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THE HEREDITY PROBLEM FOR WEAKLY COMPACTLY GENERATED BANACH SPACES

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Introduction

A Banach space is said to be weakly compactly generated (WCG) if it has a weakly compact subset which generates the space, i.e. its linear span is dense in the space. Perhaps the main unsolved problem in this class of Banach spaces was the heredity problem: Is every subspace of a WCG Banach space also WCG? (Throughout, 'subspace' means 'closed linear submanifold'.) In Section 1 we exhibit a probability measure μ on a certain measurable space and a subspace $X_{\mathfrak{R}}$ of $L^1(\mu)$ which is not WCG thus solving the heredity problem in the negative. ($\{f \in L^1(\mu) : \int|f|^2 d\mu \leqq 1\}$ is a generating weakly compact subset of $L^1(\mu)$).

$X_{\mathfrak{R}}$ is obtained as the span of a certain family $\mathscr{F}$ of independent random variables which, using a classical criterion given in Lemma 1.4, is immediately seen to have the property that $\mathscr{F} \cup \{0\}$ is not σ-weakly compact (i.e. a denumerable union of weakly compact sets). Each element of $\mathscr{F}$ is of mean zero; hence as observed in [17], $\mathscr{F}$ is an unconditional basis for $X_{\mathfrak{R}}$. A simple, elegant result due to W. Johnson (Proposition 1.3) asserts that if a Banach space is WCG and has an unconditional basis, then the basis, together with 0, must be σ-weakly compact. Johnson's result also yields that $X^*_{\mathfrak{R}}$ is not isomorphic to the dual of a WCG Banach space.

Proposition 1.3 and Lemma 1.4 thus yield a conceptual proof that $X_{\mathfrak{R}}$ is not WCG; at the same time, they lead to a satisfactory characteri-

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zation of those families of independent random variables which have WCG closed lineaı spans (Theorem 1.5).

In Section 2 we present deeper properties of WCG subspaces of $L^1(\mu)$-spaces. This section has three main results; the first is that there is a non-separable space Y spanned by a family of independent random variables so that every weakly compact subset of Y has density character less than that of Y (Theorem 2.1).

It happens that $X_{\mathfrak{R}}$ contains Y as a complemented subspace; in an earlier version of our results, we used this method to show that $X_{\mathfrak{R}}$ is not WCG. In addition to using the tools of Section 1, we require a result concerning lower orders of magnitude of families of (not-strictly) increasing unbounded functions defined on the set of all positive integers (the family of all such functions is denoted by $\mathscr{M}$ for 'monotone'). Defining $f < g$ if $g = o(f)$ for all $f, g \in \mathscr{M}$, we prove in Lemma 2.3 that there is a subfamily $\mathscr{G}$ of $\mathscr{M}$ which is well-ordered by $<$, has no upper bound in $\mathscr{M}$, and has order-type a cardinal number.

The second main result of Section 2, Theorem 2.7, yields that every non-separable subspace of $L^1(\mu)$ for some probability measure μ, contains a sequence of elements of norm one tending weakly to zero, while it is consistent with set theory that there exists a non-separable $L^1(\mu)$-space such that all of its subspaces are WCG. Moreover it is consistent that every non-separable subspace of an $L^1(\mu)$-space contains a non-separable weakly compact subset. On the other hand, it is also consistent that this is false; indeed assuming the Continuum Hypothesis, the non-separable space Y of Theorem 2.1 is such that all of its weakly compact sets are separable. To prove Theorem 2.7 we require a striking result of J. Silver (Lemma 2.5) which shows that a) every uncountable subset of $\mathscr{M}$ contains an infinite subset with an upper bound in $\mathscr{M}$ and b) it is consistent with set theory that $\aleph_1 < 2^{\aleph_0}$ and that every subset of $\mathscr{M}$ of cardinality less than the continuum has an upper bound in $\mathscr{M}$. Our proof of Lemma 2.3 also shows that it is consistent with set theory that the assertion of b) is false. (We present only the proof of a); we are grateful to J. Silver for the communication of this result.)

The third main result of Section 2, Theorem 2.9, shows that the unit ball of $X^*_{\mathfrak{R}}$ in its weak* topology, is homeomorphic to a weakly compact subset of a Banach space. The proof of this result involves a careful use of the modulus of absolute continuity, and is the most delicate argument in the paper.

Theorem 2.9 led to the topological characterization of weakly compact sets in Banach spaces, Theorem 3.1, which is the main result of Section 3: *A compact Hausdorff space is homeomorphic to a weakly compact subset of some Banach space if and only if it has a point-separating σ-point-finite* *family of open F_σ's.* (For the definitions of these terms, see Section 3.) The proof of 3.1 is a surprisingly simple consequence of known results, including the Grothendieck characterization of weakly compact subsets of $C(K)$ spaces [7] and the work of Amir and Lindenstrauss [1]. For the sake of completeness and clarification, we present (possibly new) proofs of all of these except the consequence of the work of [1] due to Lindenstrauss (stated as Lemma 3.5). The reader familiar with these results may wish to skip over them to the proof of 3.1 following Lemma 3.5.

At the beginning of Section 3 we note the known fact that a compact Hausdorff space is metrizeable if and only if it has a strongly-point-separating σ-point-finite family of open F_σ's. Thus the word 'strongly' constitutes the dividing line between metrizeable and non-metrizeable Eberlein compacts (i.e. homeomorphs of weakly compact subsets of Banach spaces) such as the one-point compactification of an uncountable set. Section 3 concludes with a number of topological questions and remarks. We feel that the main question concerning Eberlein compacts is whether they are closed under continuous Hausdorff images (stated as problem 5 of [12]). Phrased another way, is every closed subalgebra of a WCG $C(K)$ space also WCG? (We work only with real scalars; a $C(K)$-space refers to the space of real continuous functions on a compact Hausdorff space K under the sup-norm.) A question related to this: is every compact Hausdorff space with a point-separating point-countable family of open F_σ's, an Eberlein compact? We are grateful to M.E. Rudin for the communication of an example (unpublished as of this writing) which shows that it is consistent with the axioms of set theory, that the answer to the last-stated question is negative. (See Remark 3 Section 3.)

It would certainly be desirable to determine those Banach spaces which are hereditarily WCG. Is it possible that this class of Banach spaces coincides with those which are Lindelöf in their weak topology? It is a result of Corson's (see [2]) that $c_0(\Gamma)$ is Lindelof in its weak topology for any set Γ, although its still unknown if $L^1(\mu)$-spaces have this property, for finite-measures μ. ($c_0(\Gamma)$ is the space of continuous functions on the discrete space Γ, vanishing at infinity). John and Zizler [9] and, independently, D. Friedland [6], have proved that $c_0(\Gamma)$ is hereditarily WCG for any set Γ. ($c_0(\Gamma)$ and $L^1(\mu)$ spaces are actually related by the following fact: for any set Γ, there exists a probability measure μ on some measurable space such that $c_0(\Gamma)$ is isometric to a quotient space of $L^1(\mu)$. More generally, using recent results of S. Troyanski [20], they have proved that the property of having a shrinking Markusevic basis is hereditary, thus showing that such spaces are hereditarily WCG. They have also proved that X has a shrinking M-basis iff X is WCG and has an equivalent Frechet differentiable norm iff (by the results of [20]) X is WCG and has an equivalent norm which induces on X^* a locally uniformly rotund norm. (See [9] and [6] for the definitions and proofs.) D. Friedland obtains the additional information that if X is WCG and $Y \subset X$ is such that Y has an equivalent Frechet differentiable norm, then Y is WCG.

We refer the reader to the Lindenstrauss survey-paper [12] and the recent paper [13] for many properties of WCG Banach spaces. Our negative solution to the heredity problem answers Problems 1 and 2 of [12] in the negative, while Theorem 2.9 answers Problem 5 in the negative.

We of course lean on the standard facts in functional analysis (e.g. as given in [4].) Our notation and definitions are standard for the most part. We use the notation [S] to denote the closed linear span of a subset S of a Banach space; if $S = \{x_n : n = 1, 2, \cdots\}$ for some sequence (x_n), we also denote $[S]$ by $[x_n]$. 'Isomorphic' means 'linearly homeomorphic'; 'operator' means 'bounded linear operator'. The results of Section 3 are independent of those of Sections 1 and 2, except for the proof of Lemma 2.8 which leans on (the simple) Proposition 3.3.

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1. A counter-example to the heredity problem

Let $\mathscr{R}$ denote the class of all integrable functions f defined on [0, 1] with

$$\int_0^1 |f(t)|\,\mathrm{d}t = 1 \text{ and} \int_0^1 f(t)\,\mathrm{d}t = 0.$$

Let μ denote the product-Lebesgue measure on $[0, 1]^{\mathscr{R}}$. As we noted above, $L^1(\mu)$ is WCG. For each $r \in \mathscr{R}$, let $\tilde{r}$ be the function in $L^1(\mu)$ defined by: $\tilde{r}(x) = r(x(r))$ for all $x \in [0, 1]^{\mathscr{R}}$. The main result of this section is

THEOREM 1.1: *Let $X_{\mathscr{R}}$ denote the closed linear span in $L^1(\mu)$ of the set of $\tilde{r}$'s, where r ranges over arbitrary elements of $\mathscr{R}$. Then $X_{\mathscr{R}}$ is not weakly compactly generated.*

Our proof of Theorem 1.1 yields that, in the language of probability theory, if Y is the closed linear span in L^1-norm, of a family of independent random variables each of L^1-norm one and mean zero, maximal with respect to no two distinct variables having the same distribution, then Y is not WCG.

The following simple observations will prove very useful (throughout, the letters 'X', 'Y', 'Z', 'B', shall stand for given Banach spaces, unless otherwise specified):

PROPOSITION 1.2: *If B has a σ-relatively weakly compact subset with dense linear span, then B is WCG. If B is WCG, then there is a Banach space Y and a weakly compact operator T from Y to B with range dense in B.*

PROOF: Let S be a σ-relatively weakly compact subset of B with dense linear span. By definition there exist $S_1, S_2, \cdots$ with

$$S = \bigcup_{i=1}^{\infty} S_i$$

and the weak closure $\tilde{S}_i$ of S_i is weakly compact for all i. Let

$$\lambda_n = 1 + \sup_{s \in \tilde{S}_n} \|s\|$$

for all n. Then

$$\{0\} \cup \bigcup_{n=1}^{\infty} 2^{-n} \lambda_n^{-1} \tilde{S}_n$$

is easily seen to be a weakly compact generating subset of B. Now suppose K is a weakly compact subset of B, generating B. Since the closed convex symmetric hull of K is also weakly compact, we may assume that K is itself convex and symmetric. Now let Y be the linear space generated by K and norm Y by $\|y\| = \inf\{\lambda > 0 : y \in \lambda K\}$. Then standard results yield that Y is a Banach space under this norm and the identity injection is the desired T. (For further properties of such a Y, see Remark 6 of Section 3; the argument above has the virtue that K equals the image of the unit ball of Y and T is one-one). Alternatively, define $T : l^1(K) \to B$ by $T(f) = \sum f(k)k$ for all $f \in l^1(K)$ (where by definition, $l^1(K) = \{f : K \to R$ with $\|f\| = \sum_{k \in K} |f(k)| < \infty\}$.

REMARK: It has recently been proved that Y may be chosen to be reflexive (see [4]).

We next need some properties of unconditional bases in WCG Banach spaces. A set Γ in a Banach space B is called an unconditional basis for B if the linear span of Γ is dense in B, and there exists a constant λ so that for all n, $\gamma_1, \cdots, \gamma_n$ in Γ, scalars $c_1, \cdots, c_n$, and numbers $\varepsilon_1, \cdots, \varepsilon_n$ with $\varepsilon_i = \pm 1$ for all i, $\|\sum \varepsilon_i c_i \gamma_i\| \leqq \lambda \|\sum c_i \gamma_i\|$. The best (i.e. least) possible constant λ for which the above holds, is called the unconditional constant of Γ.

If Γ is a normalized unconditional basis for B, then for each $\gamma \in \Gamma$ there is a functional $\gamma^* \in B^*$ uniquely determined by the conditions:

$$\gamma^*(\beta) = 0 \text{ if } \beta \neq \gamma$$

$$\gamma^*(\beta) = 1 \text{ if } \beta = \gamma \text{ for all } \beta \in \Gamma.$$

One has that $||\gamma^*|| \leqq \lambda$ and for all

$$x \in X, \ x = \sum_{\gamma \in \Gamma} \gamma^*(x)\gamma,$$

where $\gamma^*(x)$ equals zero for all but countably many γ's and the series converges unconditionally to x. The main property of use to us is that for all $\Lambda \in \Gamma$, there exists a projection $P_\Lambda : X \to [\Lambda]$ with $||P_\Lambda|| \leqq \lambda$ uniquely determined by the conditions: For all $x \in X$ and $\gamma \in \Gamma$;

$$(1) \qquad \begin{aligned} \gamma^*(P_\Lambda(x)) &= \gamma^*(x) \text{ if } \gamma \in \Lambda \\ \gamma^*(P_\Lambda(x)) &= 0 \qquad \text{if } \gamma \notin \Lambda. \end{aligned}$$

We are grateful to W. Johnson for the communication of the next result, which provides considerable simplification of an earlier proof of Theorem 1.1 and certain complements to it.

PROPOSITION 1.3: (W. Johnson) *Let B have an unconditional basis Γ. Then the following are equivalent*:
(1) $\{0\} \cup \Gamma$ *is σ-weakly compact.*
(2) *B is WCG.*
(3) *There is a WCG X with B^* isomorphic to X^*.*
(4) *There is a one-one weakly compact operator defined on B^* with range in some Banach space.*
(5) *There is a one-one operator defined on B^* with range in some Banach space containing no isomorph of l^∞.*

PROOF: (1) $\Rightarrow$ (2) follows from 1.2 and (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (4): If X is as in (3), there is a Y and a weakly compact operator $T : Y \to X$ with dense range. Thus $T^* : X^* \to Y^*$ is one-one and also weakly compact by standard results. The conclusion of (4) now follows immediately. The 'target' Banach space of (4) may obviously be chosen to be WCG. But l^∞ does not imbed in a WCG Banach space. Indeed, if Γ is a set of cardinality $2^{\aleph_0}$, then $l^1(\Gamma)$ imbeds in l^∞ but $l^1(\Gamma)$ does not imbed in a WCG space (c.f. page 214 of [14]). To complete the proof, it suffices to prove (5) $\Rightarrow$ (1). However for ease in readability, we prove (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) simultaneously. Let Γ^* be the functionals biorthogonal to Γ and let Y and $T : B^* \to Y$ be chosen so that T is a one-one operator. Let $\Gamma_j = \{\gamma \in \Gamma : ||T\gamma^*|| \geqq 1/j\}$. Since T is one-one, $\Gamma = \bigcup_{j=1}^\infty \Gamma_j$. Fix j and let $\gamma_1, \gamma_2, \cdots$ be a sequence of distinct elements of Γ_j (if such exists). It suffices to prove that $b^*(\gamma_n) \to 0$ as $n \to \infty$ for all $b^* \in B^*$. If not, by passing to a subsequence of the γ_n's if necessary, we can assume there is a $b^* \in B^*$ and a $\delta > 0$ with $|b^*(\gamma_n)| \geqq \delta$ for all n. Now since Γ is unconditional, $(\gamma_n)_{n=1}^\infty$ is equivalent to the usual basis for l^1. Hence again using unconditionality of the basis, $(\gamma_n^*)_{n=1}^\infty$ is equivalent to the usual basis for c_0. Now if T is weakly compact, then $T|[y_n^*]$ is compact and hence $\|T(y_n^*)\| \to 0$, a contradiction. We also have that if Z denotes the weak* closure of $[y_n^*]$, then Z is isomorphic to l^∞. Thus if Y contains no isomorph of l^∞, $T|Z$ is weakly compact by Corollary 1.4 of [16]. So again $T|[y_n^*]$ is compact and we have a contradiction. Q.E.D.

REMARKS: By the remark following 1.2 and results of Amir and Lindenstrauss [1], other equivalences to (1)–(5) are that B^* admit a one-one operator into some reflexive space or into $c_0(\Lambda)$ for some set Λ.

Let ν denote a probability measure on some measurable space. As final preparation for the proof of our main result, we need the following classical characterization of relatively weakly compact subsets of $L^1(\nu)$ (c.f. page 294 of [5]):

A bounded subset S of $L^1(\nu)$ is relatively weakly compact if and only if S is uniformly absolutely continuous; i.e. $\int_E |f| d\nu \to 0$ *as* $\nu(E) \to 0$, *uniformly in S.* It is useful to introduce a quantitative version of this result. For any $f \in L^1(\nu)$, we define the modulus of absolute continuity of f, $\omega(f, \delta)$, by $\omega(f, \delta) = \sup \int_E |f| d\nu$, the supremum being taken over all measurable sets E with $\nu(E) \leqq \delta$. The function $\delta \to \omega(f, \delta)$ shall be denoted by $\omega(f, \cdot)$. We have that fixing f, $\omega(f, \cdot)$ is a monotonically increasing function with $\omega(f, \delta) \to 0$ as $\delta \to 0$. The above result may now be reformulated as

LEMMA 1.4: *Let S be a non-empty bounded subset of $L^1(\nu)$. Then S is relatively weakly compact if and only if*

$$\lim_{\delta \to 0} \sup_{f \in S} \omega(f, \delta) = 0.$$

S is σ-relatively-weakly-compact if and only if there is an increasing function g with $\lim_{x \to 0} g(x) = 0$ so that $\omega(f, \cdot) = 0(g)$ for all $f \in S$.

PROOF: The first statement follows immediately from the classical criterion stated above. Suppose $S = \bigcup_{n=1}^{\infty} S_n$ with each S_n relatively weakly compact. Let

$$g_n(x) = \sup_{f \in S_n} \omega(f, x) \text{ and } \lambda_n = 1 + g_n(1)$$

for all $x > 0$ and $n = 1, 2, \cdots$. Then

$$g(x) = \sum_{n=1}^{\infty} \frac{g_n(x)}{2^n \lambda_n}$$

serves as the desired g. On the other hand, suppose $\omega(f, \cdot) = 0(g)$ for all $f \in S$ and $g(x) \to 0$ as $x \to 0$. Let $S_{n,m} = \{f \in S : \omega(f, x) \leqq mg(x)$ for all $x \leqq 1/n\}$. Then $S = \bigcup_{n,m} S_{n,m}$ and $S_{n,m}$ is relatively weakly compact for all n and m. Q.E.D.

Now it is easily seen that if $\mathscr{R}$ is as defined at the beginning of Section 1, there is no increasing g with $g(x) \to 0$ as $x \to 0$ and $\omega(f, \cdot) = 0(g)$ for all $f \in \mathscr{R}$. Hence 1.1 is an immediate consequence of the next and final result of this section.

THEOREM 1.5: *Let Γ be a set and $\{g_\alpha : \alpha \in \Gamma\}$ an indexed family of elements of $\mathscr{R}$. Let ν denote the product-Lebesgue measure on $[0, 1]^\Gamma$, for each $\alpha \in \Gamma$ let $\tilde{g}_\alpha \in L^1(\nu)$ be the function defined by $\tilde{g}_\alpha(x) = g_\alpha(x(\alpha))$ for all $x \in [0, 1]^\Gamma$, and let X denote the closed linear span in $L^1(\nu)$ of $\{\tilde{g}_\alpha : \alpha \in \Gamma\}$. Then X is WCG (if and) only if there is an increasing function g with $g(x) \to 0$ as $x \to 0$ so that $\omega(g_\alpha, \cdot) = 0(g)$ for all $\alpha \in \Gamma$.*

1.5 may be alternatively phrased: if $\mathscr{F}$ is a family of independent random variables, each of norm one and mean zero, defined on some probability space Ω, then the subspace of $L^1(\Omega)$ generated by $\mathscr{F}$ is WCG if and only if there is an increasing g with $g(x) \to 0$ as $x \to 0$ so that $\omega(f, \cdot) = 0(g)$ for all $f \in \mathscr{F}$.

PROOF: It is obvious that $\omega(g_\alpha, \cdot) = \omega(\tilde{g}_\alpha, \cdot)$ for all $\alpha \in \Gamma$. Thus the 'if' part follows immediately from 1.2 and 1.4. By standard results in probability theory, $\{\tilde{g}_\alpha : \alpha \in \Gamma\}$ is an unconditional basis for X, with unconditional constant less than or equal to 2 (see Lemma 2a, pg. 278 of [17]). Hence the 'only if' part follows from 1.3 and 1.4. Q.E.D.

REMARKS:

1) By 1.3, $X_{\mathscr{R}}^*$ is not isomorphic to the dual of a WCG Banach space; by the last remark of Section 3, this implies that neither the unit ball of $X_{\mathscr{R}}^*$ nor in fact any convex body in $X_{\mathscr{R}}^*$ is affinely equivalent to a weakly compact subset of some Banach space. Nevertheless by the last result of Section 2, the unit ball of $X_{\mathscr{R}}^*$ is homeomorphic to a weakly compact subset of some Banach space. Again by 1.3, $X_{\mathscr{R}}^*$ does not admit a one-one operator into $c_0(\Lambda)$ for any set Λ. On the other hand, if $l_c^\infty(\mathscr{R})$ denotes the Banach space of all bounded functions defined on $\mathscr{R}$ with countable support; then $X_{\mathscr{R}}^*$ does admit a one-one operator into $l_c^\infty(\mathscr{R})$, namely via T defined by $(Tf)(r) = f(\tilde{r})$ for all $r \in \mathscr{R}$ and $f \in X_{\mathscr{R}}^*$. Lindenstrauss has recently proved that $X_{\mathscr{R}}^*$ is strictly convexifiable.

2) $X_{\mathscr{R}}$ is isomorphic to a conjugate space; in fact if Z denotes the norm-closed linear span of $\tilde{\mathscr{R}}^*$, in $X_{\mathscr{R}}^*$, then $X_{\mathscr{R}}$ is isomorphic to Z^* (where $\tilde{\mathscr{R}} = \{\tilde{r} : r \in \mathscr{R}\}$ and $\tilde{\mathscr{R}}^*$ denotes the functionals biorthogonal to $\tilde{\mathscr{R}}$). By the results of [6] and [9], Z is hereditarily WCG.

3) Let Y be a subspace of $L^1(\mu)$ for some probability measure μ, and suppose Y^* is isomorphic to the dual of a WCG Banach space. Is Y WCG? There are many counter-examples to this question if the hypothesis that $Y \subset L^1(\mu)$ is omitted; for example, $Y = C(\omega_1)$ where ω_1

denotes all ordinals less than or equal to the first-uncountable ordinal in the order-topology. The most striking counter-example is a recent one of Lindenstrauss [13], where Y is not WCG; Y^* is isomorphic to Z^*, yet Z^* and Z are both WCG. The following result, which applies the results of [14], may be useful in this connection.

THEOREM: *If A is a subspace of $L^1(\mu)$ so that A^* is isomorphic to the dual of a WCG Banach space, then there is a sequence of measurable sets $E_1, E_2, \cdots$ with $\mu(\bigcup_{j=1}^{\infty} E_j) = 1$ and $E_j \subset E_{j+1}$ for all j, so that for all j, $\overline{\chi_{E_j} A}$ is WCG, where $\overline{\chi_{E_j} A}$ equals the norm-closure of $\{\chi_{E_j} a : a \in A\}$.*

We identify $L^\infty(\mu)$ with $C(\Omega)$ for a certain compact Hausdorff space Ω; we also identify μ with a certain finite regular positive Borel measure μ on Ω; this measure μ has the properties that for every non-empty open subset U of Ω, $\mu(U) = \mu(\overline{U}) > 0$ and also $\overline{U}$ is open;moreover $C(\Omega) = L^\infty(\mu)$; i.e. every bounded Borel-measurable function f is equal μ-almost everywhere to a continuous function on Ω. (See the proof of Theorem 3.1, page 218 of [14] for further details.) It follows from our assumptions on A that there exists a WCG subspace B of $C(\Omega)^*$ so that $B^\perp = A^\perp$(we are here regarding $A \subset C(\Omega)^*$ also). By lemma 1.3 of [14], there is a positive regular Borel measure ν_1 on Ω with $B \subset L^1(\nu_1)$; write $\nu_1 = \lambda+\rho$ where ρ is absolutely continuous with respect to μ and λ is singular with respect to μ. As shown on page 218 of [14], it then followsthat there exists a closed nowhere-dense subset F of Ω with $\mu(F) = \lambda(\sim F) = 0$. Now choose $U_1 \subset U_2 \subset \cdots$, with U_j a closed and open subset of $\sim F$ for all j and $\mu(\bigcup_{i=1}^{\infty} U_i) = 1$, and fix j. Evidently $\overline{\chi_{U_j} B}$ is WCG since B is.

We claim that $\overline{\chi_{U_j} B} = \overline{\chi_{U_j} A}$. Since ρ is absolutely continuous with respect to μ, we may regard $B \subset L^1(\lambda+\mu)$. Were the assertion false, since $\chi_{U_j} B$ and $\chi_{U_j} A$ are both contained in $L^1(\mu)$, there would exist a $\varphi \in L^1(\mu)^*$ so that for some $b \in B$, $\varphi(\chi_{U_j} b) \neq 0$ and $\varphi(\chi_{U_j} a) = 0$ all $a \in A$ (assuming that $\overline{\chi_{U_j} B}$ is not contained in $\overline{\chi_{U_j} A}$). Since $(L^1(\mu))^* = C(\Omega)$, there is a continuous ψ so that $\int \psi \chi_{U_j} b \, d\mu \neq 0$ yet $\int \psi \chi_{U_j} a \, d\mu = 0$ for all $a \in A$. But then $\psi \chi_{U_j} \notin B^\perp$ while $\psi \chi_{U_j} \in A^\perp$. The proof assuming $\overline{\chi_{U_j} A}$ is not contained in $\overline{\chi_{U_j} B}$, is exactly the same. Now simply identify U_j with a measurable set E_j; the proof of the theorem is complete.

2. Complements

Our first main result of this section shows that if we assume the Continuum Hypothesis, there exists a closed non-separable linear subspace Y of $L^1(\mu)$ for some probability measure μ, so that every weakly compact subset of Y is separable.

THEOREM 2.1: *There exists an infinite cardinal $\boldsymbol{b} \leqq 2^{\aleph_0}$ and a closed* *linear subspace Y of $L^1(\mu)$ for some probability measure μ, so that the density character of Y is $\boldsymbol{b}$ yet every weakly compact subset of Y has density character less than $\boldsymbol{b}$.*

Our proof will yield that Y may be chosen as a complemented subspace of $X_{\mathfrak{R}}$; our original proof of 1.1 (prior to Johnson's discovery of Proposition 1.3) proceeded via 2.1. We need three preliminary results.

LEMMA 2.2: *Let B have an unconditional basis Γ and an infinite weakly compact subset K of density character $\boldsymbol{b}$. Then there is a subset Λ of Γ with card $\Lambda = \boldsymbol{b}$ so that $\{0\} \cup \Lambda$ is σ-weakly compact.*

PROOF: We indicate two demonstrations; the first is due to W. Johnson. We may choose a Y and a weakly compact operator $T : Y \to B$ so that $K \subset T(Y)$. Let T^* be the functionals biorthogonal to Γ. Since T^* is weakly compact, the proof of Proposition 1.3 yields that $\Lambda = \{\gamma \in \Gamma : T^*\gamma^* \neq 0\}$ together with 0 forms a σ-weakly compact set. Now $\Lambda^* = \{\alpha^* : \alpha \in \Lambda\}$ is total over K. Indeed suppose $\alpha^*(k) = \alpha^*(k')$ for some k, $k' \in K$ and all $\alpha^* \in \Lambda^*$. Choose $y \in Y$ with $Ty = k - k'$; then for all $\gamma \in \Gamma$, $(T^*\gamma^*)y = \gamma^*(Ty) = 0$, hence $k - k' = 0$. Now we may assume without loss of generality that $||\gamma|| = 1$ for all $\gamma \in \Gamma$. Then the operator $Q : B \to c_0(\Lambda^*)$ defined by $Q(b)(\lambda^*) = \lambda^*(b)$, is one-one on K, hence $Q|K$ is a weak-homeomorphism. Thus the density character of K is less than or equal to the cardinality of Λ. An alternate proof: Assume K is convex symmetric. By Zorn's lemma, we may choose a subset D of the non-zero elements of K, maximal with respect to the following property: if $d_1 \neq d_2$, d_1, d_2 in D, then $\Gamma_{d_1} \cap \Gamma_{d_2} = \phi$, where for $d \in K$, we put $\Gamma_d = \{\gamma \in \Gamma : \gamma^*(d) \neq 0\}$. Let $\Lambda = \bigcup_{d \in D} \Gamma_d$. Then the maximality of D implies that Λ^* is total over K and the use of Q as above implies that card $\Lambda \geqq \boldsymbol{b}$. Now let $\{\gamma_1^d, \gamma_2^d, \cdots\}$ be an enumeration of Γ_d for all $d \in D$. Let $\Lambda_{i,j} = \{\gamma \in \Lambda : \gamma = \gamma_i^d$ for some i and $|(\gamma_i^d)^*(d)| \geqq 1/j\}$. Then letting $P_{i,j}$ be the natural projection onto $[\Lambda_{i,j}]$(i.e. that given by (1) for 'Λ' of (1) equal to $\Lambda_{i,j}$), we obtain that $\Lambda_{i,j} \cup \{0\}$ is weakly compact since $P_{i,j}(D)$ is relatively weakly compact. Q.E.D.

Our proof of 2.1 will involve the classical criterion given by Lemma 1.4. Rather than working with families of functions decreasing to zero as the independent variable decreases to zero, we prefer to take reciprocals and work with increasing functions tending to infinity; also we prefer to reduce to the simpliest setting, namely functions defined only on N, the set of positive integers.

DEFINITIONS: We let $\mathscr{M}$ denote the set of all (not necessarily strictly) increasing unbounded functions $f : N \to N$. i.e. $f \in \mathscr{M}$ iff $f(n) \leqq f(n+1)$ for all n and $\lim_{n\to\infty} f(n) = \infty$. For f, $g \in \mathscr{M}$, we write $f = o(g)$, if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Given a subset S of $\mathscr{M}$, we define the function inf S by

$$\inf S(n) = \inf_{f\in S} f(n) \quad \text{for all } n \in N.$$

Evidently inf S is also an increasing function; inf S thus fails to belong to $\mathscr{M}$ iff it is stationary, i.e. for some j, $\inf S(n) = \inf S(j)$ for all $n \geqq j$. We say a cardinal $\boldsymbol{b}$ is type I if $\boldsymbol{b}$ is infinite and given $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ with card $\Gamma = \boldsymbol{b}$ then card $\Gamma_n = \boldsymbol{b}$ for some n.

LEMMA 2.3: *There exists a type I cardinal $\boldsymbol{b}$ and a subfamily $\mathscr{G}$ of $\boldsymbol{b}$ with card $\mathscr{G} = \boldsymbol{b}$ so that for every $\mathscr{A} \subset \mathscr{G}$ with card $\mathscr{A} = \boldsymbol{b}$,* inf $\mathscr{A}$ *is stationary.*

Of course $\aleph_0 < \boldsymbol{b} \leqq 2^{\aleph_0}$; naturally if we assume the Continuum Hypothesis (CH), the statement as well as the proof of this result is simplified. See the remarks immediately following this proof for further comments. (Throughout this paper we denote the cardinal of the continuum by $\boldsymbol{c}$ and also $2^{\aleph_0}$, and the first uncountable cardinal by $\aleph_1$ and also ω_1.)

PROOF: We identify cardinal numbers with initial ordinal numbers, for whose standard properties we refer the reader to [19] without further reference. The basic classical fact needed for the proof, is that

(2) given a countable set $f_1, f_2, \cdots$ of elements in $\mathscr{M}$, there exists a $g \in \mathscr{M}$ with $g = o(f_i)$ for all i.

Of course this is well-known: to see it, choose $1 = n_1 < n_2 < n_3 \cdots$ so that $f_i(n_j) \geqq 2^{j-1}$ for all $1 \leqq i \leqq j$, $j = 1, 2, \cdots$; then define g by $g(m) = j$ for all m with $n_j \leqq m < n_{j+1}$, $j = 1, 2, \cdots$.

Now let $\boldsymbol{b}$ be the smallest cardinal number such that there exists a subset $\mathscr{B}$ of $\mathscr{M}$ with card $\mathscr{B} = \boldsymbol{b}$ so that there is no $g \in \mathscr{M}$ with $g = o(b)$ for all $b \in \mathscr{B}$.

It is immediate from (2) that $\boldsymbol{b}$ is a type I cardinal; thus in particular, $\aleph_0 < \boldsymbol{b} \leqq$ card $\mathscr{M} = 2^{\aleph_0}$. Now choose $\mathscr{B}$ satisfying the above condition with card $\mathscr{B} = \boldsymbol{b}$, and let $\{b_\alpha : \alpha < \boldsymbol{b}\}$ be a one-one enumeration of $\mathscr{B}$ by the cardinal $\boldsymbol{b}$. We now choose $\mathscr{G}$ as follows: let $g_0 = b_0$. Let $\beta < \boldsymbol{b}$ and suppose $\{g_\alpha : \alpha < \beta\}$ has been chosen. We have by standard results in cardinal numbers that since card $\{\alpha : \alpha < \beta\} < \boldsymbol{b}$, card $(\{g_\alpha : \alpha < \beta\} \cup \{b_A : \alpha < \beta\}) < \boldsymbol{b}$. Hence by the definition of $\mathscr{M}$, we may choose a $g_\beta \in \mathscr{M}$ so that $g_\beta = o(g_\alpha)$ and $g_\beta = o(b_\alpha)$ for all $\alpha < \beta$.

This completes the definition of $\mathscr{G} = \{g_\alpha : \alpha < \boldsymbol{b}\}$ by transfinite induction. Obviously card $\mathscr{G} = \boldsymbol{b}$ and in fact for all $\alpha, \beta < \boldsymbol{b}$, $\alpha < \beta$ iff $g_\beta = o(g_\alpha)$. (Thus $\mathscr{G}$ is well-ordered by the relation: $f < g$ if and only if $g = o(f)$.) Now suppose $\mathscr{A}$ is a subset of $\cup$ with card $\mathscr{A} = \boldsymbol{b}$. Since $\boldsymbol{b}$ is a cardinal number, it then follows that there is a transfinite increasing sequence of ordinals $\{\eta_\alpha\}_{\alpha<\boldsymbol{b}}$ with $\mathscr{A} = \{g_{\eta_\alpha} : \alpha < \boldsymbol{b}\}$. Now in fact there is no $f \in \mathscr{M}$ with $f = o(a)$ for all $a \in \mathscr{A}$. Indeed; such an f would have the property that $f = o(g)$ for all $g \in \mathscr{G}$. But given $b \in \mathscr{B}$, there is a $g \in \mathscr{G}$ with $g = o(b)$; hence $f = o(b)$ for all $b \in \mathscr{B}$, a contradiction. Q.E.D.

REMARKS: J. Silver has shown that Martin's axiom yields that there is no cardinal $\boldsymbol{b} < 2^{\aleph_0}$ satisfying Lemma 2.3. Our proof of Lemma 2.3 and results of set theory then yield: It is consistent with set theory that $\aleph_1 < 2^{\aleph_0}$ and every subset $\mathscr{B}$ of $\mathscr{M}$ with card $\mathscr{B} < 2^{\aleph_0}$ has an upper-bound in $\mathscr{M}$; i.e. there exists on $f \in \mathscr{M}$ with $f = o(b)$ for all $b \in \mathscr{B}$. On the other hand, the $\boldsymbol{b}$ defined in our proof is easily seen to be a regular cardinal (from the properties of $\mathscr{G}$); hence it is also consistent with the axioms of set theory that $\boldsymbol{b}$ can be chosen with $\boldsymbol{b} < 2^{\aleph_0}$. (Given a set Γ, card Γ is said to be regular if whenever $\Gamma = \bigcup_{\alpha\in\Lambda}\Gamma_\alpha$ then either card Λ = card Γ or card Γ_α = card Γ for some α; Martin's axiom is equivalent to the assertion that a compact Hausdorff space which has no uncountable family of disjoint open sets, cannot be the union of a family $\mathscr{F}$ of nowhere dense subsets with card $\mathscr{F} < 2^{\aleph_0}$.)

To complete the proof of Theorem 2.1, we need the final step of linking up members of $\mathscr{M}$ with appropriate functions in $L^1[0, 1]$.

LEMMA 2.4: *There is a one-one-correspondence between $\mathscr{M}$ and Lebesgue integrable functions defined on* $[0, 1]$, $M \to f_M$, *so that for all* M, $\int |f_M| \mathrm{d}t = 1$ *and* $\int f_M \mathrm{d}t = 0$, *and such that for any non-empty subset $\mathscr{A}$ of $\mathscr{M}$, if inf $\mathscr{A}$ is stationary, then*

$$\limsup_{\delta\to 0}\sup_{M\in\mathscr{A}} \omega(f_M, \delta) \neq 0.$$

PROOF: Fix M; we shall define $f = f_M$. Put $a_n = (M(n))^{-1}$. (a_n) is a (not necessarily strictly) decreasing sequence of positive numbers tending to zero. Now let f be the unique (non-strictly) decreasing function, non-negative and right continuous on $(0, \frac{1}{2})$, so that for each positive n, f is constant on the interval

$$\left(\frac{1}{n+2}, \frac{1}{n+1}\right)$$

with

$$\int_{1/(n+3)}^{1/(n+2)} f(t)\,\mathrm{d}t = \tfrac{1}{2}(a_n - a_{n+1}), \int_e^{\frac{1}{2}} f(t)\,\mathrm{d}t = \tfrac{1}{2}(1-a_1),$$

with $f(x) = -f(1-x)$ for all $\frac{1}{2} < x < 1$. Thus $\int |f(t)| \mathrm{d}t = 1$ and $\int f(t) \mathrm{d}t = 0$; moreover f is a symmetric measurable function; i.e. for all t, $m(\{x : f(x) > t\}) = m\{x : -f(x) > t\}$ where m denotes Lebesgue measure. We have that for all positive n

(3) $$\int_0^{1/(n+2)} f_M(t)\mathrm{d}t = \tfrac{1}{2}(M(n))^{-1}$$

hence

(4) $$\omega\left(f_M, \frac{1}{n+2}\right) \geqq \frac{1}{2}\,\frac{1}{M(n)}.$$

Thus if $\mathscr{A} \subset \mathscr{M}$ is stationary, then

$$\liminf_{n\to\infty}\,\inf_{M\in\mathscr{A}} M(n) \neq \infty$$

or

$$\limsup_{n\to\infty}\,\sup_{M\in\mathscr{A}} \frac{1}{M(n)} \neq 0,$$

which by (4) implies the lemma. Also, (3) shows that the correspondence is $1-1$. Q.E.D.

Completion of the proof of Theorem 2.1: Let $\boldsymbol{b}$ be the type I cardinal and $\mathscr{G}$ the subfamily of $\mathscr{M}$ given by Lemma 2.3. Let $\mathscr{B} = \{f_M : M \in \mathscr{G}\}$, let μ, $\mathscr{R}$, and $\mathscr{M}$ for $r \in \mathscr{R}$ be as defined prior to Theorem 1.1, and let Y be the closed linear span in $L^1(\mu)$ of $(\tilde{f}_M : M \in \mathscr{G}\}$. As observed in the proof of Theorem 1.1, $\tilde{\mathscr{B}}(= \{\tilde{b} : b \in \mathscr{B}\})$ is an unconditional basis for Y (in fact in this case the unconditional constant equals one). Then the density character of Y equals card $\mathscr{B} = \boldsymbol{b}$; suppose K were a weakly compact subset of Y of density character $\boldsymbol{b}$. Taking note of the one-one nature of the correspondence of 2.4 and the fact that $\mathscr{M}$ is a type I cardinal there would be a subset $\mathscr{A}$ of $\mathscr{G}$ with card $\mathscr{A} =$ and $\boldsymbol{b}$ $\{\tilde{f}_M : M \in \mathscr{A}\}$ relatively weakly compact. Hence by Lemma 1.4,

$$\limsup_{\delta\to 0}\,\sup_{M\in\mathscr{A}} \omega(\tilde{f}_M, \delta) = 0;$$

but by Lemma 2.3, inf $\mathscr{A}$ is stationary, hence by Lemma 2.4,

$$\limsup_{\delta\to 0}\,\sup_{M\in\mathscr{A}} \omega(\tilde{f}_M, \delta) \neq 0,$$

a contradiction. Q.E.D.

The next main result of Section 2 yields that it is consistent with the axioms of set theory that every subspace of $L^1([0, 1]^{\omega_1})$ is WCG. We first require the following result due to J. Silver.

LEMMA 2.5: *Let $\mathscr{A}$ be an uncountable subset of $\mathscr{M}$.*

a) *There exists a countable infinite subset* $\mathscr{B}$ *of* $\mathscr{A}$ *with* $\inf \mathscr{B} \in \mathscr{M}$.

b) *Assuming Martin's axiom, if card* $\mathscr{A} < \mathbf{c}$, *then there is an* $m \in \mathscr{M}$ *with* $m = o(a)$ *for all* $a \in \mathscr{A}$.

We note that it is consistent with the axioms of set theory, that the hypotheses of 2.5 b) are not vacuous.

PROOF: We give only Silver's proof for a). We first observe that for any uncountable subset $\mathscr{B}$ of $\mathscr{A}$ and integer k, there exists an m and an uncountable subset $\mathscr{D}$ of $\mathscr{A}$ with $d(m) \geqq k$ for all $d \in \mathscr{D}$. Indeed, $\mathscr{B} = \bigcup_{m=1}^{\infty}\{b \in \mathscr{B} : b(m) \geqq k\}$. Now let $\mathscr{A}_1 = \mathscr{A}$, let $f_1 \in \mathscr{A}$, and let $n_1 = 1$. Suppose $k \geqq 1$, $n_1 < n_2 \cdots < n_k, f_1, \cdots, f_k$, and $\mathscr{A}_k$ have been chosen so that $\mathscr{A}_k$ is an uncountable subset of $\mathscr{A}$, so that for all $1 \leqq i \leqq k, f(n_i) \geqq i$ for all $f \in \mathscr{A}_k$ and also $f_j(n_i) \geqq i$ for all $1 \leqq j \leqq k$. Now choose $l > n_k$ with $f_j(l) \geqq k+1$ for all $i \leqq j \leqq k$. Then by our initial observation, we may choose $\mathscr{A}_{k+1}$ an uncountable subset of $\mathscr{A}_k$ and m, so that $f(m) \geqq k+1$ for all $f \in \mathscr{A}_{k+1}$. Finally, let $n_{k+1} = \max\{m, l\}$ and choose $f_{k+1} \in \mathscr{A}_{k+1}$ with $f_{k+1} \neq f_j$ for all $1 \leqq j \leqq k$.

We have thus constructed a sequence $f_1, f_2, \cdots$ of distinct elements of $\mathscr{A}$ and a sequence of positive integers $n_1 < n_2 < \cdots$ so that $f_j(n_i) \geqq i$ for all j and i. Hence

$$\liminf_{i\to\infty}\inf_j f_j(i) \geqq \liminf_{i\to\infty}\inf_j f_j(n_i) = \infty,$$

thus $\inf\{f_1, f_2, \cdots\} \in \mathscr{M}$. Q.E.D.

It is useful to have a sort of inverse for the correspondence of Lemma 2.4. This isn't completely possible, since for any $f \in L^1[0, 1]$ with $\|f\|_1 = 1$

$$\omega(f, x) \geqq x \text{ for all } 0 \leqq x \leqq 1. \tag{5}$$

(To see this assume that f is a decreasing positive function; then were

$$\int_0^x f(t)\,dt < x, \int_x^1 f(t)\,dt > 1-x;$$

but

$$\int_x^1 f(t)\,dt \leqq f(x)(1-x),$$

hence $f(x) > 1$, but then

$$\int_0^x f(t)\,dt > f(x)x > x,$$

a contradiction.) However, by restricting one's self to the members M of $\mathscr{M}$ with $M(n) = 0(n)$ as $n \to \infty$, one can verify that the correspondence defined below is inverse in the sense that for such M, putting $f = f_M$, then M_f and M have the same order of magnitude at infinity.

PROPOSITION 2.6: *Given μ a probability measure and*

$$f \in L^1(\mu) \text{ with } 0 < \int |f| \, d\mu \leqq 1, \text{ put } M_f(n) = \left[\frac{1}{\omega\left(f, \frac{1}{n}\right)} \right]$$

for all $n \in N$ (where $[x]$ denotes the greatest integer less than or equal to x). Now let $\mathscr{F}$ be a family of non-zero functions in $L^1(\mu)$ of norm at most one. Then $\mathscr{F}$ is relatively weakly compact if and only if inf $\{M_f : f \in \mathscr{F}\} \in \mathscr{M}$, while $\mathscr{F}$ is σ-relatively weakly compact if and only if there is a $g \in \mathscr{M}$ with $g = 0(M_f)$ for all $f \in \mathscr{F}$.

The proof is an easy consequence of Lemma 1.4 and shall be omitted. We are now prepared for the second main result of this section.

THEOREM 2.7: *Let μ be a probability measure on some measurable space, and let Y be a closed non-separable linear subspace of $L^1(\mu)$.*

a) *There exists a sequence (y_n) of elements of Y with $||y_n|| = 1$ for all n and $y_n \to 0$ weakly.*

b) *Assuming Martin's axiom together with the hypothesis that $\omega_1 < \mathbf{c}$, there exists a non-separable weakly compact set contained in Y. Moreover if the density character of Y is less than $\mathbf{c}$, then Y is WCG.*

REMARK: Theorem 2.7. b) together with Theorem 2.1 shows that the following question is undecideable: Given a closed non-separable linear subspace Y of $L^1(\mu)$ for some probability measure μ, does Y contain a non-separable weakly compact subset?

PROOF OF 2.7: We may choose by Zorn's Lemma a maximal subset S of Y of elements of norm 1, so that if $y \neq y'$, y, y' in S, then $||y-y'|| \geqq \frac{2}{3}$. By standard reasoning, it follows that S must be uncountable; in fact $[S] = Y$, for else there would exist a $y^* \in Y^*$ with $y^*(s) = 0$ for all $s \in S$ Assuming $||y^*|| = 1$, simply choose $y \in Y$ with $||y|| = 1$ and $y^*(y) \geqq \frac{2}{3}$; then $||y-s|| \geqq (y^*(y-s)) \geqq \frac{2}{3}$ for all $s \in S$, thus $S \cup \{y\}$ contradicts the maximality of S. Now suppose first that $\mathscr{A} = \{M_s : s \in S\}$ is uncountable (where M_s is defined in Proposition 2.6). Then by Lemma 2.5 a), there exists an infinite sequence $s_1, s_2, \cdots$ of elements of S with inf $\{M_{s_i} : i = 1, 2, \cdots\} \in \mathscr{M}$. Hence by Proposition 2.6, $s_1, s_2 \cdots$ is a relatively weakly compact subset of Y. Let $\{s_{n_j}\}_{j=1}^{\infty}$ be a weakly convergent subsequence. Since $\frac{2}{3} \leqq ||s_{n_{j+1}} - s_{n_j}|| \leqq 2$ for all j, letting

$$y_j = \frac{s_{n_{j+1}} - s_{n_j}}{||s_{n_{j+1}} - s_{n_j}||} \text{ for all } j,$$

we have that $y_j \to 0$ weakly. Now assume the hypotheses of 2.7 b), and let D be a subset of S with card $\{M_d : d \in D\} = \omega_1$. By 2.5 b) there exists a subset E of D with card $\{M_e : e \in E\} = \omega_1$ and $\inf \{M_e : e \in E\} \in \mathcal{M}$. Hence by Proposition 2.6, E is a relatively-weakly-compact subset of Y. But E is non-separable in the norm-topology, so also the weak-closure of E is non-separable in the weak-topology.

Now if the density character of Y is less than the continuum, then also card $\mathcal{A} < c$, hence by 2.5 (6) and Propositions 1.2 and 2.6, since S is σ-weakly-relatively-compact, Y is WCG. Finally, suppose $\mathcal{A}$ is countable. Then since S is uncountable, there exists an uncountable subset D of S and a function f so that $M_d = f$ for all $d \in D$. Thus the weak-closure of D is a non-separable weakly compact subset of Y. Also choosing $s_1, s_2, \cdots$ distinct elements of D, we have that $\inf \{M_{s_i} : i = 1, 2 \cdots\} = f \in \mathcal{M}$, so as before, Y contains a sequence of elements of norm one, tending to zero weakly. Q.E.D.

REMARKS:

1) Theorem 2.7 naturally leads to the following questions: Let Y be a non-separable closed linear subspace of a WCG Banach space. Does Y contain a sequence tending to zero weakly but not in norm? Is it consistent with the axioms of set theory, that every such Y contains a non-separable weakly compact set? Of course the answer to both questions is trivially yes, if Y is itself WCG.

2) *Theorem* 2.7 a may be phrased as follows: *Let Y be a subspace of $L^1(\mu)$ for some probability measure μ, and suppose that every sequence in Y which tends to zero weakly, also tends to zero in norm. Then Y is isomorphic (in fact isometric) to a subspace of* $L^1[0, 1]$. However there exists a Y which satisfies these hypotheses, yet Y is not isomorphic to a subspace of l^1. Indeed, for each n let Y_n be the Banach space consisting of all sequences (c_j) in l^1 under the norm

$$||(c_j)||_n = \frac{1}{n} \sum_{j=1}^{\infty} |c_j| + (\sum_{j=1}^{\infty} (c_j)^2)^{\frac{1}{2}},$$

and let $Y = (\sum Y_n)_{l^1}$. Since l^2 is isometric to a subspace of $L^1([0, 1])$, so is Y. On the other hand, if a space Z is isomorphic to a subspace of l^1, there is a constant K so that given any sequence (b_j) in Z equivalent to the usual l^1-basis, there is a subsequence (b_{j_i}) so that $[b_{j_i}]_{i=1}^m$ is K-isomorphic to l_m^1 for all m. (X and Y are K-isomorphic if there is a linear bijection T between them with $||T|| \, ||T^{-1}|| \leqq K$.) But for each n, if (b_j) is the natural basis for Y_n, then (b_j) is isometrically equivalent to all of its subsequences and $[b_j]_{j=1}^n$ is $(1+1/\sqrt{n})$-isomorphic to l_n^2.

The final main result of this section, combined with Theorem 1.1, shows that there exists a non-WCG Banach space with the unit ball of its dual an Eberlein compact in the weak* topology. This answers a question of Lindenstrauss in the negative [12]. We recall that a compact Hausdorff space is called an Eberlein compact if it is homeo morphic to a weakly compact subset of some Banach space. We first require a criterion for a compact Hausdorff space to be an Eberlein compact. The proof of the criterion makes use of a simple result from the next section.

LEMMA 2.8: *Let K be a compact Hausdorff space and let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a family of continuous functions on K, separating the points of K, with $\|f_\alpha\| \leqq 1$ for all α. Suppose for each α there corresponds a decreasing sequence $(\delta_j^\alpha)_{j=1}^\infty$ of positive numbers, tending to zero, so that for each $k \in K$ and j, there are at most finitely many α's in Γ with $|f_\alpha(k)| \geqq \delta_j^\alpha$. Then K is an Eberlein compact.*

PROOF: For each α, j, and $j' > j$, let $\varphi_{j,j'}^\alpha : [-1, 1] \to [-1, 1]$ be a continuous function supported on $\{t : |t| \geqq \delta_{j'}^\alpha\}$ with $\varphi_{j,j'}^\alpha(t) = t$ for all t with $|t| \geqq \delta_j^\alpha$ if $\delta_{j'}^\alpha < \delta_j^\alpha \leqq 1$; otherwise let $\varphi_{j,j'}^\alpha \equiv 0$. Now for each j and $j' > j$, let $L_{j,j'} = \{\varphi_{j,j'}^\alpha \circ f_\alpha : \alpha \in \Gamma\}$; then let $L = \bigcup_{j,j'} L_{j,j'}$. We have that L separates the points of K. Indeed let $x \neq y$ in K; then choose α with $f_\alpha(x) \neq f_\alpha(y)$. Suppose $f_\alpha(x) \neq 0$; then we may choose a j so that $|f_\alpha(x)| \geqq \delta_j^\alpha$ so that also $|f_\alpha(y)| \geqq \delta_j^\alpha$ if $f_\alpha(y) \neq 0$; with $\delta_j^\alpha < 1$. Then choose $j' > j$ with $\delta_{j'}^\alpha < \delta_j^\alpha$; we obtain that $\varphi_{j,j'}^\alpha(f_\alpha(x)) = f_\alpha(x)$ and $\varphi_{j,j'}^\alpha(f_\alpha(y)) = f_\alpha(y)$.

Now fix $j, j' > j$, and $k \in K$. We claim that there are at most finitely many members l of $L_{j,j'}$ with $l(k) \neq 0$. If not, we could choose $\alpha_1, \alpha_2, \cdots$ distinct with $\varphi_{j,j'}^{\alpha_i} \circ f_{\alpha_i}(k) \neq 0$ for all i. By our definition of the $\varphi_{j,j'}^\alpha$'s it follows that $|f_{\alpha_i}(k)| \geqq \delta_{j'}^{\alpha_i}$ for all i, contradicting the hypotheses of 2.8.

We thus have that $L_{j,j'}$ is a relatively weakly compact subset of $C(K)$ for all j, j'; hence L is a σ-relatively weakly compact subset of $C(K)$, separating the points of K. It now follows from the proof of Proposition 1.2 that there is a weakly compact subset of $C(K)$ separating the points of K; thus by (the simple) Proposition 3.3 of the next section, $C(K)$ is WCG; hence by a result of Amir and Lindenstrauss (proved in Corollary 3.4 of the next section), K is an Eberlein compact. Q.E.D.

THEOREM 2.9: *Let Γ, $\{g_\alpha : \alpha \in \Gamma\}$, v, $\tilde{g}_\alpha$ for $\alpha \in \Gamma$, and X be as defined in Theorem 1.5. Then the unit ball of X^*, in its weak* topology, is an Eberlein compact.*

PROOF: We first observe that we may assume without less of generality that each g_α is a symmetric measurable function (as defined directly preceding equation (3)). Indeed, for each α, let h_α be defined on $[0, 1]\times[0, 1]$ by $h_\alpha(x, y) = g_\alpha(x) - g_\alpha(y)$; then its easily seen that the closed linear span of $\{\tilde{h}_\alpha : \alpha \in \Gamma\}$ in $L^1(\rho)$ is isomorphic to X, where ρ equals the product-Lebesgue measure on $([0, 1]\times[0, 1])^\Gamma$ (c.f. the proof of Corollary 4.3, page 166 of [14]). But then by standard results, we may choose f_α defined on [0, 1] so that for all real r, $m\{x : f_\alpha(x) > r\} = m\times m\{(x, y): h_\alpha(x, y) > r\}$, i.e. so that f_α and h_α have the same distribution; then in fact the closed linear span of the $\tilde{f}_\alpha$'s is isometric to the closed linear span of the $\tilde{h}_\alpha$'s whence $[\tilde{f}_\alpha]_{\alpha\in\Gamma}$ is isomorphic to X. Thus the unit ball of X^* is weak*-homeomorphic to a subset of a multiple of the unit ball of the dual of $[\tilde{f}_\alpha]_{\alpha\in\Gamma}$; hence once the latter is proved to be an Eberlein compact, so is the unit ball of X^*.

To complete the proof that S, the unit ball of X^* in its weak*-topology, is an Eberlein compact, it suffices to prove

(6) for each positive integer j and $\varphi \in S$, there are at most finitely many α's in Γ with $|\varphi(\tilde{g}_\alpha)| \geqq 2\,\omega(g_\alpha, 1/j)$. (the modulus of absolute continuity, $\omega(g_{\alpha_j}\cdot)$, is defined preceding Lemma 1.4).

Indeed, for each $\alpha\in\Gamma$, let $f_\alpha(\varphi) = \varphi(\tilde{g}_\alpha)$ for all $\varphi\in S$, and let $\delta_j^\alpha = 2\,\omega(g_\alpha, 1/j)$ for all $j = 1, 2, \cdots$. Then the f_α's and δ_j^α's satisfy the hypotheses of Lemma 2.8, hence (6) implies that S is an Eberlein compact.

Now fix j and $\varphi\in S$, and suppose to the contrary that

$$|\varphi(\tilde{g}_{\alpha_n})| \geqq 2\omega\left(g_{\alpha_n}, \frac{1}{j}\right) \tag{7}$$

for an infinite sequence $\alpha_1, \alpha_2, \cdots$ of members of Γ.

Let

$$\lambda = \lim_{\delta\to 0}\sup_n \omega(g_{\alpha_n}, \delta).$$

Now if $\lambda = 0$, then by Lemma 1.4 and the fact that the $\tilde{g}_{\alpha_n}$'s are an unconditional basic sequence, $\tilde{g}_{\alpha_n}\to 0$ weakly. But then $\varphi(\tilde{g}_{\alpha_n})\to 0$, yet by (5) and (7), $|\varphi(\tilde{g}_{\alpha_n})| \geqq 2\omega(g_{\alpha_n}, 1/j) \geqq 2/j$ for all n, a contradiction. Now suppose $\lambda > 0$. Then we may choose an infinite subsequence $(g_{\alpha'_n})$ of the g_{α_n}'s and a sequence (E_n) of measurable subsets of [0, 1] with $m(E_n)\to 0$, so that

$$\int_{E_n} |g_{\alpha'_n}|\,dm \to \lambda \text{ as } n\to\infty, \tag{8}$$

and so that for each n, letting $u_n = g_{\alpha'_n}\chi_{E_n}$, u_n is a symmetric measurable function (this last statement is possible by the fact that the g_α's are symmetric.) Now putting $g_{\alpha'_n} = u_n + v_n$ for all n, we have by the definition of λ that

$$\lim_{\delta\to 0}\sup_n \omega(v_n, \delta) = 0,$$

hence $\{v_n\}_{n=1}^{\infty}$ is relatively weakly compact by Lemma 1.4 and also

$$\int_0^1 v_n \, \mathrm{d}t = 0$$

for all n. Hence letting $\tilde{u}_n(x) = u_n(x(\alpha'_n))$ and $\tilde{v}_n(x) = v_n(x(\alpha'_n))$ for all n, we have that $\tilde{g}_{\alpha'_n} = \tilde{u}_n + \tilde{v}_n$ for all n and $\tilde{v}_n \to 0$ weakly; moreover by (8), $\lim_{n\to\infty} ||\tilde{u}_n|| = \lambda$. Hence

$$\overline{\lim_{n\to\infty}} |\varphi(\tilde{g}_{\alpha'_n})| \leqq \lambda. \tag{9}$$

But by (7) and the fact that $m(E_n) \to 0$, we have that for all n sufficiently large, $|\varphi(\tilde{g}_{\alpha'_n})| \geqq 2\,\omega(g_{\alpha'_n}, 1/j) \geqq 2\,\omega(g'_{\alpha_n}, m(E_n))$ and by (8), $\lim_{n\to\infty} 2\omega(g_{\alpha'_n}, m(E_n)) = 2\lambda$, contradicting (9).

REMARK: The use of the modulus of absolute continuity in isomorphism problems concerning subspaces of L^1, may be found in the work of Kadec and Pelczynski (see [10]).

3. A topological characterization of weakly compact subsets of Banach spaces

We recall our terminology; a topological space is called an Eberlein compact if it is homeomorphic to a weakly compact subset of some Banach space. We regard the following question as the main unsolved problem in the theory of Eberlein compacts: *Is every Hausdorff continuous image of an Eberlein compact, also an Eberlein compact*? (This question has been raised before as problem 5 of [12].) We feel that our main result of this section should be useful in answering this question; further remarks are given at the end of Section 3.

The main theorem of this section is as follows:

THEOREM 3.1: *A compact Hausdorff space K is an Eberlein compact if and only if K admits a denumerable sequence $\mathscr{G}_1, \mathscr{G}^2, \cdots$ of families of open F_σ's, so that for all $k \in K$, if $k' \in K$ and $k' \neq k$, then there exists an n and a $G \in \mathscr{G}_n$ with $\chi_G(k) \neq \chi_G(k')$; and so that for all m, there are at most finitely many G's in $\mathscr{G}_m$ with $k \in G$.*

It is convenient to introduce some terminology, most of which is standard. Let a set S and a family $\mathscr{F}$ of subsets of S be given. Say that $\mathscr{F}$ separates the points of S if given $s \neq s'$ in S, then for some $F \in \mathscr{F}$, $\chi_F(s) \neq \chi_F(s')$; i.e. $s \in F$ and $s' \notin F$ or $s' \in F$ and $s \notin F$; (similarly, a family $\mathscr{G}$ of real-valued functions on S separates the points of S if given $s \neq s'$ in S, then $g(s) \neq g(s')$ for some g in $\mathscr{G}$); $\mathscr{F}$ strongly separates the points of S if given $s \neq s'$ in S, then for some $F \in \mathscr{F}$, $\chi_F(s) = 1$ and $\chi_F(s') = 0$;

i.e. $s \in F$ and $s' \notin F$; $\mathscr{F}$ is point-finite if each point of S belongs to at most finitely many members of $\mathscr{F}$; $\mathscr{F}$ is σ-point-finite if there exists a denumerable sequence $\mathscr{F}_1, \mathscr{F}_2, \cdots$ with $\mathscr{F} = \bigcup_{n=1}^{\infty} \mathscr{F}_n$ and $\mathscr{F}_n$ point-finite for all n; $\mathscr{F}$ is point-countable if each point of S belongs to at most countably many members of $\mathscr{F}$. (It's evident that a σ-point-finite family is point-countable.) Finally, we recall that an F_σ-set in a topological space, is one which equals a denumerable union of closed sets.

Theorem 3.1 may thus be rephrased: *a compact Hausdorff space K is an Eberlein compact if and only if K admits a point-separating σ-point-finite family of open F_σ's*. We note before proceeding that the insertion of the word 'strongly' before 'point-separating' constitutes the precise dividing line between compact metric spaces and the general class of Eberlein compacts. That is, *a compact Hausdorff space K is metrizable if and only if K admits a strongly-point-separating σ-point-finite family of open F_σ's*. Indeed, if K is metrizable then K has a countable base consisting of open F_σ's; from which K immediately has the above properties. Now suppose $\mathscr{G}$ is a strongly-point-separating σ-point-finite family of open subsets of K. Then there is a point-countable famely $\mathscr{F}$ of open subsets of K with the property that given x, y in K with $x \neq y$, there is an F in $\mathscr{F}$ with $x \in F$ and $y \notin \bar{F}$ (for each G in $\mathscr{G}$, choose G_i open with $\bar{G}_i \subset G_i$ for all i and $G = \bigcup_{i=1}^{\infty} G_i$; then let $\mathscr{F} = \{G_n : G \in \mathscr{G}, n = 1, 2, \ldots\}$). Now $\mathscr{F}^*$, the class of all finite intersections of elements of $\mathscr{F}$, is also point-countable. A standard compactness argument yields that $\mathscr{F}^*$ is a base for the topology of K. But it is known that a compact Hausdorff space with a point-countable base, is metrizable. (See Corollary 2.3 of [3]; we are grateful to E. Michael for this reference.) We shall discuss further variations of the hypotheses of 3.1 later. We now proceed to its proof, which is actually a fairly simple consequence of known results. For the sake of completeness, we wish to sketch the proofs of all but the most difficult of these known results; it will then be evident that the complete proof of the 'only if' part of 3.1 lies deeper the proof of the 'if' part. (The knowledgeable reader may wish to read Proposition 3.3 and then pass to the proof of 3.1 given after Lemma 3.5.).

We introduce one last notation: Throughout this section, 'K' shall stand for a compact Hausdorff space.

LEMMA 3.2. (Grothendieck [7]): *A bounded subset L of $C(K)$ is weakly compact if and only if L is compact in the topology of point-wise convergence on K.*

Suppose first that L is weakly compact. Since the weak-topology on $C(K)$ is stronger than the topology of point wise convergence, it follows immediately that L is compact in the topology of point-wise convergence.

Now suppose that L is compact in the topology of point-wise convergence, and let (f_n) be a sequence of elements in L. We then claim that there is a denumerable subset D of K so that if g and g' are elements of $C(K)$, each in the point-wise closure of $\{f_n : n = 1, 2, \cdots\}$ with $g(d) = g'(d)$ for all d in D, then $g = g'$. To see this, define an equivalence relation on K by $k \sim k'$ if and only if $f_n(k) = f_n(k')$ for all n. Let S be the set of equivalence classes and $\varphi : K \to S$ the natural quotient map; then topologize S by: U is open in S if and only if $\varphi^{-1}(U)$ is open. It then follows that S is compact metrizable and moreover if $g \in C(K)$ is such that $g(k) = g(k')$ whenever $k \sim k'$, then the function $\tilde{g}$ defined on S by $\tilde{g}(\varphi(k)) = g(k)$, is in $C(S)$. Thus S is separable, so we may choose a denumerable subset D of K with $\varphi(D)$ dense in $C(S)$. Now suppose g and g' are each in the point-wise closure of $\{f_n : n = 1, 2, \cdots\}$. It then follows that if $k \sim k'$, then $g(k) = g(k')$ and $g'(k) = g'(k')$. But then if g and g' are in $C(K)$ and agree on D, $\tilde{g}$ agrees with $\tilde{g}'$ on a dense subset of S, hence $\tilde{g} = \tilde{g}'$ on all of S, which implies that $g = g'$.

By a standard diagonal procedure, we may choose a subsequence (f'_j) of (f_n) so that (f'_j) converges point-wise on the set D. Let $g \in L$ be a cluster-point of the sequence (f'_j) in the topology of point-wise convergence, and let g' also be a cluster-point. Then $g'(d) = g(d)$ for all $d \in D$, hence $g' = g$. But a sequence in a compact Hausdorff space with exactly one cluster point, converges to it. Hence f'_j converges point-wise to g. Since L is assumed to be bounded, we have that

$$\lim_{j \to \infty} \int f'_j \mathrm{d}\mu = \int g \mathrm{d}\mu$$

for all signed regular Borel measures μ on K, by the bounded convergence theorem; hence $f'_j \to g$ weakly by the Riesz representation theorem.

Q.E.D.

Although our next proposition may be new, its proof is a very simple consequence of classical results.

PROPOSITION 3.3: *$C(K)$ is WCG if (and only if) there is a weakly compact subset L of $C(K)$, separating the points of K.*

PROOF: The 'only if' assertion is trivial, since any subset of $C(K)$ with linear span dense in $C(K)$, must separate the points of K. Now given U and V subsets of $C(K)$, let $U \cdot V = \{u \cdot v : u \in U \text{ and } v \in V\}$. If U and V are weakly-compact, so is $U \cdot V$. Indeed, given $(u_n \cdot v_n)$ a sequence of elements in $U \cdot V$ with u_n in U and v_n in V, we may choose a subsequence $n_1 < n_2 \cdots$ of the positive integers and u and v in U and V respectively, with

$u_{n_i} \to u$ and $v_{n_i} \to v$ point-wise. Then $u_{n_i} \cdot v_{n_i} \to uv$ point-wise; since $U \cdot V$ is obviously bounded, $u_{n_i} \cdot v_{n_i} \to uv$ weakly by the bounded-convergence and Riesz-representation theorems. (A 'quicker' argument, using 3.2, is simply that $U \cdot V$ is bounded and compact in the topology of point-wise convergence.)

We now suppose that L has the desired properties. We may assume without loss of generality that $||l|| \leqq 1$ for all $l \in L$, and also that 1, the 'identically-one' function, belongs to L. Define L^n inductively by $L^1 = L$ and $L^{n+1} = L^n \cdot L$. Then we have that L^n is weakly compact for all n; hence so is the set

$$W = \{0\} \cup \bigcup_{n=1}^{\infty} \frac{1}{2^n} L^n.$$

But the linear span of W is a subalgebra of $C(K)$, separating the points of K and containing the constant functions. Hence W generates $C(K)$ by the Stone-Weierstrauss theorem. Q.E.D.

We only need the a) part of the following assertion; the b) part is given for the sake of completeness.

COROLLARY 3.4:

a) (Lindenstrauss and Amir [1]): *K is an Eberlein compact if and only if $C(K)$ is WCG.*

b) (Lindenstrauss [12]): *a Banach space X is WCG if and only if the unit ball S^* of X^* in its weak*-topology is affinely homeomorphic to a weakly compact subset of some Banach space.*

PROOF: Suppose first that L is a weakly compact subset of the Banach space X, and consider the map $T : X^* \to C(L)$ defined by $(Tx^*)(l) = x^*(l)$ for all $l \in L$. Then $T(S^*)$ is a bounded subset of $C(L)$, compact in the topology of point-wise convergence. Indeed, suppose that f is a function on L, in the point-wise closure of $T(S^*)$. Let $\{s_\alpha^*\}_{\alpha \in D}$ be a net in S^* with $T(s_\alpha^*) \to f$ pointwise on L, and let s^* be a weak* cluster point of this net. Then we obtain easily that $Ts^* = f$. Hence by the Grothendieck Lemma 3.2, $T(S^*)$ is weakly compact. If L generates X, then T is $1-1$; moreover $T|S^*$ is continuous from the weak*-topology of S^* to $T(S^*)$ endowed with its weak-topology. (Indeed, the topology of point-wise convergence on $T(S^*)$ coincides with the weak topology.) Hence then S^* is affinely homeomorphic to $T(S^*)$.

In any case, $T(S^*)$ separates the points of L by the Hahn-Banach theorem, so $C(L)$ is WCG by Proposition 3.3. We have thus proved that if K is an Eberlein compact, then $C(K)$ is WCG. But if $C(K)$ is WCG, we also obtain that S^*, the unit ball of $C(K)^*$ in its weak* topology, is an

Eberlein compact; hence K being homeomorphic to a subset of S^* is also an Eberlein compact.

For the 'if' assertion of b), we observe that the hypotheses imply that there is a Banach space Y, a weakly compact subset L of Y generating Y, and a one-one linear map $T: X^* \to Y$ so that $T(S^*) = L$ and T is continuous from the ω^*-topology on S^* to the weak topology on Y. It follows that T is a weakly compact map, hence so is $T^*: Y^* \to X^{**}$; we also obtain that if S denotes the unit ball of Y^*, then $T^*(S)$ is a weakly compact subset of X^{**}; the fact that T is one-one implies that $T^*(S)$ is total with respect to X^*. We also have that for each $y^* \in Y^*$, if $\{x_\alpha^*\}_{\alpha \in D}$ is a uniformly bounded net in X^* with $x_\alpha^* \to 0$, ω^*, then $Tx_\alpha^* \to 0$ weakly, and hence for $y^* \in Y^*$; $y^*(Tx_\alpha^*) = (T^*y^*)(x_\alpha^*) \to 0$. Thus by the Krein-Smulian theorem $T^*(S) \subset X$(where we regard X as canonically imbedded in X^{**}). Thus $T^*(S)$ is a weakly compact subset of X, generating X by the Hahn-Banach theorem and the fact that $T^*(S)$ is total with respect to X^*. Q,E.D.

REMARK: The last result of the preceding section shows that there is a non-WCG Banach space X so that the unit ball of the dual of X^* is an Eberlein compact; thus the word 'affinely' cannot be deleted in the 'if' part of 3.4 b).

LEMMA 3.5 (Lindenstrauss, pages 249–260 of [12]): *A WCG Banach space is generated by a subset which is homeomorphic, in its weak topology, to the one-point compactification of a discrete set.*

This result strikes me as being the most difficult of everything stated so far in this section; the proof given in [12] uses all the machinery given in [1]. It is not difficult to see that it is equivalent to the assertion that every weakly compact convex set in a Banach space is affinely homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some discrete set Γ. By the recent factorization theorem for weakly compact operators (see [4]), every weakly compact convex set in a Banach space is affinely homeomorphic to a weakly compact subset of a reflexive Banach space; the fact that every such may be imbedded in an appropriate $c_0(\Gamma)$-space may then be deduced from the earlier work of Lindenstrauss given in [11], which is somewhat simpler than the arguments of [1].

PROOF OF THEOREM 3.1: Suppose first that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ are families of open F_σ's satisfying the conditions at the end of the statement of 3.1. Fix n and for each $\alpha \in \mathscr{G}_n$, let f_α^n be a continuous function on K with $0 \leqq f_\alpha^n \leqq 1$ and $\alpha = \{x : f_\alpha^n(x) \neq 0\}$. Now let $L_n = \{0\} \cup \{f_\alpha^n : \alpha \in \mathscr{G}_n\}$. Then L_n is weakly-compact; in fact L_n is homeomorphic in the weak topology, to the one-point compactification of a discrete set. Indeed, let $\alpha_1, \alpha_2, \cdots$ be an infinite sequence of distinct elements of $\mathscr{G}_n$ (if such exists, of course). Then for each $k \in K$, k belongs to at most finitely many a_j's, hence $\lim_{j\to\infty} f^n_{\alpha_j}(k) = 0$; thus $f^n_{\alpha_j} \to 0$ weakly as $j \to \infty$. Now let

$$L = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n}\right) L_n .$$

Then L is also a weakly compact set (in fact again weakly homeomorphic to the one-point compactification of a discrete set) and L separates the points of K. For given $k \neq k'$, there is an n and an $\alpha \in \mathscr{G}_n$ with $\chi_\alpha(k) \neq \chi_\alpha(k')$. Thus e.g. if $\chi_\alpha(k) = 1$, $f^n_\alpha(k) \neq 0$ but $f^n_\alpha(k') = 0$, so $f^n_\alpha(k) \neq f^n_\alpha(k')$, hence also

$$\frac{1}{2^n} f^n_\alpha(k) \neq \frac{1}{2^n} f^n_\alpha(k').$$

Thus by Proposition 3.3, $C(K)$ is WCG, so by Corollary 3.4, K is an Eberlein compact.

Now suppose that K is an Eberlein compact, hence $C(K)$ is WCG by Corollary 3.4. By Lemma 3.5, there exists a family $\{f_\alpha\}_{\alpha \in \Gamma}$ of continuous functions on K, with linear span dense in $C(K)$, so that $\{f_\alpha\}_{\alpha\in\Gamma} \cup \{0\}$ is weakly homeomorphic to the one-point compactification of the discrete set Γ, with 0 the compactification point. (It follows from the proof on page 250 of [12] that the set generating X in the statement of 3.5, may be chosen with 0 as its compactification point; a simple translation argument also allows this deduction directly from the statement of 3.5.) We may and shall assume that $0 < \|f_\alpha\|_\infty \leqq 1$ for all α; the properties of the f_α's of interest are that they separate the points of K, and for any infinite sequence $\alpha_1, \alpha_2, \cdots$ of distinct α's, $f_{\alpha_i} \to 0$ pointwise on K.

Now fix n and let $\mathscr{G}_n$ be the family of all sets

$$U^n_{\alpha, j} = \left\{x : \frac{j-2}{n} < f_\alpha(x) < \frac{j}{n}\right\}$$

where $3 \leqq j \leqq n+1$ or $-n+1 \leqq j \leqq -1$ and $\alpha \in \Gamma$. Then we have

$$\bigcup_j U^n_{\alpha, j} = \left\{x : |f_\alpha(x)| > \frac{1}{n}\right\}. \tag{10}$$

Since each $U^n_{\alpha,j}$ is an open F_σ, it suffices to prove that each family $\mathscr{G}_n$ is point finite, and $\bigcup_{n=1}^\infty \mathscr{G}_n$ separates the points of K. First fix n and suppose to the contrary that for some $k \in K$, k belonged to infinitely many members of $\mathscr{G}_n$. Since for each α, there are only finitely many sets of the form $U^n_{\alpha,j}$, we must have that for some j and for an infinite sequence of distinct α's, $\alpha_1, \alpha_2, \cdots$, $k \in \cup_i U^n_{\alpha_i, j}$. Thus by (10), $|f_{\alpha_i}(k)| > 1/n$ for all i, contradicting the fact that $f_{\alpha_i} \to 0$ point-wise.

Finally, suppose $x \neq y$, x, y in K. Then there is an α with $f_\alpha(x) \neq f_\alpha(y)$. We may assume without loss of generality that $f_\alpha(x) < f_\alpha(y)$. Now suppose $f_\alpha(y) > 0$. Then we may choose a positive integer n so that $f_\alpha(y) > 1/n$ and $f_\alpha(y) - f_\alpha(x) > 2/n$. Then for some j with $3 \leqq j \leqq n+1$, $y \in U^n_{\alpha,j}$ and $x \notin U^n_{\alpha,j}$; the case $f_\alpha(x) < 0$ is handled in an entirely analogous fashion. Thus $\bigcup_{n=1}^\infty \mathscr{G}_n$ separates the points of K, and the proof of Theorem 3.1 is complete. Q.E.D.

We close with several remarks and questions.

1) Suppose that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ satisfy the properties of the last part of the statement of 3.1. For each $\alpha \in \mathscr{G}_j$, choose $\alpha_1, \alpha_2, \cdots$ so that for all i, α_i is an open F_σ with $\overline{\alpha_i} \subset \alpha_{i+1}$ and $\alpha = \bigcup_{i=1}^\infty \alpha_i$; (if K is totally disconnected, we can also choose α_i to be closed and open for all i). It is then easily seen that $\mathscr{H} = \{\alpha_i : \alpha \in \mathscr{G}_j$ for some j and $i = 1, 2, \cdots\}$ is also a σ-point finite family of sets. We have the further property that given $x \neq y$ in K, then for some $H \in \mathscr{H}$, $x \in H$ and $y \in {\sim}\overline{H}$ or $y \in H$ and $x \in {\sim}\overline{H}$. It follows that $\mathscr{H} \cup \{{\sim}\overline{H} : H \in \mathscr{H}\}$ is a subbase for the topology of K. We have also arrived at the following result: *If K is totally disconnected, then K is an Eberlein compact if and only if there exists a point-separating family of closed and open subsets of K, which is σ-point-finite*. If K is totally disconnected, we thus have a characterization of whether or not K is an Eberlein-compact, in purely algebraic terms of the Boolean algebra of closed and open subsets of K.

2) There are a number of properties of Eberlein compacts for which the only proofs presently available, are analytic. It would be desirable to have purely topological proofs of these properties, based on the properties given in Theorem 3.1. For example, it is known that if K is an Eberlein compact, then the closure of any subset of K equals its sequential closure, and also any separable subset of K has metrizeable closure. (Incidentally, both of these properties are obviously preserved under continuous images.) It is also known (see [12]) that K has a dense subset of G_δ-points; (a G_δ-subset of a topological space is by definition a denumerable intersection of open sets).

3) There are a number of possible ways of weakening the topological conditions of Theorem 3.1. We pose a number of questions concerning these. Throughout this remark, $\mathscr{F}$ denotes a point-separating family of open subsets of K.

Is K an Eberlein compact if any of the following are true?

A) *$\mathscr{F}$ is point-countable and consists of F_σ's.*

B) *$\mathscr{F}$ is σ-point-finite.*

C) *$\mathscr{F}$ is σ-point-finite and $\mathscr{F} \cup \{{\sim}\overline{F} : F \in \mathscr{F}\}$ is strongly point-separating.*

Theorem 3.1 and the preceding remark show that an Eberlein compact admits a family $\mathscr{F}$ satisfying both conditions A and C. It has recently been proved by M. E. Rudin that it is consistent with set theory that the answer to A is negative. Precisely, she has proved (unpublished as of this writing) that it is consistent that there exists a non-metrizeable K with a family $\mathscr{F}$ satisfying A, such that K has no uncountable family of disjoint open sets. However, *any non-metrizeable Eberlein compact has an uncountable family of disjoint open sets*, by Corollary 4.6, page 230 of [14].

We wish also to point out that if K is a continuous image of an Eberlein compact and $C(K)$ admits an isometrically norming Markusevic basis, then K admits a family satisfying A. Indeed, $X = C(K)$ isometrically imbeds as a closed linear subspace of a WCG Banach space Y. Let $\{x_\alpha, f_\alpha) : \alpha \in \Gamma\}$ be an isometrically norming M-basis for X; that is, for all $\alpha, \beta \in \Gamma$, $f_\alpha \in x^*$, $x_\alpha \in x$, $f_\alpha(x_\beta) = \delta_{\alpha,\beta}$, the linear span of the x_α's is dense in X, and Z, the linear-span of the f_α's, is isometrically norming over X; i.e. for all $x \in X$, $||x|| = \sup |g(x)|$, the supremum taken over all g in Z of norm one. It follows that if S denotes the unit ball of X^* in its ω^* topology then $S \cap Z$ is dense in S, by the Hahn-Banach theorem. Thus since S is a continuous image of an Eberlein compact, namely the unit ball of Y^* in its weak* topology, $S \cap Z$ is sequentially dense in S. Now for each g in Z, $g(x_\alpha) \neq 0$ for at most finitely many α's; it then follows that for each s in S, $s(x_\alpha) \neq 0$ for at most countably many α's. If we now define h_α on S by $h_\alpha(s) = s(x_\alpha)$ for all α, we have, since the linear span of the x_α's is dense in X, that $\{h_\alpha : \alpha \in \Gamma\}$ is a point-separating family of continuous functions on S so that for each s in S, s belongs to the non-zero points of at most countably many h_α's. An argument somewhat simpler than the one for the proof of Lemma 2.8 yields a point-countable point-separating family $\mathscr{F}$ of open F_σ-subsets for S. It then follows that any closed subset of S also admits such a family, and K is homeomorphic to a subset of S. It is apparently unknown if every WCG space or every subspace there-of, admits a norming M-basis; the work of John and Zizler [8] seems relevant to this question.

4) There are further variations possible on the topological conditions. The following example is instructive in this connection. Let K denote the space of all ordinals less than or equal to ω_1, in the order topology. Then K is not an Eberlein compact, since $K \sim \{\omega_1\}$ is sequentially closed but not closed. Now $\{[\alpha, \omega_1) : \alpha < \omega_1\}$ is a point-countable point-separating family of open subsets of K.

One can also consider strengthening the conditions. As mentioned at the beginning of this section, demanding that the family be strongly point-separating is equivalent to K being metrizeable.

QUESTION: *What are the compact Hausdorff spaces K so that K admits a point-separating family $\mathscr{F}$ of open F_σ-subsets, such that $\mathscr{F}$ is point-finite?*

We note that *if K admits such a family $\mathscr{F}$, then K must be dispersed, i.e. K has no non-empty perfect subsets.*

Indeed, since admitting such a family is a hereditary property, we may suppose to the contrary that K has such a family and is itself perfect (non-empty of course). Now for each non-empty $F \in \mathscr{F}$, choose a sequence $F_1, F_2, \cdots$ of compact subsets of F with $F = \bigcup_{j=1}^{\infty} F_j$ and $\mathscr{I}F_j$ non-empty for all j, where $\mathscr{I}\mathrm{F}_j$ denotes the interior of the set F_j. Let $j_1 = 1$ and F^1 be a non-empty member of $\mathscr{F}$. Suppose $F^1, \cdots, F^k$ distinct members of $\mathscr{F}$ and $j_1, \cdots, j_k$ have been chosen with $U = \bigcap_{i=1}^{k} \mathscr{I}(F^i)_{j_i}$ non-empty. Since K is perfect, we may choose x and y in U with $x \neq y$. We may choose F^{k+1} with $\chi_{F^{k+1}}(x) \neq \chi_{F^{k+1}}(y)$. Then $F^{k+1} \neq F^i$ for any $1 \leqq i \leqq k$, since $\chi_{F^i}(x) = 1 = \chi_{F^i}(y)$ for all such i. Now suppose that $x \in F^{k+1}$; then choose j_{k+1} so that $x \in (F^{k+1})_{j+1}$. We then have that $\bigcap_{i=1}^{k+1} \mathrm{W}(F^i)_{j_i}$ is non-empty. Finally, since then $\bigcap_{i=1}^{k}(F^i)_{j_i}$ is non-empty for all k, so is $\bigcap_{i=1}^{\infty}(F^i)_{j_i}$; this contradicts the point-finiteness of the family. We note also that the one-point compactification of any discrete set does admit such a family.

5) We wish to mention a purely topological question concerning Eberlein compacts themselves. *If K is an Eberlein compact and every compact metrizeable subset of K is a G_δ, is K itself metrizeable?* There are examples given in [12] of non-metrizeable Eberlein compacts in which every point is a G_δ-point. These examples seem to admit non-G_δ separable closed subsets. In this connection, we wish to point out that *if K is a non-metrizeable Eberlein compact and every element of K is a G_δ-point, then K has an infinite perfect subset.* To see this, by Corollary 4.6 of [14] we may choose an uncountable family $\{U_\alpha : \alpha < \omega_1\}$ of non-empty disjoint open subsets of K (which, without loss of generality, we have indexed by the first uncountable ordinal ω_1). Choose $l_\alpha \in U_\alpha$, let $L = \overline{\{l_\alpha : \alpha \in \omega_1\}}$, and let $D = \bigcap_{\beta<\omega_1} \overline{\{l_\alpha : \beta \leqq \alpha\}}$. D is non-empty by compactness; we shall refer to its elements as co-countable cluster points. Now if U is an open set containing D, then for some $\beta < \omega_1$, $U \subset \{l_\alpha : \beta \leqq \alpha\}$. Since no $l_\alpha \in D$, it follows that D cannot be a G_δ-set. Now if x is an isolated point of D, we can choose an open neighborhood U of x with $D \sim \{x\} \cap U = \phi$. It still must be the case that for all $\beta < \omega_1$, there is a $\beta < \alpha < \omega_1$ with $l_\alpha \in U$. Hence $\bigcap_{\beta<\omega_1} \overline{\{l_\alpha : l_\alpha \in U \text{ and } \beta \leqq \alpha\}}$ is also non-empty; but then this set equals $\{x\}$, hence $\{x\}$ is itself the set of co-countable cluster points of $U \cap \{l_\alpha : \alpha \in \omega_1\}$, so $\{x\}$ cannot be a G_δ-set.

6) *A Banach space B is isometric (resp. isomorphic) to the dual of a WCG Banach space if and only if the unit ball of B (resp. some convex*

body in B) is affinely equivalent to a weakly subset of some Banach space. (By a convex body, we mean a closed convex bounded set with non-void interior; subsets K and L of two real linear spaces are called *affinely equivalent* if there is a bijection $T : K \to L$ so that for all x, y in K and $0 \leqq \lambda \leqq 1$, $T(\lambda x+(1-\lambda)y) = \lambda Tx+(1-\lambda)Ty$.) We thus obtain for the space $X_{\mathfrak{R}}$ that although the unit ball of $X_{\mathfrak{R}}^*$ is homeomorphic to an Eberlein compact, neither it nor any convex body in $X_{\mathfrak{R}}^*$ is affinely equivalent to a weakly compact subset of a Banach space. To see the assertion mentioned at the beginning of this remark, it suffices to consider only the isometric case, to which the isomorphic one easily reduces. The 'only if' part follows from 3.4 b). If B satisfies the hypotheses of the 'if' part, then there is a Banach space Y, a weakly compact subset K of Y, and an operator $T : B \to Y$ so that K equals $T(S)$, where S denotes the unit ball of B. Now it follows by standard reasoning (e.g. that of the proof of the 'Goldstine theorem', p. 424 of [5]) that B is isometric to the dual of T^*Y^* via the canonical map $\pi : B \to (T^*Y^*)^*$ defined by $(\pi b)(T^*y^*) = y^*(Tb)$ for all $b \in B$ and $y^* \in Y^*$. But then $T|S$ is an affine homeomorphism between S and K where S is endowed with the 'weak* topology' induced by T^*Y^*. Hence the 'if part' also follows from 3.4 b). (This also yields that if K is a weakly compact symmetric subset of some Banach space and Y is the Banach space with K as its unit ball, then Y is isometric to the dual of a WCG Banach space).

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