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TRANSCENDENCE MEASURES OF CERTAIN NUMBERS WHOSE TRANSCENDENCY WAS PROVED BY A. BAKER

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1. Introduction

In the subsequent paper we continue the investigation of transcendence measures of certain transcendental numbers σ, i.e. positive lower bounds for $|P(\sigma)|$ in terms of the degree N and height H of P, where P is an arbitrary polynomial with integral coefficients. For more information about transcendence measures and the type of transcendence measures we will look for, see the earlier paper [4]; see also the authors thesis [3], which includes the results of the present paper.

Let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers such that, for any (fixed) values of the logarithms, $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\boldsymbol{Q}$. In this paper, transcendence measures are derived for numbers which can be written in one of the following ways:

(i) $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$ with $n \geqq 2$, where $\beta_1, \cdots, \beta_n$ are algebraic numbers, not all zero

(ii) $e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ with $n \geqq 1$ and

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i}$$

where $\beta_0, \beta_1, \cdots, \beta_n$ are algebraic numbers such that at least one of $\beta_1, \cdots, \beta_n$ is irrational. We prove the transcendence measure

$$\exp\{-C_1 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}$$

for numbers of the form (1), and the transcendence measure

$$\exp\{-C_2 N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon}\}$$

for numbers of the form (2). Here $S = N + \log H$, ε is an arbitrary positive number and C_1 and C_2 are effectively computable numbers, depending on ε, n, the α's and their logarithms and the β's.

We remark, that these transcendence measures are the first explicit ones to be published for these numbers in which both the dependence on H and N is expressed. If one is interested merery in the height H, better results can be given. For numbers of the form (i), N. I. Fel'dman [6] proved the transcendence measure $\exp\{-CS\}$, where C is a positive number, depending on N, the α's and their logarithms and the β's. For numbers of the form (ii), a recent result of A. BAKER [2] implies the transcendence measure $\exp\{-C\log H\log\log H\}$ for $H \geqq 4$, where C again is a positive number depending on N, the α's and their logarithms and the β's. An earlier result for the special case of numbers of the form $e^{\beta_0}\alpha_1^{\beta_1}$ can be found in [7].

The method of proof of our transcendence measures is A. BAKER's one with some improvements introduced by N. I. FEL'DMAN. In the proof we firstly derive a measure for the approximability of numbers of the types (i) and (ii); after that, the transcendence measures are given.

The transcendence of numbers of the form (i) follows immediately from e.g. Theorem 1 of A. BAKER's paper [1]. The transcendence of numbers of the form (ii) was proved by A. BAKER, distinguishing the cases $\beta_0 = 0$ and $\beta_0 \neq 0$; see the same paper.

2. Lemmas

We shall use the same notations (especially for the degree; height and size) as in [4]. For shortness, we use without reformulation the lemmas 3, 6, 7, 8 and 9 of [4]. Further, we need the following lemmas:

LEMMA 1: *Let $\alpha_1, \cdots, \alpha_n$ be non-zero algebraic numbers such that, for any fixed values of the logarithms,* $\log\alpha_1, \cdots, \log\alpha_n$ *are linearly independent over Q. Let ε be positive and let d be a positive integer. Then there exists an effectively computable positive number*

$$\theta(\varepsilon, d) = \theta(\varepsilon, d, \log\alpha_1, \cdots, \log\alpha_n)$$

such that

$$(1)\quad |\beta_0+\beta_1\log\alpha_1+\cdots+\beta_n\log\alpha_n| > \theta(\varepsilon, d)\exp\{-(\log h)^{1+\varepsilon}\}$$

for all algebraic numbers $\beta_0, \beta_1, \cdots, \beta_n$, not all zero, of degrees at most d and of heights at most h.

PROOF: See Theorem 1 of [5].

LEMMA 2: *Let α and β be algebraic numbers, $\beta \neq 0$. Then*

$$(2)\qquad s(\alpha\beta^{-1}) \leqq 2d(\alpha)d(\beta)+d(\alpha)s(\beta)+d(\beta)s(\alpha).$$

PROOF: From Lemma 3 of [5] it follows that

$$h(\alpha\beta^{-1}) \leqq 2^{d(\alpha)d(\beta)}\{h(\alpha)(d(\alpha)+1)\}^{d(\beta)}\{h(\beta)(d(\beta)+1)\}^{d(\alpha)}.$$

Using $d(\alpha)+1 < e^{d(\alpha)}$ and $d(\beta)+1 < e^{d(\beta)}$ we get

$$\begin{aligned} s(\alpha\beta^{-1}) &= d(\alpha\beta^{-1})+\log h(\alpha\beta^{-1}) \\ &\leqq d(\alpha)d(\beta)(1+\log 2)+d(\beta)s(\alpha)+d(\alpha)s(\beta) \end{aligned}$$

from which (2) follows.

LEMMA 3: *Let β be an algebraic number, and let k and ℓ be integers. Then*

$$h(k+\ell\beta) \leqq h(\beta)|2k\ell|^{d(\beta)}. \tag{3}$$

PROOF: If $a_n z^n + \cdots + a_1 z + a_0$ is the minimal polynomial of β, then

$$a_n(z-k)^n + \cdots + a_1 \ell^{n-1}(z-k) + a_0 \ell^n$$

is a constant multiple of the minimal polynomial of $k+\ell\beta$. Thus, the coefficient of z^i $(i = 0, 1, \cdots, n)$ of this minimal polynomial is in absolute value at most

$$\left| a_n \binom{n}{i} (-k)^{n-i} + a_{n-1}\ell \binom{n-1}{i} (-k)^{n-1-i} + \cdots + a_i \ell^{n-i} \right|$$
$$\leqq h(\beta)|k|^n|\ell|^n \left\{ \binom{n}{i} + \binom{n-1}{i} + \cdots + 1 \right\}.$$

From the obvious inequality $\binom{m}{i} \leqq 2^{m-1}$ for all positive integers m, it follows that

$$\binom{n}{i} + \binom{n-1}{i} + \cdots + 1 \leqq 2^n,$$

by which the proof is completed.

3. The case $\beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n$

First we give a measure for the approximability for numbers of the form (i), in the special case in which $\beta_n = -1$.

THEOREM 1: *Let, for $n \geqq 2$, $\alpha_1, \cdots, \alpha_n$ and $\gamma_1, \cdots, \gamma_{n-1}$ be non-zero algebraic numbers such that, for any fixed values of the logarithms, $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\boldsymbol{Q}$. Let ε be a positive number. Then there exists an effectively computable number $S_1 = S_1(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \gamma_1, \cdots, \gamma_{n-1})$ such that*

$$|\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| > \exp\{-N^{n^2+n+\varepsilon} S(\log S)^{n+1+\varepsilon}\} \tag{4}$$

for all algebraic numbers η of degree N and size $S \geqq S_1$.

PROOF: Put $\delta = (2n^3+4n^2+3n+7)^{-1}\varepsilon$. For abbreviation, put

$$\sigma = \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n$$

and

$$U = N^{n^2+n+(2n^3+4n^2+2n+1)\delta} S(\log S)^{n+1+(2n^2+3n+7)\delta}.$$

It is sufficient to prove that

$$|\sigma-\eta| > \exp\{-U\}$$

if $S \geqq S_1$. In this proof we may restrict ourselves to the case in which δ is rather small. By $c_1, c_2, \cdots$ we denote positive numbers which depend only on $n, \log\alpha_1, \cdots, \log\alpha_n, \gamma_1, \cdots, \gamma_{n-1}$.

Suppose that

$$|\sigma-\eta| \leqq \exp\{-U\} \tag{5}$$

for some algebraic number η of degree N and size S. We prove that this leads to a contradiction if S is sufficiently large.

Choose the following integers:

$$\begin{aligned}
K &= [N^{n+(n^2-1)\delta}(\log S)^{1+(2n+1)\delta}],\\
M &= [N^{n+(n^2+n)\delta}S(\log S)^{1+(2n+3)\delta}],\\
C &= 2[\tfrac{1}{2}\exp\{N^{n+(n^2+n)\delta}S(\log S)^{2+(2n+4)\delta}\}],\\
T &= [N^{n+1+(n^2+n)\delta}(\log S)^{1+(2n+3)\delta}],\\
P &= [S(\log S)^{1+2\delta}],\\
R &= \left[\frac{n-1}{\delta}+2n^2+n\right],\\
T' &= [2^{-R}T] \text{ and}\\
P' &= [N^{n^2-1+(n^3-n)\delta}S(\log S)^{n+(2n^2+n+1)\delta}].
\end{aligned}$$

Put

$$F(z) = \sum_{k_1=0}^{K-1}\cdots\sum_{k_n=0}^{K-1}\sum_{m=0}^{M-1}\sum_{v=0}^{N-1} C_{k_1\cdots k_n m v}\eta^v z^m \times\exp\{-k_n\eta z+\sum_{i=1}^{n-1}(k_i+k_n\gamma_i)(\log\alpha_i)z\},$$

where the numbers $C_{k_1\cdots k_n m v}$ are integers of absolute values at most C; they will be specified later.

For $t = 0, 1, 2, \cdots$ it is easily seen that

$$F^{(t)}(z) = \sum_{\tau+\tau_1+\cdots+\tau_{n-1}=t}\frac{t!}{\tau!\tau_1!\cdots\tau_{n-1}!}\times(\log\alpha_1)^{\tau_1}\cdots(\log\alpha_{n-1})^{\tau_{n-1}}F_{\tau\tau_1\cdots\tau_{n-1}}(z) \tag{6}$$

where

$$\begin{aligned}
F_{\tau\tau_1\cdots\tau_{n-1}}(z) &= \sum_{k_1}\cdots\sum_{k_n}\sum_{m}\sum_{v} C_{k_1\cdots k_n m v}\eta^v\\
&\times\sum_{\kappa=0}^{m}\binom{\tau}{\kappa}\frac{m!}{(m-\kappa)!}z^{m-\kappa}(-k_n)^{\tau-\kappa}\eta^{\tau-\kappa}\prod_{i=1}^{n-1}(k_i+k_n\gamma_i)^{\tau_i}\\
&\times\exp\{(\sum_{i=1}^{n}k_i\log\alpha_i)z+k_n(\sigma-\eta)z\}.
\end{aligned} \tag{7}$$

Put

$$\Phi_{\tau\tau_1\cdots\tau_{n-1}}(z) = \sum_{k_1}\cdots\sum_{k_n}\sum_{m}\sum_{\nu} C_{k_1\cdots k_n m\nu}\eta^{\nu}$$
$$\times \sum_{\kappa=0}^{m}\binom{\tau}{\kappa}\frac{m!}{(m-\kappa)!}z^{m-\kappa}(-k_n)^{\tau-\kappa}\eta^{\tau-\kappa}\prod_{i=1}^{n-1}(k_i+k_n\gamma_i)^{\tau_i}$$
$$\times \exp\{(\sum_{i=1}^{n} k_i \log \alpha_i)z\}.$$

We estimate the difference

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)-\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p)|$$

as follows: by $|e^z-1| \leqq |z|e^{|z|}$, one has for $k_n = 0, 1, \cdots, K-1$ and

$$p = 0, 1, \cdots, [N^{n^2-1+(2n^3+3n^2+n)\delta}S(\log S)^{n+(2n^2+n+2)\delta}]$$

the inequality

$$|e^{k_n(\sigma-\eta)p}-1| \leqq \exp\{-\tfrac{1}{2}U\}.$$

Hence,

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)-\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\} \tag{8}$$

for $\tau, \tau_1, \cdots, \tau_{n-1} = 0, 1, \cdots, T-1$ and

$$p = 0, 1, \cdots, [N^{n^2-1+(2n^3+3n^2+n)\delta}S(\log S)^{n+(2n^2+n+2)\delta}].$$

Let

$$P_{\tau\tau_1\cdots\tau_{n-1}pk_1\cdots k_n m}(z_0, z_1, \cdots, z_{2n-1})$$

($\tau, \tau_1 \cdots, \tau_{n-1} = 0, 1, \cdots, T-1$; $p = 0, 1, \cdots, P-1$; $k_1, \cdots, k_n = 0, 1, \cdots, K-1$; $m = 0, 1, \cdots, M-1$) be the polynomials, chosen in the appropriate way, such that

$$\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p) = \sum_{k_1=0}^{K-1}\cdots\sum_{k_n=0}^{K-1}\sum_{m=0}^{M-1}\sum_{\nu=0}^{N-1} C_{k_1\cdots k_n m\nu}\eta^{\nu}$$
$$\times P_{\tau\tau_1\cdots\tau_{n-1}pk_1\cdots k_n m}(\eta, \alpha_1, \cdots, \alpha_n, \gamma_1, \cdots, \gamma_{n-1}).$$

We apply Lemma 6 of [4] to these polynomials in the specified points. If r, s and B have the same meaning as in this lemma we have

$$r = T^nP \leqq N^{n^2+n+(n^3+n^2)\delta}S(\log S)^{n+1+(2n^2+3n+2)\delta},$$
$$s = K^nM \geqq \tfrac{1}{2}N^{n^2+n+(n^3+n^2)\delta}S(\log S)^{n+1+(2n^2+3n+3)\delta}.$$

Hence, $s \geqq 4rd$ where $d = [Q(\alpha_1, \cdots, \alpha_n, \gamma_1, \cdots, \gamma_{n-1}) : Q]$. Further,

$$B \leqq \exp\{c_1 N^{n+(n^2+n)\delta}S(\log S)^{2+(2n+3)\delta}\}.$$

Hence, the right hand side of the second condition in Lemma 6 of [4] is at most

$$\exp\{c_2 N^{n+1+(n^2+n)\delta} S(\log S)^{2+(2n+3)\delta}\} \leqq C^N.$$

It follows that the numbers $C_{k_1\cdots k_n m\nu}$ can be chosen as integers, not all zero, of absolute values at most C, such that $\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p) = 0$ for $\tau, \tau_1, \cdots, \tau_{n-1} = 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$. Doing so, we certainly have $\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p) = 0$ for $0 \leqq \tau+\tau_1+ \ldots +\tau_{n-1} \leqq T-1$ and $0 \leqq p \leqq P-1$. From (8) we now get

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\} \tag{9}$$

for $0 \leqq \tau+\tau_1+ \cdots +\tau_{n-1} \leqq T-1$ and $0 \leqq p \leqq P-1$.

Define T_r and P_r for $r = 0, 1, \cdots, R$ by

$$T_r = [2^{-r}T]$$

and

$$P_r = [(N^{n+1}\log S)^{r\delta}P].$$

Remark that

$$P_R \leqq [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}]. \tag{10}$$

LEMMA: *For $r = 0, 1, \cdots, R$ the inequality*

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\} \tag{11}$$

holds for all non-negative integers $\tau, \tau_1, \cdots, \tau_{n-1}$ and p with $0 \leqq \tau+\tau_1+\cdots +\tau_{n-1} \leqq T_r-1$ and $0 \leqq p \leqq P_r-1$.

PROOF: We proceed by induction on r. For $r = 0$ the statement is proved in (9). Let r be an integer with $0 \leqq r \leqq R-1$ for which

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\} \tag{12}$$

for $0 \leqq \tau+\tau_1+ \cdots +\tau_{n-1} \leqq T_r-1$ and $0 \leqq p \leqq P_r-1$. Since

$$F_{\tau\tau_1\cdots\tau_{n-1}}(z) = \sum_{k_1}\cdots\sum_{k_n}\sum_{m}\sum_{\nu} C_{k_1\cdots k_n m\nu}\eta^{\nu}\prod_{i=1}^{n-1}(\log\alpha_i)^{-\tau_i} \times (z^m e^{-k_n z})^{(\tau)}\prod_{i=1}^{n-1}(\exp\{(k_i+k_n\gamma_i)(\log\alpha_i)z\})^{(\tau_i)}$$

we have for $t = 0, 1, 2, \cdots$

$$F^{(t)}_{\tau\tau_1\cdots\tau_{n-1}}(z) = \sum_{\mu+\mu_1+\cdots+\mu_{n-1}=t}\frac{t!}{\mu!\mu_1!\cdots\mu_{n-1}!} \times \prod_{i=1}^{n-1}(\log\alpha_i)^{\mu_i} \times F_{\tau+\mu,\tau_1+\mu_1,\cdots,\tau_{n-1}+\mu_{n-1}}(z).$$

Together with (12) we obtain

$$(13) \qquad |F^{(t)}_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{4}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T_{r+1}-1, 0 \leqq t \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_r-1$.

From (7) we know that

$$(14) \qquad \max_{|z|\leqq 6P_{r+1}} |F_{\tau\tau_1\cdots\tau_{n-1}}(z)| \leqq \exp\{c_3 N^{n+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+4)\delta}\}$$

for $\tau+\tau_1+\cdots+\tau_{n-1} \leqq T_{r+1}-1$. We apply Lemma 7 of [4] to $F_{\tau\tau_1\cdots\tau_{n-1}}$ with $R = P_{r+1}$, $A = 6$, $T = T_{r+1}$ and $P = P_r$. From (13), (14) and the inequality

$$N^{n+1}\log S \leqq \exp\{N^\delta(\log S)^\delta\}$$

we then obtain

$$\max_{|z|\leqq P_{r+1}} |F_{\tau\tau_1\cdots\tau_{n-1}}(z)| \leqq \exp\{-2^{-(r+3)}N^{n+1+(n^2+n+nr+r)\delta}S(\log S)^{2+(2n+r+5)\delta}\}.$$

In particular,

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-2^{-(r+3)}N^{n+1+(n^2+n+nr+r)\delta}S(\log S)^{2+(2n+r+5)\delta}\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$.

From (8) and (10) it follows that

$$(15) \qquad |\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-2^{-(r+4)}N^{n+1+(n^2+n+nr+r)\delta}S(\log S)^{2+(2n+r+5)\delta}\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$. But for these values of $\tau, \tau_1, \cdots, \tau_{n-1}$ and p, we can consider $\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p)$ as a polynomial in $\eta, \alpha_1, \cdots, \alpha_n, \gamma_1, \cdots, \gamma_{n-1}$, of degree less than $T_{r+1}+N$ in η, KP_{r+1} in $\alpha_1, \cdots, \alpha_n$ and T_{r+1} in $\gamma_1, \cdots, \gamma_{n-1}$. If B denotes the sum of the absolute values of the coefficients, then we have

$$B \leqq \exp\{2N^{n+(n^2+n)\delta}S(\log S)^{2+(2n+4)\delta}\}.$$

According to Lemma 3 of [4] we thus have either $\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p) = 0$ or

$$(16) \qquad |\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p)| \geqq \exp\{-c_4 N^{n+1+(n^2+n+nr+r)\delta}S(\log S)^{2+(2n+r+4)\delta}\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$. Hence, $\Phi_{\tau\tau_1\cdots\tau_{n-1}}(p) = 0$ for these $\tau, \tau_1, \cdots, \tau_{n-1}$ and p. Again from (8) and (10) we obtain

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$. The lemma has been proved.

From (11) with $r = R$ we get

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\} \tag{17}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T_R-1$ and $0 \leqq p \leqq P_R-1$. From their definitions we have $T_R = T'$. Since

$$R \geqq \frac{n-1}{\delta}+2n^2+n-1$$

we see that

$$P_R \geqq \tfrac{1}{2}N^{n^2-1+(2n^3+3n^2-1)\delta}S(\log S)^{n+(2n^2+n+1)\delta} \geqq P'.$$

Hence,

$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leqq \exp\{-\tfrac{1}{3}U\} \tag{18}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_{n-1} \leqq T'-1$ and $0 \leqq p \leqq P'-1$. From (6) it now follows that

$$|F^{(t)}(p)| \leqq \exp\{-\tfrac{1}{4}U\} \tag{19}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots, P'-1$.

We apply Lemma 8 of [4] to F with K replaced by K^n. Let Ω and ω have the same meaning as in this lemma. Since the exponents of F have the form

$$k_1 \log\alpha_1+\cdots+k_n\log\alpha_n+k_n(\sigma-\eta),$$

we see that

$$\Omega \leqq c_5\, N^{n+(n^2-1)\delta}(\log S)^{1+(2n+1)\delta}. \tag{20}$$

With this, the condition

$$T'P' \geqq 2K^nM+13\Omega P'$$

is easily checked. Further, we know from Lemma 1, applied with $\varepsilon = \delta$, that

$$\begin{aligned}|k_1\log\alpha_1+\cdots+k_n\log\alpha_n| &\geqq \exp\{-(\log K)^{1+2\delta}\}\\ &\geqq \exp\{-N^\delta(\log S)^\delta\}\end{aligned} \tag{21}$$

for all integers $k_1, \cdots, k_n$, not all zero, with $|k_1| \leqq K-1, \cdots, |k_n| \leqq K-1$. From (21) and (5) it follows that

$$\omega \geqq \exp\{-N^\delta(\log S)^\delta\}-\exp\{-\tfrac{1}{2}U\} \geqq \exp\{-2N^\delta(\log S)^\delta\}. \tag{22}$$

From (20) we have

$$\Omega \leqq \exp\{N^{\delta}(\log S)^{\delta}\}. \tag{23}$$

From lemma 8 of [4], with (19), (22) and (23) we obtain

$$\begin{aligned}(24)\quad & \Big|\sum_{\nu=0}^{N-1} C_{k_1\cdots k_n m\nu}\eta^{\nu}\Big| \\ & \leqq \exp\{c_6 N^{n^2+n+(n^3+n^2+1)\delta} S(\log S)^{n+1+(2n^2+3n+4)\delta} - \tfrac{1}{4}U\} \\ & \leqq \exp\{-\tfrac{1}{5}U\}\end{aligned}$$

for $k_1, \cdots, k_n = 0, 1, \cdots, K-1$ and $m = 0, 1, \cdots, M-1$.

But according to Lemma 3 of [4] we have either

$$\sum_{\nu=0}^{N-1} C_{k_1\cdots k_n m\nu}\eta^{\nu} = 0$$

or

$$\Big|\sum_{\nu=0}^{N-1} C_{k_1\cdots k_n m\nu}\eta^{\nu}\Big| \geqq \exp\{-2N^{n+1+(n^2+n)\delta} S(\log S)^{2+(2n+4)\delta}\} \tag{25}$$

for the same values of $k_1, \cdots, k_n$ and m. Hence,

$$\sum_{\nu=0}^{N-1} C_{k_1\cdots k_n m\nu}\eta^{\nu} = 0 \text{ for } k_1, \cdots, k_n = 0, 1, \cdots, K-1$$

and $m = 0, 1, \cdots, M-1$. Since η has the degree N, it follows that all integers $C_{k_1\cdots k_n m\nu}$ are zero, in contradiction to their choice. This contradiction proves Theorem 1.

We have the following

COROLLARY: *Under the conditions of Theorem* 1, *there exists an effectively computable, number* $C_3 = C_3(\varepsilon, \log\alpha_1, \cdots, \log\alpha_n, \gamma_1, \cdots, \gamma_{n-1}) > 0$ *such that*

$$\begin{aligned}(26)\quad & |\gamma_1 \log\alpha_1 + \cdots + \gamma_{n-1}\log\alpha_{n-1} - \log\alpha_n - \eta| \\ & \qquad > \exp\{-C_3 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}\end{aligned}$$

for all algebraic numbers η *of degree at most N and size at most S.*

PROOF: There are only finitely many algebraic numbers η of size $s(\eta) < S_1$. Choose $C_3 \geqq 1$ such that (26) holds for these finitely many numbers.

THEOREM 2: *Let, for* $n \geqq 2$, $\alpha_1, \cdots, \alpha_n$ *and* $\beta_1, \cdots, \beta_n$ *be non-zero algebraic numbers such that, for any fixed values of the logarithms,* $\log\alpha_1, \cdots, \log\alpha_n$ *are linearly independent over* $\boldsymbol{Q}$. *Let* ε *be a positive number. Then there exists an effectively computable positive number* $C_4 = C_4(\varepsilon, \log\alpha_1, \cdots, \log\alpha_n, \beta_1, \cdots, \beta_n)$ *such that*

$$(27)\qquad |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi| > \exp\{-C_4 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}$$

for all algebraic numbers ξ *of degree* N *and size* S.

PROOF: Put $\gamma_i = -\beta_i/\beta_n$ ($i = 1, \cdots, n-1$) and $\eta = -\xi/\beta_n$. Then

$$(-1/\beta_n)(\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi) = \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta.$$

We have $d(\eta) \leqq c_7 N$ with $c_7 = d(\beta_n)$ and, by Lemma 2, $s(\eta) \leqq c_8 S$ with $c_8 = 3d(\beta_n) + s(\beta_n)$. From (26) we now obtain

$$\begin{aligned} &|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi| \\ &\quad = |\beta_n||\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| > \\ &\quad > |\beta_n| \exp\{-C_3(c_1 N)^{n^2+n+\varepsilon} c_2 S(1+\log c_2 S)^{n+1+\varepsilon}\} \\ &\quad \geqq \exp\{-C_4 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\} \end{aligned}$$

for some effectively computable positive number C_4.

THEOREM 3: *Under the conditions of Theorem* 2, *there exists an effectively computable number* $C_1 = C_1(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \beta_1, \cdots, \beta_n) > 0$, *such that*

$$\exp\{-C_1 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}$$

is a transcendence measure of $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$.

PROOF: Apply Lemma 9 of [4] to the result of Theorem 2 and put $C_1 = 6C_4(1+\log 2)^{n+1+\varepsilon}$.

4. The case $e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$

THEOREM 4: *Let* n *be a positive integer. Let* β_0 *be algebraic and let* $\alpha_1, \cdots, \alpha_n$ *be non-zero algebraic numbers such that, for any fixed values of the logarithms,* $\log \alpha_1, \cdots, \log \alpha_n$ *are linearly independent over* Q. *Let* $\beta_1, \cdots, \beta_n$ *be algebraic numbers, not all rational. Put*

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i} \textit{ for } i = 1, \cdots, n.$$

Let ε *be a positive number. Then there exists an effectively computable number* $S_2 = S_2(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \beta_0, \beta_1, \cdots, \beta_n) > 0$ *such that*

$$(28)\qquad |e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} - \xi| > \exp\{-N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon}\}$$

for all algebraic ξ *of degree* N *and size* $S \geqq S_2$.

PROOF: Put $\delta = (2n^3 + 8n^2 + 10n + 4)^{-1}\varepsilon$. For the sake of brevity, put

$$\sigma = e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

and

$$U = N^{n^2+2n+2+(2n^3+8n^2+10n+4)\delta}S^{n+1+(2n^2+7n+11)\delta}.$$

It is sufficient to prove that

$$|\sigma-\xi| > \exp\{-U\}$$

if $S \geqq S_2$; in this proof we may assume that δ is rather small. By $c_1, c_2, \cdots$ we shall denote positive numbers which depend on n, $\log \alpha_1, \cdots, \log \alpha_n$, $\beta_0, \beta_1, \cdots, \beta_n$ only.

Suppose that

$$(29) \qquad |\sigma-\xi| \leqq \exp\{-U\}$$

for some algebraic number ξ of degree N and size S. We prove that this is impossible if S is sufficiently large.

Choose the following integers:

$$\begin{aligned}
K &= [N^{n+(n^2+n)\delta}S^{1+(2n+3)\delta}],\\
L &= [N^{n+1+(n^2+n)\delta}S^{(2n+3)\delta}],\\
M &= [N^{n+1+(n^2+2n+1)\delta}S^{1+(2n+5)\delta}],\\
C &= 2[\tfrac{1}{2}\exp\{N^{n+1+(n^2+2n+1)\delta}S^{1+(2n+6)\delta}\}],\\
T &= [N^{n+1+(n^2+2n+1)\delta}S^{1+(2n+5)\delta}] \ (= M),\\
P &= [NS^{2\delta}],\\
R &= \left[\frac{n}{\delta}+2n^2+5n+3\right],\\
T' &= [2^{-R}T], \text{ and}\\
P' &= [\tfrac{1}{2}N^{n^2+n+1+(n^3+2n^2+n)\delta}S^{n+(2n^2+5n+4)\delta}].
\end{aligned}$$

Put

$$F(z) = \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{k_1\cdots k_n lmv}\xi^v z^m \times \exp\{\ell\beta_0 z + \sum_{i=1}^{n}(k_i+\ell\beta_i)(\log\alpha_i)z\},$$

where the numbers $C_{k_1\cdots k_n lmv}$ are integers of absolute values at most C; they will be specified later.

For $t = 0, 1, 2, \cdots$ we have

$$(30) \qquad F^{(t)}(z) = \sum_{\tau+\tau_1+\cdots+\tau_n=t} \frac{t!}{\tau!\tau_1!\cdots\tau_n!}\prod_{i=1}^{n}(\log\alpha_i)^{\tau_i}F_{\tau\tau_1\cdots\tau_n}(z)$$

where

$$(31)\qquad F_{\tau\tau_1\cdots\tau_n}(z) = \sum_{k_1}\cdots\sum_{k_n}\sum_{l}\sum_{m}\sum_{v} C_{k_1\cdots k_nlmv}\,\xi^v \sum_{\kappa=0}^{m}\binom{\tau}{\kappa}\frac{m!}{(m-\kappa)!}$$
$$\times z^{m-\kappa}\ell^{\tau-\kappa}\beta_0^{\tau-\kappa}\prod_{i=1}^{n}(k_i+\ell\beta_i)^{\tau_i}\exp\{\sum_{i=1}^{n}k_i(\log\alpha_i)z\}\sigma^{lz}.$$

Define $\Phi_{\tau\tau_1\cdots\tau_n}$ by

$$\Phi_{\tau\tau_1\cdots\tau_n}(z) = \sum_{k_1}\cdots\sum_{k_n}\sum_{l}\sum_{m}\sum_{v} C_{k_1\cdots k_nlmv}\,\xi^v \sum_{\kappa=0}^{m}\binom{\tau}{\kappa}\frac{m!}{(m-\kappa)!}$$
$$\times z^{m-\kappa}\ell^{\tau-\kappa}\beta_0^{\tau-\kappa}\prod_{i=1}^{n}(k_i+\ell\beta_i)^{\tau_i}\exp\{\sum_{i=1}^{n}k_i(\log\alpha_i)z\}\xi^{lz}.$$

For $\ell = 0, 1, \cdots, L-1$ and

$$p = 0, 1, \cdots, [N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta}S^{n+(2n^2+5n+5)\delta}]$$

one has

$$|\sigma^{lp}-\xi^{lp}| \leqq lp\,|\sigma-\xi|\,(|\sigma|+1)^{lp} \leqq \exp\{-\tfrac{1}{2}U\}.$$

Hence,

$$(32)\qquad |F_{\tau\tau_1\cdots\tau_n}(p)-\Phi_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $\tau, \tau_1, \cdots, \tau_n = 0, 1, \ldots, T-1$ and

$$p = 0, 1, \cdots, [N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta}S^{n+(2n^2+5n+5)\delta}].$$

We apply Lemma 6 of [4] to the polynomials

$P_{\tau\tau_1\cdots\tau_npk_1\cdots k_nlm}$ $(\tau, \tau_1, \cdots, \tau_n = 0, 1, \cdots, T-1;\ p = 0, 1, \cdots, P-1;$ $k_1, \cdots, k_n = 0, 1, \cdots, K-1;\ \ell = 0, 1, \cdots, L-1$ and $m = 0, 1, \cdots, M-1)$, chosen in the appropriate way such that

$$\Phi_{\tau\tau_1\cdots\tau_n}(p) = \sum_{k_1}\cdots\sum_{k_n}\sum_{l}\sum_{m}\sum_{v} C_{k_1\cdots k_nlmv}\,\xi^v$$
$$\times P_{\tau\tau_1\cdots\tau_npk_1\cdots k_nlm}(\xi, \alpha_1, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_n).$$

If r, s and B denote the same numbers as in Lemma 6 of [4], we have

$$r = T^{n+1}P \leqq N^{n^2+2n+2+(n^3+3n^2+3n+1)\delta}S^{n+1+(2n^2+7n+7)\delta},$$
$$s = K^nLM \geqq \tfrac{1}{2}N^{n^2+2n+2+(n^3+3n^2+3n+1)\delta}S^{n+1+(2n^2+7n+8)\delta}$$

and

$$B \leqq \exp\{c_1 N^{n+1+(n^2+2n+1)\delta}S^{1+(2n+5)\delta}\log S\}.$$

From these inequalities it is easy to check the conditions of this lemma. Hence, we can fix the numbers $C_{k_1\cdots k_nlmv}$ as integers, not all zero, of absolute values at most C, such that $\Phi_{\tau\tau_1\cdots\tau_n p} = 0$ for $\tau, \tau_1, \cdots, \tau_n = 0, 1, \cdots, T-1$ and $p = 0, 1, \cdots, P-1$. With (32) this implies

$$(33) \qquad |F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T-1$ and $0 \leqq p \leqq P-1$.

Define T_r and P_r for $r = 0, 1, \cdots, R$ by

$$T_r = [2^{-r}T]$$

and

$$P_r = [(N^{n+1}S)^{r\delta}P].$$

Observe that

$$(34) \qquad P_R \leqq N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta}S^{n+(2n^2+5n+5)\delta}.$$

LEMMA: *For $r = 0, 1, \cdots, R$ the inequality*

$$(35) \qquad |F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

holds for all non-negative integers $\tau, \tau_1, \cdots, \tau_n$ and p with $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_r-1$ and $0 \leqq p \leqq P_r-1$.

PROOF: We use induction on r. For $r = 0$ the inequality has already been proved in (33). Let r be an integer with $0 \leqq r \leqq R-1$ for which

$$(36) \qquad |F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_r-1$ and $0 \leqq p \leqq P_r-1$. Since

$$F_{\tau\tau_1\cdots\tau_n}(p) = \sum_{k_1}\cdots\sum_{k_n}\sum_{l}\sum_{m}\sum_{v} C_{k_1\cdots k_n lmv}\xi^v \times \prod_{i=1}^{n}(\log\alpha_i)^{-\tau_i} \times (z^m e^{l\beta_0 z})^{(\tau)} \times \prod_{i=1}^{n}(e^{(k_i+l\beta_i)(\log\alpha_i\alpha(\tau_i))})^{(\tau_i)}$$

it follows that for $t = 0, 1, 2, \cdots$

$$F^{(t)}_{\tau\tau_1\cdots\tau_n}(z) = \sum_{\mu+\mu_1+\cdots+\mu_n=t}\frac{t!}{\mu!\mu_1!\cdots\mu_n!} \times \prod_{i=1}^{n}(\log\alpha_i)^{\mu_i}F_{\tau+\mu,\tau_1+\mu_1,\cdots,\tau_n+\mu_n}(z).$$

Hence, (36) implies

$$(37) \qquad |F^{(t)}_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{4}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_{r+1}-1$, $0 \leqq t \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_r-1$.

For the same values of $\tau, \tau_1, \cdots, \tau_n$ we obtain from (31)

$$(38)\qquad \max_{|z|\leqq 6P_{r+1}} |F_{\tau\tau_1\cdots\tau_n}(z)| \leqq \exp\{c_2 N^{n+1+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+6)\delta}\}.$$

We apply Lemma 7 of [4] to $F_{\tau\tau_1\cdots\tau_n}$ with $R = P_{r+1}$, $A = 6$, $T = T_{r+1}$ and $P = P_r$. From (37), (38) and (34) we then obtain

$$\max_{|z|\leqq P_{r+1}} |F_{\tau\tau_1\cdots\tau_n}(z)| \leqq \exp\{-2^{-(r+3)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}.$$

Consequently,

$$|F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-2^{-(r+3)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$. From (32) and (34), it follows that

$$(39)\qquad |\Phi_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-2^{-(r+4)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$.

However, for these values of $\tau, \tau_1, \cdots, \tau_n$ and p, we can consider $\Phi_{\tau\tau_1\cdots\tau_n}(p)$ as a polynomial in $\xi, \alpha_1, \cdots, \alpha_n, \beta_0, \beta_1, \cdots, \beta_n$, of degree less than $LP_{r+1}+N$ in ξ, KP_{r+1} in $\alpha_1, \cdots, \alpha_n$ and T_{r+1} in $\beta_0, \beta_1, \cdots, \beta_n$. The sum of the absolute values of its coefficients is at most

$$\exp\{2N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+6)\delta}\}.$$

According to Lemma 3 of [4] we have either $\Phi_{\tau\tau_1\cdots\tau_n}(p) = 0$ or

$$(40)\qquad |\Phi_{\tau\tau_1\cdots\tau_n}(p)| \geqq \exp\{-c_3 N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+6)\delta}\}.$$

Hence,

$$\Phi_{\tau\tau_1\cdots\tau_n}(p) = 0 \text{ for } 0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_{r+1}-1 \text{ and } 0 \leqq p \leqq P_{r+1}-1.$$

From (32) and (34) we see

$$|F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_{r+1}-1$ and $0 \leqq p \leqq P_{r+1}-1$, which proves the lemma.

From (35) with $r = R$ we get

$$|F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T_R-1$ and $0 \leqq p \leqq P_R-1$. We have $T_R = T'$. From $R \geqq n/\delta+2n^2+5n+2$ we see

$$P_R \geqq [\tfrac{1}{2}N^{n^2+n+1+(2n^3+7n^2+7n+2)\delta}S^{n+(2n^2+5n+4)\delta}] \geqq P'.$$

Thus,

$$|F_{\tau\tau_1\cdots\tau_n}(p)| \leqq \exp\{-\tfrac{1}{3}U\}$$

for $0 \leqq \tau+\tau_1+\cdots+\tau_n \leqq T'-1$ and $0 \leqq p \leqq P'-1$. From (30) we now obtain

(41) $$|F^{(t)}(p)| \leqq \exp\{-\tfrac{1}{4}U\}$$

for $t = 0, 1, \cdots, T'-1$ and $p = 0, 1, \cdots P'-1$.

The exponents of F have the form

$$\ell\beta_0+(k_1+\ell\beta_1)\log\alpha_1+\cdots+(k_n+\ell\beta_n)\log\alpha_n.$$

Let Ω and ω have the same meaning as in Lemma 8 of [4]. Then

(42) $$\Omega \leqq c_4 N^{n+(n^2+n)\delta}S^{1+(2n+3)\delta},$$

from which the condition

$$T'P' \geqq 2K^nLM+13\Omega P'$$

follows by direct computation.

The difference of two exponents of F is of the form

$$\ell\beta_0+(k_1+\ell\beta_1)\log\alpha_1+\cdots+(k_n+\ell\beta_n)\log\alpha_n$$

with integral $k_1, \cdots, k_n, \ell$, not all zero, and $|k_i| \leqq K-1$ for $i = 1, \cdots, n$ and $|\ell| \leqq L-1$. Moreover, at least one of the numbers $k_i+\ell\beta_i$ $(i = 1, \cdots, n)$ is non-zero, since $\beta_1, \cdots, \beta_n$ are not all rational. The degrees of $\ell\beta_0$, $k_1+\ell\beta_1, \cdots, k_n+\ell\beta_n$ are constants. We estimate their heights by means of Lemma 3; we then see that these heights do not exceed

$$c_5(2KL)^{c_6} \leqq S^{c_7}$$

in which c_5 and c_6 are upper bounds for the heights and degrees resp. of $\beta_0, \beta_1, \cdots, \beta_n$. From Lemma 1 with $\varepsilon = \delta$ it follows that

$$|\ell\beta_0+(k_1+\ell\beta_1)\log\alpha_1+\cdots+(k_n+\ell\beta_n)\log\alpha_n| > \exp\{-(\log S)^{1+2\delta}\}.$$

Hence, the exponents of F are distinct and

(43) $$\omega > \exp\{-(\log S)^{1+2\delta}\} > \exp\{-S^\delta\}.$$

From Lemma 8 of [4], using (41), (42) and (43) we obtain the inequality

(44) $$|\sum_{\nu=0}^{N-1} C_{k_1\cdots k_n lm\nu}\xi^\nu| \leqq \exp\{-\tfrac{1}{5}U\}$$

for $k_1, \cdots, k_n = 0, 1, \cdots, K-1$; $\ell = 0, 1, \cdots, L-1$ and $m = 0, 1, \cdots, M-1$.

According to Lemma 3 of [4] we have either

$$\sum_{\nu=0}^{N-1} C_{k_1 \cdots k_n l m \nu} \xi^\nu = 0 \text{ or}$$

$$(45) \qquad \left|\sum_{\nu=0}^{N-1} C_{k_1 \cdots k_n l m \nu} \xi^\nu\right| > \exp\left\{-2N^{n+2+(n^2+2n+1)\delta} S^{1+(2n+6)\delta}\right\}$$

for the same values of $k_1, \cdots, k_n$, ℓ and m. It follows that

$$\sum_{\nu=0}^{N-1} C_{k_1 \cdots k_n l m \nu} \xi^\nu = 0$$

for all of these values. Since ξ has the degree N, this implies that all integers $C_{k_1 \cdots k_n l m \nu}$ are zero, in contradiction to their choice. The theorem has been proved.

Using the fact, that there are only finitely many algebraic numbers ξ of size $S < S_2$, and using Lemma 9 of [4], one immedeately obtains the following theorem:

THEOREM 5: *Under the conditions of Theorem* 4, *there exists an effectively computable, number* $C_2 = C_2(\varepsilon, \log \alpha_1, \cdots, \log \alpha_n, \beta_0, \beta_1, \cdots, \beta_n) > 0$ *such that*

$$\exp\left\{-C_2 N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon}\right\}$$

is a transcendence measure of $e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$.

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