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Compositio Mathematica, tome 29, nº 1 (1974), p. 1-7
<http://www.numdam.org/item?id=CM_1974__29_1_1_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 1, 1974, pag. 1–7
Noordhoff International Publishing
Printed in the Netherlands

INVERSE LIMITS OF SIMPLICIAL COMPLEXES

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Summary

Eilenberg and Steenrod have shown that any compact space may be expressed as an inverse limit in $\mathscr{T}op$, the category of topological spaces, of a diagram of simplicial complexes. [1]

We show that any paracompact space may be expressed as a limit of a diagram of nerves; further that $[-, K] \circ \mathscr{H} : \mathscr{T}op \to \mathscr{E}ns$ preserves such limits when K is a complex, where $[-, K]$ denotes the contravariant hom-functor from $\mathscr{H}tp$ (the category of spaces and homotopy classes of maps) to $\mathscr{E}ns$ the category of sets, and $\mathscr{H} : \mathscr{T}op \to \mathscr{H}tp$ is the canonical quotient functor.

1. Introduction

Let Δ denote a small category having the property that for A, B objects of Δ, there exists a C in Δ and maps $f: C \to A$, $g: C \to B$ respectively.

1.1 Definition: A diagram in a category $\mathscr{M}$ with scheme Δ will be said to be a semi inverse diagram.

1.2. Definition: $\Gamma: \mathscr{M} \to \mathscr{B}$ is a *semi-directly continuous* contravariant functor if it takes the limit of every semi inverse diagram $(\Delta, \Phi: \Delta \to \mathscr{M})$ in $\mathscr{M}$ to the colimit of $(\Delta, \Gamma \circ \Phi: \Delta \to \mathscr{B})$ in $\mathscr{B}$.

1.3 Remark: We assume familiarity with the diagram of nerves of a space arising from locally finite partitions of unity, and with the natural map which we shall denote by $\pi_j: X \to N_j$ from X a space admitting a locally finite partition of unity having nerve N_j. We recall that if one locally finite partition of unity with nerve N_j refines another with nerve N_k, then there is a map $\alpha_{jk}: N_j \to N_k$ which homotopy commutes with the canonical maps from X.

1.4 Remark: Our first proposition shows that when X is paracompact we may choose maps between the nerves such that X is the limit of the resulting semi-inverse diagram. This is the content of §2. In §3 we show that if K is a complex, $[-, K] \circ \mathscr{H} : \mathscr{T}op \to \mathscr{E}ns$ preserves limits of this type. Finally in §4 we conclude with remarks on some applications of these results.

2.

2.1 DEFINITION: Let $\{f_u\}_U$ and $\{g_v\}_V$ denote locally finite partitions of unity on a space X, with corresponding nerves N and M. The *product partition* has maps the non-zero functions $\{f_u \cdot g_v\}_{U \times V}$, multiplying values in $[0, 1]$ pointwise, and the *product nerve* $N \times M$ is the nerve of this cover. It is trivial to verify that the product partition is a locally finite partition of unity.

2.2 PROPOSITION: *There are projections $p_n : N \times M \to N$, $p_m : N \times M \to M$ which commute in $\mathcal{T}op$ with the canonical maps from X, $\pi_{N \times M}$, π_N and π_M.*

PROOF: Let $\{f_u\}_U$ and $\{g_v\}_V$ be the partitions of unity on X giving rise to N, M respectively. Let $\{i_u\}_U$, $\{j_v\}_V$ be the sets of vertices of N, M respectively, corresponding to f_u, g_v, and let (i_u, j_v) be the vertex of $N \times M$ corresponding to the map $f_u \cdot g_v : X \to [0, 1]$, the vertex existing only when the map is not the zero map. Define $p_N : N \times M \to N$ on vertices by $(i_u, j_v) \mapsto i_u$. Then it is clear that when any finite collection

$$(i_1, j_1), (i_2, j_2) \cdots (i_r, j_r)$$

determines a simplex of $N \times M$, then the set $i_1, i_2, \cdots, i_r$ determines a simplex of N, and we may extend the vertex map linearly. Hence p_N is well defined. We show it commutes with $\pi_{N \times M}$ and π_N. Take x in X. If $\pi_N(x)$ has non-zero co-ordinates the set $f_1(x), f_2(x), \cdots, f_t(x)$ in N, with respect to vertices i_1 to i_t and if $\pi_M(x)$ has non-zero co-ordinates the set of maps $g_1(x), g_2(x), \cdots, g_s(x)$ in M, with respect to vertices j_1 to j_s, then $\pi_{N \times M}(x)$ in $N \times M$ is in the simplex with vertices the (i_a, j_b) for $1 \leqq a \leqq t$, $1 \leqq b \leqq s$, some of which may be missing, and co-ordinates the $f_a \cdot g_b$ respectively. Now we have by linearity:

$$\begin{aligned} p_N(\pi_{N \times M}(x)) &= p_N \sum_{\substack{1 \leqq a \leqq t \\ 1 \leqq b \leqq s}} (f_a \cdot g_b)(x) \cdot (i_a, j_b) \\ &= \sum_{1 \leqq a \leqq t} p_N \sum_{1 \leqq b \leqq s} f_a(x) \cdot g_b(x) \cdot (i_a, j_b) \\ &= \sum_{1 \leqq a \leqq t} f_a(x) \cdot p_N \sum_{1 \leqq b \leqq s} g_b(x) \cdot (i_a, j_b) \\ &= \sum_{1 \leqq a \leqq t} f_a(x) \cdot \sum_{1 \leqq b \leqq s} g_b(x) \cdot i_a \\ &= \sum_{1 \leqq a \leqq t} f_a(x) \cdot i_a \quad \text{Since } \Big(\sum_{1 \leqq b \leqq s} g_b(x) = 1 \Big) \\ &= \pi_N(x). \end{aligned}$$

2.3 DEFINITION: Let N be a nerve arising from a partition of unity on X, and M a subdivision of N. Then π_N regarded as a map to M defines a partition of unity on X which has as its nerve the subset of M containing simplices intersecting $\pi_N(X)$, and π_M composed with the inclusion is π_N. We shall call such a refinement a *proper refinement*, and refer to the nerve M as a *proper refinement* of N.

2.4 Definition: For any space X, the *proper diagram of nerves* is the diagram containing all nerves of locally finite partitions of unity on X, and having maps either the inclusions of 2.3 or else the projections of 2.2.

2.5 Remark: It is clear that the diagram is a semi-inverse diagram, but in general for N a proper refinement of M, and $N \times M$ the product, $i: N \to M$ the inclusion, it is not the case that $i \cdot p_N$ is equal to p_M. Hence the diagram is not an inverse system. However, we do have $i \cdot p_N$ homotopic to p_M, as is well known.

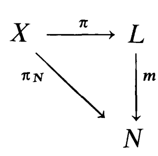
2.6 Proposition: *A paracompact space X is the limit in $\mathcal{T}op$ of its proper diagram of nerves.*

Proof: The family of natural maps $\pi_N: X \to N$ is a compatible family of maps into the diagram, and hence determines a unique map $\pi: X \to L$, where L is the limit of the diagram.We show that π is a homeomorphism.

It is immediate that π is continuous, and evident that it is injective, since any two points of X may be sent to 0 and 1 in a nerve which is a copy of the unit interval. This only requires X to be completely regular, which of course it is.

We show that π is surjective: suppose there is a y in $L - \pi X$. Since L is a completely regular space (all the nerves are, and L is a subset of their product) we take all the closed neighbourhoods of y in L and observe that their intersection is y.

The family of complements is an open cover of πX, and hence determines an open cover of X. Since X is paracompact, there is a subordinates partition of unity, and a nerve rising from it which we shall call N. We consider the diagram:



where m is the canonical map from the limit into the element of the diagram. By construction the diagram commutes.

Now if $y \in L$, my is in some simplex of N, say $(v_1, v_2, \cdots, v_s)$, and by the continuity of m, there is some neighbourhood U of y in L such that $mU \subset \mathrm{st}\, v_1$, where st denotes the open star on the vertex. If V_1 is the open set on X corresponding to the vertex v_1, then it follows that $\pi_N^{-1}(mU) \subset V_1$, and by commutativity that $(m \cdot \pi)^{-1}(mU) \subset V_1$, that is to say we have that

$$\pi^{-1}(m^{-1}(mU)) \subset V_1 \subset X$$

whence

$$\pi^{-1}U \subset V_1 \subset X,$$

and

$$\pi \cdot \pi^{-1}U \subset \pi V_1 \subset L.$$

By hypothesis, πV_1 is disjoint from y; moreover, by our construction of the cover of X, we can find a W open in L and containing y such that $W \subset \pi$, $x^{-1}U = \emptyset$. Without loss of generality, we may take $W \subset U$. Now it follows that W is disjoint from πX in L, for if $w \in W \cap \pi X$, $\exists w' \in X : \pi w' = w$, and $w \in W \cap \pi \cdot \pi^{-1}U$, contra. We have shown therefore that if $y \in L - \pi X$, $y \in L - \overline{\pi X}$ also, hence that πX is closed in L. Now an open set W on y in L contains on y the intersection of some open sets with L

$$L \cap m_1^{-1}(W_1) \cap m_2^{-1}(W_2) \cap \cdots, m_t^{-1}(W_t)$$

for some open sets $W_1, W_2, \cdots, W_t$ in nerves $N_1, N_2, \cdots, N_t$, where the maps $m_1, m_2, \cdots, m_t$ are the canonical maps from the limit.

Form $M = N_1 \times N_2 \times \cdots \times N_t$, the nerve product. Let $p_1, p_2, \cdots, p_t$ denote the projections from M to $N_1, N_2, \cdots, N_t$ respectively. Then $p_1^{-1}(W_1) \cap p_2^{-1}(W_2) \cap \cdots \cap p_t^{-1}(W_t)$ is an open set in M, containing $m_M(y)$, and is clearly disjoint from $\pi_M X$. But in this event, we can take a subdivision of M excluding $m_M(y)$, contradicting y being an element of L. Hence there are no elements in $L - \pi X$, i.e. π is surjective.

Finally to prove that π is open, take A open in X and $a \in A$. If $A = X$ there is nothing to prove, so take $A \neq X$. Then there is a map $\alpha : X \to [0, 1]$ such that $\alpha(a) = 1$ and $\alpha^{-1}(0, 1] \subset A$. This yields a nerve I of a covering with α as canonical map, and if m_I is the map from L to this nerve, then $B = m_I^{-1}(0, 1]$ is open in L. Clearly $\pi x \in B$, and since π is onto, $B = \pi \cdot \pi^{-1}B$, but $\pi^{-1}B = (\pi \cdot m_I)^{-1}(0, 1] \subset A$, hence $B \subset \pi A$ and the result follows.

3.

3.1 PROPOSITION: *With the notation of* 2.6, *if K is a simplicial complex, and a paracompact space X is a limit of its proper diagram of nerves, Δ, then*

$$[-, K] \cdot \mathscr{H}(X) = \varinjlim [-, K] \cdot \mathscr{H}\Delta.$$

PROOF: For all N_δ, N_ε objects of Δ, and all $h_{\delta\varepsilon}: N_\delta \to N_\varepsilon$ the corresponding maps of Δ, we have the set

$$\{[-, K] \circ \mathscr{H}(N_\delta): N_\delta \in \Delta\}$$

i.e. the set of homotopy classes of maps, $\{[N_\delta, K]: N_\delta \in \Delta\}$. On this set we have the relation $\sim$ given by

$$[g_\delta] \sim [g_\varepsilon] \text{ iff } [g_\delta] = [g_\varepsilon \circ h_{\delta\varepsilon}]$$

i.e. iff

$$([-, K] \circ \mathscr{H}(h_{\delta\varepsilon}))[g_\varepsilon] = [g_\delta]$$

Then $t <$ e set $\lim_{\to}[-, K] \circ \mathscr{H}\Delta$ is the set $\{[-, K] \circ \mathscr{H}(N_\delta): N_\delta \in \Delta\}$ factored by the smallest equivalence containing the relation $\sim$.

We write $\{g_\varepsilon\}$ for the class containing $[g_\varepsilon] \in [N_\varepsilon, K]$. Since the canonical maps $\pi_\delta: X \to N_\delta$ commute with the maps of the diagram Δ we may define

$$\ell: \varinjlim [-, K] \cdot \mathscr{H}\Delta \to [-, K] \cdot \mathscr{H}X \quad \text{by} \quad \{g_\varepsilon\} \mapsto [g_\varepsilon \cdot \pi_\varepsilon]$$

and this is well defined, i.e. it is independent of the choice of representative in $\{g_\varepsilon\}$.

ℓ is surjective. For take $[g] \in [X, K]$. Choosing a representative $g \in [g]$, $g: X \to K$ in $\mathscr{T}op$, we note that g factors through the canonical map to a nerve N (pull back the star open cover of K; we get not only an open cover of X but also a natural partition of unity. Then N is a subcomplex of K, viz. that complex having simplices of K intersecting gX) and we write $g = \pi_N \cdot j$ where $j: N \to K$ is the inclusion. Then j certainly defines a class $\{j\}$ in the limit, and $\ell(\{j\}) = [g]$. Choosing a different representative of $[g]$ might conceivably give us a different $\{j\}$, but any one suffices. In fact this does not happen, because: ℓ is injective. Suppose $\ell(\{g_\varepsilon\}) = \ell(\{f_\delta\})$, that is to say, we have $[g_\varepsilon]$ and $[f_\delta]$ such that $[g_\varepsilon \cdot \pi_\varepsilon] = [f_\delta \cdot \pi_\delta]$. This assures us of a homotopy $H: X \times I \to K$ between $g_\varepsilon \circ \pi_\varepsilon$ and $f_\delta \circ \pi_\delta$, which determines a partition of unity on $X \times I$, arising from the star open cover of K. We consider the cover defined by the portion of unity. For each x in X and each t in I, there are some finite number of sets of the cover, say $V_t^1, V_t^2, \cdots, V_t^n$ on x, t. Let V_t be their intersection. Cover $x \times I$ by such sets. Since it is compact, there is a finite subcover, say $V_1, V_2, \cdots, V_s$. Project each set down to X and take V_x to be the intersection.

Doing this for each x in X we obtain a cover of X, χ such that $\forall t \in [0, 1]$, H_t^{-1} st.o.c. K is refined by χ, where st.o.c. K is of course the star open cover of K. Let N be a nerve arising from a partition of unity subordinate to χ; then $H_t: X \to K$ factors through N for all t, i.e. H factors through

$N \times I$. Since we have refined the partition of unity corresponding to N_δ (and N_ε) in the process, there is a map from N to each of them, say α, β respectively, with $g_\varepsilon \circ \beta$ homotopic to $f_\delta \circ \alpha$ by the map from $N \times I$ to K. It is to be remembered that α, β will not in general be in Δ, but we may take then a nerve product $N \times N_\varepsilon \times N_\delta$ in place of N, and the result follows.

3.2 REMARK: The two preceding propositions admit the following partial converse: if P is paracompact and $[-, P] \cdot \mathscr{H}$ preserves limits of proper diagrams of nerves then P has the homotopy type of a complex. This follows immediately from Milnor [2] where it is shown that a space dominated by a CW complex has the homotopy type of one, and the observation that the identity map on P must factor through a nerve.

4.

4.1 DEFINITION: Let $\mathscr{C}$ denote the full subcategory of simplicial complexes; $\mathscr{H}\mathscr{C}$ we take to be either the image of $\mathscr{C}$ under $\mathscr{H}$ in $\mathscr{H}tp$ or the full subcategories of $\mathscr{H}tp$ having as objects those spaces having the homotopy type of a simplicial complex (this includes the CW complexes, by [2]).

4.2 PROPOSITION: *Let $\Gamma : \mathscr{T}op \to \mathscr{E}ns$ be a contravariant functor preserving limits of proper diagrams of simplicial complexes, and factoring through $\mathscr{H}tp$ to give $[\Gamma]$, with $[\Gamma]|\mathscr{H}\mathscr{C}$ representable by a complex K. Then the representation extends to spaces having the homotopy type of a paracompact space.*

PROOF: We have $\sigma : [-, K] \to [\Gamma]$ a natural transformation arising from some $s \in \Gamma K$ by the Yoneda lemma, and $\sigma|\mathscr{H}\mathscr{C}$ is an equivalence.

If P is a paracompact space we express it as the limit of the appropriate diagram of nerves in $\mathscr{T}op$. Now

$$[P, K] \approx [\lim \Delta, K] \approx \operatorname{colim} [\Delta, K] \quad \text{by continuity of } [-, K] \cdot \mathscr{H}$$

and

$$\operatorname{colim} [\Delta, K] \approx \operatorname{colim} [\Gamma](\Delta) \approx \operatorname{colim} \Gamma\Delta$$

$$\operatorname{colim} \Gamma\Delta \approx \Gamma \lim \Delta \quad \text{by continuity of } \Gamma$$

$$\approx \Gamma P$$

where the $\approx$ signs denote various natural equivalences, and some minor abuses of language have taken place.

4.3 REMARK: It follows by the same kind of argument that the Čech cohomology theory is, as is well known, representable on paracompacta.

More generally, if a suitable functor is given on $\mathscr{HC}$ and extended by the generalized Čech process, then we can again expect to obtain an extension of representability.

I wish to express my gratitude to Professor C. T. C. Wall for his help and supervision of the doctoral thesis of which this is a part.

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(Oblatum 10–II–1972 & 23–IV–1974)

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