2}, 4\}^{m_{\mathscr{P}}/3})_3 = ((1+3\sqrt{2}, 4)^{m_{\mathscr{P}}/3})_{\mathscr{P}}^2 = 1$$

hence $(1+3\sqrt{2}, 4)_{\mathscr{P}}^{m_{\mathscr{P}}/3} = 1$, and similarly

$$(Tr_{k_{\mathscr{P}}/\mathbf{Q}_3(\sqrt{-6})}\{4, 1+\sqrt{-6}\}^{m_{\mathscr{P}}/3})_3 = ((4, 1+\sqrt{-6})^{m_{\mathscr{P}}/3})_{\mathscr{P}}^2 = 1$$

hence $(4, 1+\sqrt{-6})_{\mathscr{P}}^{m_{\mathscr{P}}/3} = 1$. We are then left with

$$(x, y)_{\mathscr{P}}^{m_{\mathscr{P}}/3} = ((1+3\sqrt{2}, 1+\sqrt{-6})_{\mathscr{P}}^{m_{\mathscr{P}}}/3)^{x_1 y_1}$$

We check directly that $(1+3\sqrt{2}, 1+\sqrt{-6})_{\mathscr{P}}^{m_{\mathscr{P}}/3} = \zeta$ where

$$\zeta = \frac{-1+\sqrt{-3}}{2}$$

and note that $x_1 = \varphi(x)$ and$y_1 = \psi(y)$.

To compute $\varphi(x)$ and $\psi(y)$ we note the following: If $x \equiv 1 \pmod 3$ we may write $x = 1+3(a+b\sqrt{2})$ and then $\varphi(x) = b \pmod 3$. In general, we may adjust x by a cube without changing $\varphi(x)$. We note that $x^4 \equiv \pm 1 \pmod 3$. Let $x \equiv a+b\sqrt{2} \pmod 9$. Then

$$x^4 = (a^4+12a^2b^2+4b^4+\sqrt{2}(4a^3b+8ab^3)) \pmod 9$$

and, since $x^4 \equiv \pm 1 \pmod 3$, we have $3|(4a^3b+8ab^3)$ and $a^4+12a^2b^2+4b^4 \equiv \pm 1 \pmod 3$. Then we have

$$\varphi(x) = \varphi(x^4) = \frac{1}{3}\,\frac{4a^3b-ab^3}{a^4+b^4} \pmod 3$$

For y, we must have $y \equiv \pm 1 \bmod \mathscr{P}'$, the maximal ideal in $\boldsymbol{Q}_3(\sqrt{-6})$. In $\boldsymbol{Q}_3(\sqrt{-6})$ we have the isomorphism $\log_{\mathscr{P}'}\colon U^{(1)} \to \mathscr{P}'$ which exists because $\mu_3 \not\subset \boldsymbol{Q}_3(\sqrt{-6})$. Since -1 is a cube, we may assume $y \equiv 1 \pmod{\mathscr{P}'}$. Let $y = 1+z$, $z \in \mathscr{P}'$. Then $\log y = \sum_{n=1}^{\infty}(-1)^{n+1}z^n/n$. We note that $v_{\mathscr{P}}(z^n/n) > 1$ when $n = 2$ or $n > 3$, hence that $\log y \equiv z+z^3/3 \pmod 3$. Let $z = a(\sqrt{-6})+3b$. Then

$$z^3 = -6a^3\sqrt{6}-6\cdot 3^2a^2b+3^3ab^2\sqrt{-6}+3^3b^3,$$

so $z^3/3 \equiv a\sqrt{-6} \pmod 3$. Hence $\log y \equiv -a(\sqrt{-6}) \pmod 3$. Since $y = 1+a\sqrt{-6}+3b$, we have $\psi(y) \equiv a \pmod 3$, hence $\log y \equiv -\psi(y)(\sqrt{-6}) \pmod 3$.

Returning to our example, we must compute $(\varepsilon, \gamma)_{\mathscr{P}}^{m_{\mathscr{P}}/3}$. Assume that $(k^+)_3 = \boldsymbol{Q}_3(\sqrt{2})$ (Otherwise $(k^-)_3 = \boldsymbol{Q}_3(\sqrt{2})$ and the calculation is similar.) Then we have

$$(\varepsilon, \gamma)_{\mathscr{P}}^{m_{\mathscr{P}}/3} = \zeta^{\varphi(\varepsilon)\psi(\gamma)}.$$

We must write ε in $\boldsymbol{Q}_3(\sqrt{2})$ by expressing $\sqrt{a}$ in terms of $\sqrt{2}$. We use the convention $\sqrt{a}/\sqrt{2} \equiv 1 \pmod 3$. We compute the following congruences (mod 9) for $\sqrt{a}$ when $a \equiv x \pmod 9$

x	2	5	8
$\sqrt{a}$	$\sqrt{2}$	$\sqrt{2}(1+3)$	$\sqrt{2}(1-3)$

We have $\gamma = \beta^{1/3^r}$, so $\log\gamma = 1/3^r \log\beta$. For example, let $a = 443$. Then $|A_0^-| = 9$ and $\mathfrak{P}^9 = (\beta) = (99037+774\sqrt{-3\cdot 443})$ where $\mathfrak{P}$ is a prime ideal dividing 13 in k^-. We have $\log_{\mathscr{P}'}\beta \equiv -9\sqrt{-6}-27 \pmod{(\mathscr{P}')^7}$, hence $\beta \in U^9$ and $r = 2$. $\mathrm{Log}_{\mathscr{P}'} \gamma = \frac{1}{9} \log_{\mathscr{P}'} \beta \equiv -\sqrt{-6} \pmod 3$, so $\psi(\gamma) = 1$. We have $\varepsilon = 442 + 21\sqrt{443} \equiv 1 + 3\sqrt{2} \pmod 9$, so $\varphi(\varepsilon) = 1$. Thus $(\varepsilon, \gamma)^{m_{\mathscr{P}}/3} = \zeta$. Finally we calculate straightforwardly the tame symbol $(\varepsilon, 13)_{\mathfrak{P}}^{m_{13}/3}$. We observe that 13 splits completely in k, so $m_{13} = 12$, and thet $\varepsilon^2 \equiv -1 \pmod{13}$. Hence $(\varepsilon, 13)_{\mathfrak{P}}^4 = 1$ and we have $(\varepsilon, I_2) = \zeta$. Then $k(\sqrt[3]{\varepsilon})$ is not embeddable in a $\boldsymbol{Z}_3$-extension of k and $\mu_c^+ = \lambda_c^+ = 0$. Tabulated below is the relevant information for the other fields.

a	254	443	473	761
ε	$255+16\sqrt{254}$	$442+21\sqrt{443}$	$87+4\sqrt{473}$	$800+29\sqrt{761}$
$\varphi(\varepsilon)$	2	1	1	2
β	$89+12\sqrt{-3-254}$	$99037+774\sqrt{-3\cdot 443}$	$\dfrac{223+3\sqrt{-3\cdot 473}}{2}$	$\dfrac{1901-39\sqrt{-3\cdot 761}}{2}$
$N_{k^-/Q}(\beta)$	7^6	13^9	5^6	11^6
r	1	2	1	1
$\psi(\gamma)$	1	1	2	2
tame symbol	ζ	1	ζ	ζ
(ε, I_2)	1	ζ	1	ζ^2

We conclude that when $k = Q(\sqrt{a}, \sqrt{-3})$ and $a = 443$ *or* 761, $k(\sqrt{\varepsilon})$ is not embeddable in a $\boldsymbol{Z}_3$-extension of k, so $\mu_c^+ = \lambda_c^+ = 0$ for these fields. For the cases $a = 254$ and $a = 473$, $k(\sqrt[3]{\varepsilon})$ is embeddable in a $\boldsymbol{Z}_3$-extension. We note that when $a = 443$ and 761, the elements $\alpha = 10 + \sqrt{443}$ and $(27 + \sqrt{761})/2$ have the property that $\mathfrak{P}^3 = (\alpha)$ and $\mathfrak{P}$ represents a generator of the 3-primary part of the ideal class group of $\boldsymbol{Q}(\sqrt{a})$. In these cases we check directly that $k(\sqrt[3]{\alpha\varepsilon^{-1}})$ for $a = 443$ and $k(\sqrt[3]{\alpha\varepsilon})$ for $a = 761$ are embeddable in $\boldsymbol{Z}_3$-extensions of k.

Computation of the order of $(R_2 k)_3$ for quadratic fields

Let $k = \boldsymbol{Q}(\sqrt{d}, \sqrt{-3})$ where d is a square-free positive integer > 1. Then $k^+ = k \cap \boldsymbol{R} = \boldsymbol{Q}(\sqrt{d})$ and we define $k^- = \boldsymbol{Q}(\sqrt{-3d})$. $G(k/\boldsymbol{Q})$ is generated by automorphisms σ and τ where σ fixes k^+ and τ fixes k^- and $\sigma^2 = \tau^2 = 1$. Clearly, if F is any quadratic number field, then unless $F = Q(\sqrt{-3})$ for which $R_2 F = (0)$ [12], $F(\sqrt{-3})$ is a field of the form k. When F is real, we have $F = k^+$ and when F is imaginary we have $F = k^-$. Tate [13] has shown that when $\mu_3 \subset k$, the map $\alpha: k/k^3 \to (K_2 k)_3$ given by $\alpha(x \bmod k^3) = \{x, \zeta\}$, where ζ is a primitive cube root of unity, is surjective. Then we have the exact sequence

$$0 \to X \to k/k^3 \xrightarrow{\alpha} (K_2 k)_3 \to 0,$$

where $X = \operatorname{Ker} \alpha$. Let $\bar{k}$ be an algebraic closure of k (we may take $\bar{k} \subset \boldsymbol{C}$), $G_k = G(\bar{k}/k)$ and $\mathscr{T} = \varprojlim \mu_{3^n}$. Then from the exact sequence

$$0 \to \mathscr{T} \otimes \mathscr{T} \xrightarrow{3} \mathscr{T} \otimes \mathscr{T} \to \mu_3 \otimes \mu_3 \to 0$$

we obtain the following sequence by taking Galois cohomology with continuous cochains

$$0 \to H^1(G_k, \mathcal{T} \otimes \mathcal{T})/H^1(G_k, \mathcal{T} \otimes \mathcal{T})^3 \to H^1(G_k, \mu_3 \otimes \mu_3) \to [H^2(G_k, \mathcal{T} \otimes \mathcal{T})]_3 \to 0.$$

Since $\mu_3 \subset k$, G_k acts trivially on μ_3, we may identify $H^1(G_k, \mu_3 \otimes \mu_3)$ with $\mathrm{Hom}(G_k, \mu_3)$ (as groups). We then construct the following diagram

$$\begin{array}{ccccccccc} 0 \to & H^1(G_k, \mathcal{T} \otimes \mathcal{T})/H^1(G_k, \mathcal{T} \otimes \mathcal{T})^3 & \to & H^1(G_k \mu_3 \otimes \mu_3) & \to & & [H^2(G_k, \mathcal{T} \otimes \mathcal{T})]_3 & \to 0 \\ & \uparrow & & \wr\uparrow\beta & & & \uparrow h & \\ 0 \longrightarrow & X & \longrightarrow & k \otimes \mu_3 & \xrightarrow{\alpha} & & (K_2 k)_3 & \longrightarrow 0 \end{array}$$

where $(\beta(x \otimes \zeta))(\sigma) = \sigma(x^{1/3})/x^{1/3}$ for $\sigma \in G_k$. Tate [12], [13] has shown the existence of the isomorphism $h: (K_2 k)_3 \to [H^2(G_k, \mathcal{T} \otimes \mathcal{T})]_3$ so that the above diagram commutes. Then

$$X \simeq H^1(G_k, \mathcal{T} \otimes \mathcal{T})/H^1(G_k, \mathcal{T} \otimes \mathcal{T})^3$$

and since $H^1(G_k, \mathcal{T} \otimes \mathcal{T})$ is isomorphic to $B \oplus \mathbf{Z}_3^{r_2(k)}$ where B is a cyclic group of order

$$w_{k'} = |\mu(k)|^2 \prod_{[E:k]=2} \frac{|\mu(E)|}{|\mu(k)|},$$

we have $X \simeq (\mathbf{Z}/3\mathbf{Z})^{1+r_2(k)}$[10].

Recall that $R_2 k$ is the 3-primary part of the tame kernel of k. Then, since $R_2 k = (R_2 k)^+ \otimes (R_2 k)^-$, we have $(R_2 k)_3 = (R_2 k)_3^+ \otimes (R_2 k)_3^-$. Suppose k' is a quadratic subextension of k. Then let $G_{k'} = G(\bar{k}'/k') = G(\bar{k}/k')$. We have $G_k \subset G_{k'}$ and $|G_{k'}/G_k| = [k : k'] = 2$. Since $\mathcal{T}$ is a pro-3 group, we have

$$H^1(G_{k'}, \mathcal{T} \otimes \mathcal{T}) = H^1(G_k, \mathcal{T} \otimes \mathcal{T})^{G_{k'}/G_k}.$$

But $H^1(G_{k'}, \mathcal{T} \otimes \mathcal{T}) \simeq B' \oplus \mathbf{Z}_3^{r_2(k')}$ where B' is a cyclic group of order $w'_{k'}$ [10]. Thus, since $G_{k'}/G_k \simeq G(k/k')$ and $3|w'_{k'}$ for any k', we have

$$\dim_{\mathbf{Z}/3\mathbf{Z}} X^{G(k'/k)} = 1 + r_2(k')$$

Thus if $k' = k^-$, we have $r_2(k') = 1$, so $\dim_{\mathbf{Z}/3\mathbf{Z}}(X^\tau) = 2$, and if $k' = k^+$, $r_2(k') = 0$, so $\dim_{\mathbf{Z}/3\mathbf{Z}}(X^\sigma) = 1$. Since $(R_2 k^+)_3 \simeq (R_2 k)_3^\sigma$ and $(R_2 k^-)_3 \simeq (R_2 k)_3^\tau$ we may now determine the orders of $(R_2 k^+)_3$ and $(R_2 k^-)_3$.

Notation: If T is an abelian group of exponent 3, we will write $d(T) = \dim_{\mathbf{Z}/3\mathbf{Z}}(T)$. Let $Y = \{y \in k \otimes \mu_3 : \alpha(y) \in (R_2 k)_3\}$. We have $\alpha \otimes \zeta \in Y$ if and only if $v_{\mathscr{P}}(a)$ is divisible by 3 for all $\mathscr{P}$ not dividing 3. We have the exact sequence

$$0 \to X \to Y \to (R_2 k)_3 \to 0$$

where the maps are compatible with the action of $G(k/\mathbf{Q})$, so to determine $(R_2 k)_3^\sigma$ and $(R_2 k)_3^\tau$ we need only determine Y^σ and Y^τ.

Let A be the 3-primary part of the ideal class group of k
A' be the 3-primary part of the 3-ideal class group of k
B be the subgroup of A generated by the classes of the ideals dividing 3 in k.

Then $A' = A/B$ and we have the exact sequence

$$(**) \qquad 0 \to B_3 \to A_3 \xrightarrow{\pi} A'_3 \xrightarrow{f} B/B^3 \xrightarrow{g} A/A^3$$

Case 1: There is only one prime dividing 3 in k.

In this case, if $\mathscr{P}$ is the prime above 3 in k, we have $\mathscr{P} = (\sqrt{-3})$, so $B = 0$ and $\pi: A_3 \to A'_3$ is an isomorphism. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ be representatives of independent generators of A_3. For each i, let $\alpha_i \in k$ be so that $\mathfrak{P}_i^3 = (\alpha_i)$. Since $A_3 \cong (A^+)_3 \oplus (A^-)_3$, we may choose the elements α_i so that $\alpha_1, \cdots, \alpha_s \in k^+$ where $s = d(A_3^+)$, and $\alpha_{s+1}, \cdots, \alpha_{s+t} \in k^-$ where $t = d(A_3^-)$ and $s+t = r$. Suppose $a \otimes \zeta \in Y$. Then $(a) = (\sqrt{-3})^e \prod_{v_i|3} v_i^{e_i}$. Since $a \otimes \zeta \in Y$, we may write $e_i = 3f_i$ for each i, and we observe that $\prod (v_i^{f_i})^3$ is principal, hence in A

$$\prod_i v_i^{f_i} = \prod_{i=1}^r \mathfrak{P}_i^{m_i}$$

and we have

$$(a) = (\sqrt{-3}—^e(\prod_{i=1}^r \alpha_i^{m_i})x^3).$$

Then

$$a = \sqrt{-3}^e(\prod_{i=1}^r \alpha_i^{m_i})x^3 y$$

where y is a unit in k. Thus

$$a \otimes \zeta = (\sqrt{-3} \otimes \zeta)^e(\prod_{i=1}^r (\alpha_i \otimes \zeta)^{m_i})(y \otimes \zeta).$$

Let U be the group of units in k. Then $U \otimes \mu_3$ is generated by $\varepsilon \otimes \zeta$ and $\zeta \otimes \zeta$ where ε is the fundamental unit of k^+. Thus Y is generated by

$$\{\sqrt{-3} \otimes \zeta, \varepsilon \otimes \zeta, \zeta \otimes \zeta\} \cup \{\alpha_i \otimes \zeta\}_{i=1}^r.$$

It is clear that these elements are linearly independent so we have $d(Y) = 3+r = 3+s+t$.

Let Y_1 be the subgroup of Y generated by $\{\alpha_i \otimes \zeta\}_{i=1}^s$
Y_2 be the subgroup of Y generated by $\{\alpha_i \otimes \zeta\}_{i=s+1}^r$.

LEMMA 10: *For $y \in Y_1$, we have $\tau(y) = y$, $\sigma(y) = y^{-1}$. For $y \in Y_2$, we have $\sigma(y) = y$ and $\tau(y) = y^{-1}$. Furthermore, $\tau(\varepsilon \otimes \zeta) = (\varepsilon \otimes \zeta)$ and $\sigma(\varepsilon \otimes \zeta) =$*

$(\varepsilon \otimes \zeta)^{-1}$; $\sigma(\zeta \otimes \zeta) = \tau(\zeta \otimes \zeta) = \zeta \otimes \zeta$; *and* $\sigma(\sqrt{-3} \otimes \zeta) = \tau(\sqrt{-3} \otimes \zeta) = (\sqrt{-3} \otimes \zeta)^{-1}$.

PROOF: First we have $\varepsilon = \varepsilon^{\sigma}$ and $\varepsilon^{\tau} = \pm\varepsilon^{-1}$. $\zeta^{\sigma} = \zeta^{\tau} = \zeta^{-1}$ and $(\sqrt{-3})^{\sigma} = (\sqrt{-3})^{\tau} = -\sqrt{-3}$. From this we conclude, since -1 is a cube, that $\tau(\varepsilon \otimes \zeta) = (\varepsilon \otimes \zeta), \sigma(\varepsilon \otimes \zeta) = (\varepsilon \otimes \zeta)^{-1}; \sigma(\zeta \otimes \zeta) = \tau(\zeta \otimes \zeta) = (\zeta \otimes \zeta)$; and $\sigma(\sqrt{-3} \otimes \zeta) = \tau(\sqrt{-3} \otimes \zeta) = (\sqrt{-3} \otimes \zeta)^{-1}$.

Now, consider $\{\alpha_i \otimes \zeta\}$ for $1 \leqq i \leqq s$. Then $\alpha_i \in k^+$, so $\alpha_i = \alpha_i^{\sigma}$ and $\sigma(\alpha_i \otimes \zeta) = (\alpha_i \otimes \zeta)^{-1}$. Also, we have $\mathfrak{P}_i^3 = (\alpha_i)$, so $(\alpha_i \alpha_i^{\tau}) = (\mathfrak{P}_i \mathfrak{P}_i^{\tau})^3 = (\mathscr{A})^3$ where $\mathscr{A}$ is an ideal of Q. Then $\alpha_i \alpha_i^{\tau} = x^3$ for some $x \in Q$. From this it follows that $\tau(\alpha_i \otimes \zeta) = \alpha_i \otimes \zeta$. Similarly, if $s+1 \leqq i \leqq r$, we have $\alpha_i^{\tau} = \alpha_i$, so $\tau(\alpha_i \otimes \zeta) = (\alpha_i \otimes \zeta)^{-1}$ and $\alpha_i \alpha_i^{\sigma} = y^3$, $y \in Q$. Hence $\sigma(\alpha_i \otimes \zeta) = (\alpha_i \otimes \zeta)$.

We observe that, since $(R_2 k)^{\sigma\tau} = R_2(Q(\sqrt{-3})) = (0)$, we have $\zeta \otimes \zeta \in X$ and $\sqrt{-3} \otimes \zeta \in X$. By the above lemma, we have $d(Y^{\tau}) = 2+s$ and $d(Y^{\sigma}) = 1+t$. Since $d(X^{\tau}) = 2$ and $d(X^{\sigma}) = 1$ (in fact X^{σ} is generated by $\zeta \otimes \zeta$), we have $d(R_2 k)_3^{\tau} = d(R_2 k^-)_3 = s$ and $d(R_2 k)_3^{\sigma} = d(R_2 k^+)_3 = t$. That finishes Case 1.

Suppose there are two ideals above 3 in k. Then 3 must split in either k^+ or k^-. Let $\mathscr{P}$ and $\mathscr{P}'$ be the primes above 3 in k. Then $\mathscr{P}\mathscr{P}' = (\sqrt{-3})$, so $\mathscr{P}' = \mathscr{P}^{-1}$ in the ideal class group of k. Furthermore $\mathscr{P}' = \mathscr{P}^{\tau}$ or $\mathscr{P}^{\sigma}$ when 3 splits in k^+ or k^- respectively. Let c be the order of $\mathscr{P}$ in the ideal class group of k.

Case 2: 3 does not divide c.

In this case we have $B = 0$, so again $\pi: A_3 \to A_3'$ is an isomorphism. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ be representatives for a set of independent generators of A_3. We may assume that $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ are relatively prime to $\mathscr{P}$ and $\mathscr{P}'$. Let $\alpha_i \in k$ be so that $(\alpha_i) = \mathfrak{P}_i^3$. Again, if $s = d(A_3^+)$ and $t = d(A_3^-)$, we may choose $\alpha_1, \cdots, \alpha_s \in k^+$ and $\alpha_{s+1}, \cdots, \alpha_r \in k^-$. Now let $a \otimes \zeta \in Y$. Then

$$(a) = \mathscr{P}^{e}\mathscr{P}'^{e'} \prod_{v_i \nmid 3} v_i^{e_i}.$$

Again we have $e_i = 3\delta_i$ for each i and, since $\mathscr{P}\mathscr{P}' = (\sqrt{-3})$, we have

$$(a) = (\sqrt{-3})^n(\mathscr{P}^m) \prod_{v_i \nmid 3} (v_i{}^{\delta_i})^3.$$

Raising to the power c, we have $(a^c) = (y)\prod_{v_i \nmid 3} (v_i^{c\delta_i})^3$ where y is a 3-unit in k. Then $\prod_{v_i \nmid 3} (v_i^{c\delta_i})^3$ is principal, so we conclude that

$$(a^c) = (y)(\prod_{i=1}^{r} \alpha_i^{m_i})(x^3)$$

and hence that

$$(a^c) = z(\prod_{i=1}^{r} \alpha_i^{m_i})x^3$$

where z is a 3-unit in k. Thus, if Y' is the subgroup of Y generated by $\{\alpha_i \otimes \zeta\}_{i=1}^r$ and U is the group of 3-units of k, we have

$$a^c \otimes \zeta \in (Y')(U \otimes \mu_3),$$

and, since $3 \nmid c$,

$$a \otimes \zeta \in (Y')(U \otimes \mu_3).$$

The Galois action on Y' is described in Case 1. To determine the structure and Galois action on $U \otimes \mu_3$, we observe the following. Let $\mathscr{P}^c = (\gamma)$. If 3 splits in k^+, we have $(\gamma\gamma^\tau) = (\sqrt{-3})^c$. Then $\gamma\gamma^\tau = u(\sqrt{-3})^c$ where u is a unit in k. Replacing γ with γ^2 if necessary, we may assume $\gamma \in k^+$. Let $x = \gamma^2\gamma^\tau$. We then have the following:

$U \otimes \mu_3$is generated by $\{\zeta \otimes \zeta, \varepsilon \otimes \zeta, \sqrt{-3} \otimes \zeta, x \otimes \zeta\}$ where ε is the fundamental unit of k^+. The Galois action on the first three is given in Case 1. Since $x^\sigma = x$ and $xx^\tau = (\gamma\gamma^\tau)^3$, we have $\tau(x \otimes \zeta) = x \otimes \zeta$ and $\sigma(x \otimes \zeta) = (x \otimes \zeta)^{-1}$. We observe finally that

$$\{\zeta \otimes \zeta, \varepsilon \otimes \zeta, \sqrt{-3} \otimes \zeta, x \otimes \zeta\} \cup \{\alpha_i \otimes \zeta\}_{i=1}^r$$

is linearly independent, so we have $d(Y^\tau) = s+3$ and $d(Y^\sigma) = t+1$. Then we conclude that $d(R_2 k^+)_3 = d(R_2 k)_3^\sigma = t$ and $d(R_2 k^-)_3 = d(R_2 k)_3^\tau = s+1$. By an analogous argument, if 3 splits in k^-, we let $x = \gamma^2\gamma^\sigma$ and conclude that $d(R_2 k^+)_3 = t+1$ and $d(R_2 k^-)_3 = s$.

Case 3: 3 divides c, but the class of $\mathscr{P}$ is not a cube.

Let $c = 3^n c'$ where $3 \nmid c'$. Then B is cyclic, generated by $\mathscr{P}^{c'}$, and B_3 is generated by $\mathscr{P}^{3^{n-1}c'}$. In the exact sequence (**), we now have $B \not\subset A^3$, so g is injective. Then f is zero and we get

$$0 \to B_3 \to A_3 \xrightarrow{\pi} A'_3 \to 0.$$

Let $\mathfrak{P}'_1, \cdots, \mathfrak{P}'_r$ be independent generators of A'_3. Then they may each be lifted to classes A_3, which may be represented by ideals $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ which are relatively prime to $\mathscr{P}$ and $\mathscr{P}'$. Let $\alpha_i \in k$ be such that $\mathfrak{P}_i^3 = (\alpha_i)$. Suppose 3 splits in k^+. Then we may assume $\alpha_1, \cdots, \alpha_{s-1} \in k^+$ where $s = d(A_3^+)$. (The extra term is accounted for by the fact that $B_3 \subset A_3^+$), and $\alpha_s, \cdots, \alpha_r \in k^-$ where $t = r-s+1 = d(A_3^-)$. We note that $s = d(A_3^+) = 1 + d(A'_3)^+$ and $t = d(A_3^-) = d(A'_3)^-$.

Now let $a \otimes \zeta \in Y$. Then

$$(a) = (\sqrt{-3})^{m'}(\mathscr{P}^m) \prod_{v_i \nmid 3} (v_i^{\delta_i})^3$$

as in Case 2. In the ideal class group, we must then have

$$\prod_{v_i \nmid 3} \mathrm{v}_i^{\delta_i} = (\prod_{i=1}^{r} \mathfrak{P}_i^{m_i})(\mathscr{P}^u).$$

Hence we may write

$$(a) = (y)(\prod_{i=1}^{r} \alpha_i^{m_i})(x^3)$$

where y is a 3-unit in k. Again $a \otimes \zeta \in (Y')(U \otimes \mu_3)$ with Y' and U defined as in Case 2. Again let $\mathscr{P}^c = (\gamma)$. Then $U \otimes \mu_3$ is generated by $\{\zeta \otimes \zeta, \varepsilon \otimes \zeta, \sqrt{-3} \otimes \zeta, \gamma \otimes \zeta\}$. Replacing γ by γ^2 if necessary, we may assume $\gamma \in k^+$. Then $\gamma^\sigma = \gamma$ and $\gamma\gamma^\tau = (\sqrt{-3})^c$ is a cube because $3|c$. Hence $\sigma(\gamma \otimes \zeta) = (\gamma \otimes \zeta)^{-1}$ and $\tau(\gamma \otimes \zeta) = \gamma \otimes \zeta$. Again the given generators of Y' together with those of $U \otimes \mu_3$ are linearly independent and the Galois action on the other generators has already been described. Putting it all together, we have $d(Y^\tau) = s+2 = d(A'_3)^+ + 3$, and $d(Y^\sigma) = t+1 = d(A'_3) + 1$, so $d(R_2 k^+)_3 = t = d(A'_3)^-$ and $d(R_2 k^-)_3 = s = 1 + d(A'_3)^+$. By an exactly analogous argument, if 3 splits in k^- rather than k^+, we get $d(R_2 k_3^+) = 1 + d(A'_3)^-$ and $d(R_2 k^-)_3 = d(A'_3)^+$.

Case 4: 3 divides c and the class of $\mathscr{P}$ is a cube.

Again we write $c = 3^n c'$ where $3 \nmid c'$ and again B is cyclic, generated by the class of $\mathscr{P}^{c'}$, and B_3 is generated by the class of $\mathscr{P}^{3^{n-1}c'}$. This time we have $B \subset A^3$, so in (**) the map $g: B/B^3 \to A/A^3$ is zero. Then f is surjective and we have $0 \to B_3 \to A_3 \xrightarrow{\pi} (A')_3 \to B/B^3 \to 0$, hence $\pi(A_3)$ has index 3 in A'_3. Again assume 3 splits in k^+. Then we may write $\mathscr{P}\mathfrak{P}^3 = 1$ in A for some ideal $\mathfrak{P}$. Let $(\beta) = \mathscr{P}\mathfrak{P}^3$. We may choose $\beta \in k^+$. Then let $\mathfrak{P}'_2, \cdots, \mathfrak{P}'_r$ be independent generators of $\pi(A_3)$ in A'_3. These may be lifted to classes in A_3, represented by ideals $\mathfrak{P}_2, \cdots, \mathfrak{P}_r$, relatively prime to $\mathscr{P}$ and $\mathscr{P}'$. Choose $\alpha_2, \cdots, \alpha_s$ in k^+ and $\alpha_{s+1}, \cdots, \alpha_r \in k^-$ so that $(\alpha_i) = \mathfrak{P}_i^3$, where $s = d(A'_3)^+ = d(A_3)^+$ and $r - s = t = d(A'_3)^- = d(A_3)^-$. Let $a \otimes \zeta \in Y$. Then we have

$$(a) = (\sqrt{-3})^{m'}(\mathscr{P}^m) \prod_{v_i \nmid 3} (v_i^{f_i})^3.$$

In the ideal class group,

$$1 = \mathscr{P}^m \prod_{v_i \nmid 3} (v_i^{f_i})^3 = ((\prod_{v_i \nmid 3} v_i^{f_i})\mathfrak{P}^{-m})^3.$$

Then we conclude that

$$(a) = (y\beta^m(\prod_{i=2}^{r} \alpha_i^{m_i})x^3)$$

where y is a 3-unit in k. Finally we have

$$a = z\beta^m\left(\prod_{i=2}^{r} \alpha_i^{m_i}\right)x^3$$

where z is a 3-unit in k. Let Y' be the subgroup of Y generated by $\{\alpha_i \otimes \zeta\}_{i=2}^r$. Let $Y'' = U \otimes \mu_3$ and Y''' be generated by $\beta \otimes \zeta$. Again, if $\mathscr{P}^c = (\gamma)$, then Y'' is generated by $\{\zeta \otimes \zeta, \varepsilon \otimes \zeta, \sqrt{-3} \otimes \zeta, \gamma \otimes \zeta\}$ with Galois action previously described. The given set of generators of Y', Y'', and Y''' taken together are linearly independent. To get a more convenient Galois action on Y''', we let $\alpha_1 = \beta^2\beta^\tau$. We have $\alpha_1^\sigma = \alpha_1$ and $\alpha_1\alpha_1^\tau \in k^3$. Since $(\beta\beta^\tau) = \mathscr{P}\mathscr{P}^\tau\mathfrak{P}^3(\mathfrak{P}^\tau)^3 = (\sqrt{-3}x^3)$ for some $x \in k$, we see that the group generated by $\{\sqrt{-3} \otimes \zeta, \beta \otimes \zeta\}$ is the same as that generated by $\{\sqrt{-3} \otimes \zeta, \alpha_1 \otimes \zeta\}$. Then we have $\tau(\alpha_1 \otimes \zeta) = \alpha_1 \otimes \zeta$ and $\sigma(\alpha_1 \otimes \zeta) = (\alpha_1 \otimes \zeta)^{-1}$. Putting it all together and counting, we find $d(Y^\tau) = s+3$ and $d(Y^\sigma) = t+1$. Then $d(R_2k^+)_3 = t$ and $d(R_2k^-)_3 = s+1$. Again, an analogous argument shows that if 3 splits in k^-, we have $d(R_2k^+)_3 = t+1$ and $d(R_2k^-)_3 = s$. We now collect the results of this discussion as follows: For any field F, let $\Delta_F = \{x \,\varepsilon \dot{F} : 3|v(x)$ for all non-archimedean valuations v not dividing $3\}$. Then $\dot{F}^3 \subset \Delta_F$. Let $Y_F = \Delta_F/\dot{F}^3$.

THEOREM 4: *There are surjective homomorphism* $\rho_+ : Y_{k^+} \to (R_2k^-)_3$ *and* $\rho_- : Y_{k^-} \to (R_2k^+)_3$ *given by* $\rho_+(\bar{a}) = Tr_{k/k^-}(\{a, \zeta\})$ *and* $\rho_-(\bar{a}) = Tr_{k/k^+}(\{a, \zeta\})$ *where* $\bar{a}$ *is the class of* a *in* Y_{k^+} *or* Y_{k^-}. *Furthermore* $\operatorname{Ker}(\rho_-)$ *is of order* 3 *and is generated by the class of* 3 *in* Y_{k^-}. $\operatorname{Ker}(\rho_+)$ *is of order* 9 *and contains the class of* 3 *in* Y_{k^+}.

PROOF: We need only remark that for any field we have an exact sequence

$$0 \to U_F/U_F^3 \to Y_F \to (A'_F)_3 \to 0$$

where U_F is the group of 3-units of F and A'_F is the 3-ideal class group of F. Then $d(Y_F) = d(U_F/U_F^3) + d(A'_F)_3$. But $d(U_{k^+}/U_{k^+}^3) = 1+n$ where n is the number of primes above 3 in k^+, and $d(U_{k^-}/U_{k^-}^3) = m$, the number of primes above 3 in k^-. Let $s = d(A')_3^+$ and $t = d(A')_3^-$. We have $d(Y_{k^+}) = 1+s+n$ and $d(Y_{k^-}) = t+m$. The previous discussion shows that in all cases ρ_+ and ρ_- are surjective and that

$$d(R_2k^+)_3 = t-1+m$$

$$d(R_2k^-)_3 = s-1+n$$

We conclude that $d(\operatorname{Ker}\rho_+) = 2$ and $d(\operatorname{Ker}\rho_-) = 1$. Finally we observe that $\{3, \zeta\} = 1$ in R_2k completing the proof.

COROLLARY: *Let* $\mathfrak{P}'_1, \cdots, \mathfrak{P}'_u$ *be independent generators of* $\pi(A_3)^-$ *in*

$(A')_3^-$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_u$ be representative of classes in A_3^- with $\pi(\mathfrak{P}_i) = \mathfrak{P}'_i$. Let $\alpha_1, \cdots, \alpha_u$ be elements of k^- with $(\mathfrak{P}_i)^3 = (\alpha_i)$. Then the elements $\{\alpha_i, \zeta\}$ are linearly independent in $(R_2 k)_3^+$. Furthermore, they form a basis for $(R_2 k)_3^+$ unless 3 splits in k^-.

We will now compute the order of $(R_2 k)_3$ for all fields $k = \boldsymbol{Q}(\sqrt{d})$ where d is the discriminant of k and $|d| \leqq 200$. We observe that $(R_2 k)_3 = (1)$ if and only if $R_2 k = (1)$ and $|(R_2 k)_3| = 3$ if and only if $R_2 k$ is a non-trivial cyclic group. When $d < 0$, we have $|(R_2 k)_3| = 3^{s-1+n}$ where n is the number of primes dividing 3 in $k^+ = \boldsymbol{Q}(\sqrt{-3d})$ and $s = d(A')_3^+$. where $(A')^+$ is the 3-primary part of the 3-ideal class group of k^+. We find that the only value of d, $0 > d \geqq -200$ for which $s \neq 0$ is $d = -107$ for which $s = 1$. We note that $n = 2$ when $d = 3(3m-1)$ for an integer $m < 0$ and $n = 1$ otherwise. Then we conclude that for $0 > d \geqq -200$

$$|R_2(\boldsymbol{Q}\sqrt{d}))_3| = \begin{cases} 3 \text{ when } d = -39, -84, -107, -111, -120, -183 \\ 1 \text{ otherwise.} \end{cases}$$

PROPOSITION 11: *Let $d = 3(3m-1)$ for an integer $m < 0$. Suppose the field $\boldsymbol{Q}(\sqrt{d})$ has class number prime to 3. Then $R_2\boldsymbol{Q}(\sqrt{d})$ is cyclic of order 3 generated by $Tr_{\boldsymbol{Q}(\sqrt{d}, \sqrt{-3})/\boldsymbol{Q}(\sqrt{d})}\{\varepsilon, \zeta\}$ where ε is the fundamental unit of the real field $\boldsymbol{Q}(\sqrt{-d/3})$, and $\mathscr{R}_2\boldsymbol{Q}(\sqrt{d}) = 1$.*

PROOF: First we note that there are two primes $\mathscr{P}$ and $\mathscr{P}'$ dividing 3 in $k^+ = \boldsymbol{Q}(\sqrt{-d/3})$. Let M be the maximal 3-ramified abelian 3-extension of k^+ and $(A')^+$ be the 3-primary part of the 3-ideal class group of k^+. Then we have the exact sequence

$$0 \to k_{\mathscr{P}}^+ \times k_{\mathscr{P}'}^+/\langle 3\text{-units of } k^+\rangle \to G(M/k^+) \to (A')^+ \to 0$$

We know that $G(M/k^+) \simeq \boldsymbol{Z}_3 \oplus T$ where T is a finite group and, by the methods of the previous section, we establish that for fields of this type, $d(T/T^3) = d(A^-/(A^-)^3)$ where A^- is the 3-primary part of the ideal class group of $\boldsymbol{Q}(\sqrt{d})$. Then in this case $T = 0$, hence M is the cyclotomic $\boldsymbol{Z}_3$-extension of k^+. Thus M is totally ramified at all primes above 3, so $(A')^+ = 0$. We know that $R_2\boldsymbol{Q}(\sqrt{d})$ is cyclic and that $|R_2\boldsymbol{Q}(\sqrt{d})/\mathscr{R}_2\boldsymbol{Q}(\sqrt{d})| = 3$ (cf. proposition 4). The completion of $\boldsymbol{Q}(\sqrt{d})$ at the prime above 3 is $\boldsymbol{Q}_3(\sqrt{-3})$ and a direct calculation shows that in $\boldsymbol{Q}_3(\sqrt{-3})$, we have $(3, \zeta)_3 = 1$ and $(4, \zeta) \neq 1$. We now consider $Tr_{\boldsymbol{Q}(\sqrt{d}, \sqrt{-3},)/\boldsymbol{Q}\sqrt{d})}\{\varepsilon, \zeta\} \in R_2\boldsymbol{Q}(\sqrt{d})$. Any unit in $\boldsymbol{Q}_3$ may be written as $(4)^r x^3$, so in order for $\{\varepsilon, \zeta\}$ to be locally trivial at the prime above 3, it would be necessary for ε to embed locally as a cube at both of those primes. Then there would be an unramified cubic extension of $\boldsymbol{Q}(\sqrt{d})$ which contradicts the hypothesis that the class number of $\boldsymbol{Q}(\sqrt{d})$ is prime to 3. Hence $Tr_{\boldsymbol{Q}(\sqrt{d}, \sqrt{-3})/\boldsymbol{Q}(\sqrt{d})}\{\varepsilon, \zeta\} \notin$

$\mathscr{R}_2\boldsymbol{Q}(\sqrt{d})$, so we conclude that $R_2\boldsymbol{Q}(\sqrt{d})$ is cyclic of order 3 and $\mathscr{R}_2\boldsymbol{Q}(\sqrt{d}) = 1$.

The hypotheses of the above proposition are verified for $d = -39$, -84, -111, -120, and -183, hence $R_2\boldsymbol{Q}(\sqrt{d})$ is determined for those fields. For $d = -107$, we know that $R_2\boldsymbol{Q}(\sqrt{d}) = \mathscr{R}_2\boldsymbol{Q}(\sqrt{d})$, a non-trivial cyclic group.

When $0 < d \leqq 200$, we have $|R_2\boldsymbol{Q}(\sqrt{d})_3| = 3^{t-1+m}$ where m is the number of primes above 3 in $k^- = \boldsymbol{Q}(\sqrt{-3d})$ and $t = d(A')_3^-$, where $(A')^-$ is the 3-primary part of the 3-ideal class group of k^-. There are ten positive values of $d \leqq 200$ for which $t = 1$, and those are $d = 29, 77, 85, 93, 109, 113, 137, 172, 173$, and 181. For all other cases, $t = 0$. Again $m = 2$ when $d = 3(3u-1)$ for a positive integer u and $m = 1$ otherwise. The values of d of the form $3(3u-1)$ are $d = 24, 33, 60, 69, 105, 141, 168, 177$. Hence we have $|(R_2\boldsymbol{Q}(\sqrt{d})_3| = 3$ and $R_2\boldsymbol{Q}(\sqrt{d}) = \mathscr{R}_2\boldsymbol{Q}(\sqrt{d})$ when $d = 29, 77, 85, 93, 109, 113, 137, 172, 173, 181$ (Coates and Lichtenbaum have determined that the order of R_2F is the power of 3 dividing $\omega_2(F)\zeta_F(-1)$ for these fields [2]). $|R_2\boldsymbol{Q}(\sqrt{d})_3| = 3$ and $|R_2\boldsymbol{Q}(\sqrt{d})/\mathscr{R}_2\boldsymbol{Q}(\sqrt{d})| = 3$ when $d = 24, 33, 60, 69, 105, 141, 161, 177$. $R_2\boldsymbol{Q}(\sqrt{d}) = 1$ for all others.

In each of the eight fields where $d = 3(3u-1)$, let $\mathscr{P}$ be one of the ideals above 3 in k^-. Let c be the order of $\mathscr{P}$ in the ideal class group of k^- and let $\mathscr{P}^c = (\gamma)$, $\gamma \in k^-$. Then, by checking the local symbols (γ, ζ) at the ideals above 3 in $\boldsymbol{Q}(\sqrt{d}, \sqrt{-3})$, we determine that for $d = 24, 33, 60, 69$ and 177, $R_2\boldsymbol{Q}(\sqrt{d})$ is cyclic of order 3. For $d = 105, 141, 168$, we find

$$Tr_{\boldsymbol{Q}(\sqrt{d}, \sqrt{-3})/\boldsymbol{Q}(\sqrt{d})}\{\gamma, \zeta\} \in \mathscr{R}_2\boldsymbol{Q}(\sqrt{d}),$$

hence $R_2\boldsymbol{Q}(\sqrt{d})$ is cyclic of order divisible by 9. This is consistent with the conjecture that the order of $R_2\boldsymbol{Q}(\sqrt{d})$ is the power of 3 dividing

$$\omega_2(F)\zeta_F(-1).$$

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