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A SHORT PROOF OF FINITE MONODROMY FOR ANALYTICALLY IRREDUCIBLE PLANE CURVES

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Introduction

E. Brieskorn posed the following problem: For a complex hypersurface having an isolated singularity at 0, is the local monodromy h of finite order? Lê Dũng Tráng [3] gave an affirmative answer to the problem in the case of analytically irreducible plane curves, utilizing Zariski's results [8] to investigate the Alexander invariants. A'Campo [1] has given a simpler proof of this result, and A'Campo's method allows explicit calculation of the matrix of the monodromy transformation for plane curves in terms of the Puiseux characteristic pairs of the singularity. A'Campo also provided in [1] a negative answer to Brieskorn's question in general by exhibiting a reducible plane curve with infinite monodromy, from which he obtained higher-dimensional singularities with infinite monodromy.

The short proof given in this paper is a direct application of results of Brauner [2], Brieskorn & Pham [4], and Shinohara [6]. Shinohara's method ([6], pp. 39–42) enables us to construct a proof with few computations, and yields a theorem about the monodromy of higher-dimensional tube knots associated with singularities.

THEOREM 1: *Let $f: C^{m+1} \to C$ be a complex polynomial such that f has an isolated critical point at* 0. *If the associated algebraic link L, where $L = (f^{-1}(0)) \cap S_\varepsilon^{2m+1}$ for ε small* [4], *is equivalent to an iterated tube knot of type $\{K_1, K_2, \cdots, K_r\}$, then the local monodromy associated with L has finite order iff the local monodromy associated with each K_i has finite order, $1 \leqq i \leqq r$.*

1. Shinohara's method–knots in tubes

DEFINITION: An m-knot K is a smooth oriented submanifold of S^{m+2} homeomorphic to S^m. Let V be a tubular neighborhood of K, and V' be a tubular neighborhood of the trivial m-knot K'. Let $f: V' \to V$

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be an orientation-preserving diffeomorphism of V' onto V which carries longitudes to longitudes. Let L' be a knot contained in the interior of V'. Then $L' \sim \gamma K'$ in V' for some integer γ (for L' a torus knot $(m = 1)$ of type (p, q), $\gamma = q$). Moreover, $L = f(L')$ is a knot in the interior of V and $L \sim \gamma K$ in V. L will be called a *tube knot of type* $\{K, L'\}$. An *interated tube knot* L_r *of type* $\{K_1, K_2, \cdots, K_r\}$ is the knot L_r where L_i is inductively defined to be a tube knot of type $\{L_{i-1}, K_i\}$, $2 \leqq i \leqq r$, and $L_1 = K_1$.

Let K, L, K', L', and γ be as above, and let U be a tubular neighborhood of L which is contained in the interior of the tubular neighborhood V of K (see Fig. 1). We introduce the following notation: $X = S^{m+2} - \text{Int } U$, $W = V - \text{Int } U$, $Y = S^{m+2} - \text{Int } V$, and $T = \partial V$. Let $\tilde{X}$ denote the

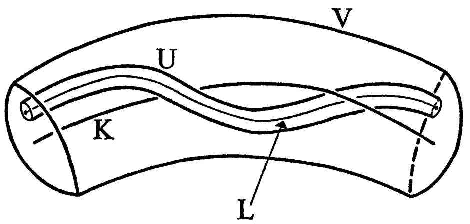

Figure 1

infinite cyclic covering space of X with t as the generator of the infinite cyclic multiplicative group of covering transformations, and $\tilde{T}$, $\tilde{W}$, $\tilde{Y}$ denote the induced (not necessarily connected) coverings of T, W, Y respectively. Let ${}_L\Delta_i^q(t)$ denote the i^{th} polynomial invariant of the knot L in dimension q, that is the i^{th} polynomial invariant of the module $H_q(\tilde{X}; Q)$[7]. Let

$${}_L\lambda_1^q(t) = {}_L\Delta_1^q(t)/{}_L\Delta_2^q(t)$$

denote the minimal polynomial of the Q-linear transformation (induced by covering translation) $t_* : H_q(\tilde{X}; Q) \to H_q(\tilde{X}; Q)$, or simply the minimal polynomial of the module $H_q(\tilde{X}; Q)$.

Shinohara shows that for $\gamma \neq 0$, the following sequence is exact (it is in fact split exact when $m = 1$ and L' is a torus knot):

(1) $\quad 0 \to H_m(\tilde{T}; Q) \to H_m(\tilde{W}; Q) \oplus H_m(\tilde{Y}; Q) \to H_m(\tilde{X}; Q) \to 0$

and that

(2) $\quad H_q(\tilde{X}; Q) \cong H_q(\tilde{W}; Q) \oplus H_q(\tilde{Y}; Q)$ for $1 \leqq q \leqq (m-1)(m > 1)$.

He also shows that

(3) order $H_m(\tilde{T}; Q) = t^\gamma - 1$, where order $H_m(\tilde{T}; Q)$ is the first polynomial invariant of the module $H_m(\tilde{T}; Q)$[5], and

$$\text{(4)} \qquad \text{order } H_q(\tilde{W}; Q) = \begin{cases} {}_{L'}\Delta_1^q(t), & 1 \leqq q \leqq (m-1) \\ (t^\gamma - 1)_{L'}\Delta_1^m(t), & q = m. \end{cases}$$

Since Shinohara further shows that

$$\text{(5)} \qquad \text{order } H_q(\tilde{Y}; Q) = {}_K\Delta_1^q(t^\gamma),$$

it follows that

$$\text{(6)} \qquad \text{order } H_q(\tilde{X}; Q) = {}_L\Delta_1^q(t) = {}_{L'}\Delta_1^q(t)_K\Delta_1^q(t^\gamma), \; 1 \leqq q \leqq m.$$

Similar results follow for $\gamma = 0$.

LEMMA 1: ${}_L\lambda_1^q(t)$ *divides the least common multiple*

$$[{}_{L'}\lambda_1^q(t), {}_K\lambda_1^q(t^\gamma)] \qquad 1 \leqq q \leqq m.$$

Furthermore, if $1 \leqq q \leqq m$ *or* L' *is a torus knot* $(m = 1)$, *then*

$${}_L\lambda_1^q(t) = l.c.m.[{}_{L'}\lambda_1^q(t), {}_K\lambda_1^q(t^\gamma)].$$

PROOF: The lemma is immediate from (1)–(5) and the fact that the minimal polynomial of the direct sum of two modules is the l.c.m. of the two minimal polynomials.

We now have the following theorem, which will, as a special case yield finite monodromy for analytically irreducible plane curves.

THEOREM 1: *Let* $f: C^{m+1} \to C$ *be a complex polynomial such that* f *has an isolated critical point at* 0. *If the associated algebraic link* L *is equivalent to an iterated tube knot of type* $\{K_1, K_2, \cdots, K_r\}$, *then the local monodromy* h *of* f *at* 0 *has finite order iff* ${}_{K_i}\lambda_1^m(t)$ *has distinct roots,* $1 \leqq i \leqq r$. *That is, the local monodromy of* L *has finite order iff the local monodromy of each* K_i *has finite order,* $1 \leqq i \leqq r$.

PROOF: It is immediate from Lemma 1 and the definition of iterated tube knot that ${}_L\lambda_1^m(t)$ has distinct roots iff each ${}_{K_i}\lambda_1^m(t)$ has distinct roots, $1 \leqq i \leqq r$. By a result of Grothendieck [4], ${}_L\lambda_1^m(t)$ is a product of cyclotomic polynomials. Since it is known that ${}_L\lambda_1^m(t)$ is the minimal polynomial of h, it follows that h has finite order iff the local monodromy of each K_i has finite order.

2. Analytically irreducible plane curves

Let $f: C^2 \to C$ be a polynomial function such that $f(0) = 0$, f has an isolated singularity at 0, and f is analytically irreducible.

THEOREM 2: *The local monodromy* h *of* f *at* 0 *is of finite order.*

PROOF: Let $\{(m_1, n_1), \cdots, (m_r, n_r)\}$ be the set of Puiseux characteristic pairs of f at 0 [3], and let $\mu_1 = m_1$ and $\mu_i = m_i - m_{i-1}n_i + \mu_{i-1}n_{i-1}n_i$, $2 \leqq i \leqq r$. Then, by Brauner's theorem [2], the associated algebraic link L is an iterated tube knot of type $\{K_1, K_2, \cdots, K_r\}$ where K_i is the torus knot of type (μ_i, n_i), $1 \leqq i \leqq r$. The Brieskorn-Pham theorem [4] and the fact that μ_i and n_i are relatively prime yield that ${}_{K_i}\lambda_1^1(t)$ has distinct roots, $1 \leqq i \leqq r$. Thus, by Theorem 1, ${}_L\lambda_1^1(t)$ has distinct roots, or equivalently h has finite order.

UNSOLVED PROBLEM: For $m \geqq 2$, when is the algebraic link L of the complex polynomial $f: C^{m+1} \to C$ with an isolated singularity at 0 equivalent to an iterated tube knot?

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