COMPOSITIO MATHEMATICA

JÁNOS GALAMBOS

On infinite series representations of real numbers

Compositio Mathematica, tome 27, nº 2 (1973), p. 197-204
<http://www.numdam.org/item?id=CM_1973__27_2_197_0>

COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 2, 1973, pag. 197-204
Noordhoff International Publishing
Printed in the Netherlands

ON INFINITE SERIES REPRESENTATIONS OF REAL NUMBERS

János Galambos

1. Summary

The major objective of the present paper is to generalize some of the results of Vervaat [12] and of the present author [1] and [6] in metric number theory by considering an algorithm which includes those investigated in the above papers. Though hints have been given for this more general expansion in the literature, metric results achieved their most general formulations in the quoted papers. Some of the results are new for the special cases of [1], [6] and [12], or even for the classical expansions of Engel, Sylvester and Cantor.

2. The algorithm

Let $\alpha_j(n) > 0, j = 1, 2, \cdots$ be a sequence of strictly decreasing functions of natural numbers n and such that, for each j, $\alpha_j(1) = 1$ and $\alpha_j(n) \to 0$ as $n \to +\infty$. Let $\gamma_j(n)$ be another sequence of positive functions of n on which some further assumptions will be imposed in the sequel. Let $0 < x \leqq 1$ be an arbitrary real number and define the integers $d_j = d_j(x)$ and the real numbers x_j by the algorithm

$$x = x_1, \alpha_j(d_j) < x_j \leqq \alpha_j(d_j-1) \tag{1}$$

and

$$x_{j+1} = \{x_j-\alpha_j(d_j)\}/\gamma_j(d_j). \tag{2}$$

In view of (1) and (2), we have to make a restriction on $\gamma_j(n)$ in order to guarantee that $0 < x_{j+1} \leqq 1$. Since by (1)

$$0 < x_j-\alpha_j(d_j) \leqq \alpha_j(d_j-1)-\alpha_j(d_j),$$

we impose the condition

$$\{\alpha_j(n-1)-\alpha_j(n)\} \leqq \gamma_j(n), \text{ for all } n \geqq 2, \tag{3}$$

on the selection of $\gamma_j(n)$ for $j = 1, 2, \cdots$. (1) and (2), under (3), yield the infinite series

$$(4) \qquad y(x) = \alpha_1(d_1)+\gamma_1(d_1)\alpha_2(d_2)+\gamma_1(d_1)\gamma_2(d_2)\alpha_3(d_3)+\ldots.$$

Note that for any N,

$$(5) \qquad x = \alpha_1(d_1)+\cdots+\gamma_1(d_1)\cdots\gamma_{N-1}(d_{N-1})\alpha_N(d_N) + \gamma_1(d_1)\cdots\gamma_N(d_N)x_{N+1},$$

and thus the infinite series in (4) always converges and

$$(6) \qquad 0 < y(x) \leqq x.$$

In this general set up, it is a very difficult question to find a criterion for an infinite series in the form of (4) to be the expansion of its sum y by the algorithm (1) and (2). We will not make an attempt to answer this question. Its difficulty will be made clear through the examples, taken from the literature, which are all special cases of (1) and (2). We shall however formulate a simple criterion for $y(x) = x$, which is actually a simple consequence of (5). For its formulation, we introduce a concept. Let $k_1, k_2, \cdots, k_N$ be positive integers and assume that there is at least one real number x such that $d_j(x) = k_j$, $j = 1, 2, \cdots, N$. Then, following Vervaat [12], we call the vector $(k_1, k_2, \cdots, k_N)$ realizable (with respect to the sequences $\alpha_j(n)$ and $\gamma_j(n)$), and an infinite sequence $k_1, k_2, \cdots$ of positive integers is called realizable if $(k_1, k_2, \cdots, k_N)$ is realizable for $N = 1, 2, \cdots$. We now have

THEOREM 1: *$y(x) = x$ for each $x \in (0, 1]$ if, and only if, for any realizable sequence $k_1, k_2, \cdots$,* as $N \to +\infty$,

$$\lim \gamma_1(k_1)\gamma_2(k_2)\cdots\gamma_{N-1}(k_{N-1})\{\alpha_N(k_N-1)-\alpha_N(k_N)\} = 0.$$

PROOF: Applying (2) in (5), we have that, for $N \geqq 2$,

$$0 < x - \sum_{j=1}^{N} \alpha_j(d_j) \prod_{m=1}^{j-1} \gamma_m(d_m) = \gamma_1(d_1)\cdots\gamma_{N-1}(d_{N-1})\{x_N-\alpha_N(d_N)\},$$

which, by (1), is smaller than

$$\gamma_1(d_1)\cdots\gamma_{N-1}(d_{N-1})\{\alpha_N(d_N-1)-\alpha_N(d_N)\},$$

and the part 'if' of the theorem is thus proved. On the other hand, assume that for each $x \in (0, 1]$, $y(x) = x$. Then the decreasing sequence

$$(A_N, B_N] = \{x : d_1 = k_1, d_2 = k_2, \cdots, d_N = k_N\}$$

of intervals, for any given realizable sequence $k_1, k_2, \cdots$, contains a single point in common as $N \to +\infty$. Since by (1) and (2),

$$A_N = \sum_{j=1}^{N} \alpha_j(k_j) \prod_{m=1}^{j-1} \gamma_m(k_m)$$

and

$$B_N = A_{N-1} + \alpha_N(k_N - 1) \prod_{m=1}^{N-1} \gamma_m(k_m),$$

$B_N - A_N \to 0$ is exactly the condition of the theorem, hence the proof is complete.

Though Theorem 1 is stated in terms of realizable sequences, it is applicable without their complete characterization as our examples below will show this. We shall even have an example when the characterization of realizable sequences is known but complicated and Theorem 1 will therefore be applied without making use of the criterion for realizability.

Let us turn to some examples of the algorithm (1) and (2).

EXAMPLE 1: Let $\alpha_j(n) = \alpha(n)$ for all $j \geqq 1$. Let further $\gamma_j(n) = \gamma(n) = \{\alpha(n-1) - \alpha(n)\}/\alpha(h(n))$, where $h(n)$ is an arbitrary integer valued function with $h(n) \geqq 1$. Our algorithm reduces to that of Vervaat [12], who termed the expansion (4) as the Balkema-Oppenheim expansion. (3) is evidently satisfied and the criterion for realizability is easily seen to be $k_1 \geqq 2$, and $k_j > h(k_{j-1})$ for $j \geqq 2$.

EXAMPLE 2: When $\alpha_j(n) = 1/n$ for all $j \geqq 1$, and $\gamma_j(n) = a_j(n)/b_j(n)$, where $a_j(n)$ and $b_j(n)$ are positive integer valued functions of n, we get back the expansion considered in Galambos [1] and called there the Oppenheim expansion. The condition (3) was overcome by the assumption that $\gamma_j(n)n(n-1) = h_j(n)$ is integer valued for all j. In this case, realizability is similarly characterized as in the case of Example 1 with $h_j(n)$ for $h(n)$. Dropping, however, this restriction, a complete solution of the problem of realizability under (3) is yet to be found; for further details, see Oppenheim [9]. The question $y(x) = x$ is, however, settled for most cases in [9], which solutions are all consequences of Theorem 1. We remark here that Oppenheim [8] recommended a much more general expansion than the one described in this example, most of his results are unpublished on that line.

Both examples include the classical expansions of Engel, Sylvester, Lüroth and the product expansion of Cantor; see the quoted references, and also [11].

EXAMPLE 3: Let $g > 1$ and let $\alpha_j(n) = \alpha(n)$ be the sequence ug^{-m}, $u = 1, 2, \cdots, [g]-1$ and $m = 1, 2, \cdots$, arranged in a decreasing order. The digits $d_k = d_k(x)$, determined in (1) and (2), when we choose $\gamma_j(n) = 1$ for all j and n, are the non-zero digits of the usual algorithm. For non-integral g, the criterion for realizability is very complicated, see [5], Theorem 1, however, immediately yields that $y(x) = x$ for each $x \in (0, 1]$.

EXAMPLE 4: Let $\alpha_j(n) = \alpha(n)$ be the sequence $u_k/q_1 q_2 \cdots q_k$, where $q_t \geqq 2$ are given integers and $u_k = 1, 2, \cdots, q_k - 1$, and let $\gamma_j(n) = 1$ again for all j and n. Then our algorithm reduces to the Cantor series, leaving out the zero terms. For a list of references on metric results for this series, see Galambos [5].

3. Metric results

We now turn to the investigation of some metric properties of sequences associated with the algorithm (1) and (2), assuming, of course, the validity of (3). These metric results will be in terms of Lebesgue measure λ. Our first results will be for the variables

$$(7) \quad z_1 = x_1 = x \text{ and } z_{n+1} = x_{n+1}\gamma_n(d_n)/\{\alpha_n(d_n - 1) - \alpha_n(d_n)\},\ n \geqq 1.$$

THEOREM 2: *For any integer t, z_t has a uniform distribution and it is independent of $(d_1, d_2, \cdots, d_{t-1})$, that is, for any integers $j_1, j_2, \cdots, j_{t-1}$ and for any real number $0 < c \leqq 1$,*

$$\lambda(z_t \leqq c) = c$$

and

$$\lambda(d_1 = j_1, d_2 = j_2, \cdots, d_{t-1} = j_{t-1}, z_t \leqq c) = c\lambda(d_1 = j_1, d_2 = j_2, \cdots, d_{t-1} = j_{t-1}).$$

PROOF: We first prove the second equation. Note that if $(j_1, j_2, \cdots, j_{t-1})$ is not realizable then both sides are zero and thus the conclusion is evidently true. Let now $(j_1, j_2, \cdots, j_{t-1})$ be realizable. Then by (1) and (2), the set

$$\{x : d_1 = j_1, d_2 = j_2, \cdots, d_{t-1} = j_{t-1}, z_t \leqq c\} = (A_{t-1}, C_{t-1}]$$

is an interval with

$$A_t = \sum_{k=1}^{t} \alpha_k(j_k) \prod_{m=1}^{k-1} \gamma_m(j_m)$$

and

$$C_t = A_t + \{\alpha_t(j_t - 1) - \alpha_t(j_t)\} c \prod_{m=1}^{t-1} \gamma_m(j_m).$$

Hence its length

$$C_t - A_t = c\{\alpha_t(j_t - 1) - \alpha_t(j_t)\} \prod_{m=1}^{t-1} \gamma_m(j_m),$$

i.e.,

$$\lambda(d_1 = j_1, \cdots, d_{t-1} = j_{t-1}, z_t \leqq c) = c\lambda(d_1 = j_1, \cdots, d_{t-1} = j_{t-1}, z_t \leqq 1),$$

which proves the second equation. The first equation immediately follows from the second one by simply adding up both sides for all positive integers $j_1, j_2, \cdots, j_{t-1}$.

We could now proceed to show that, under some restrictions on $\alpha_j(n)$ and $\gamma_j(n)$, the z's satisfy an 'almost independence' property, and thus strong laws and asymptotic normality follow. We do not go into the details of this program, which can be done on the line of Galambos [1]. We shall rather make our investigation in another direction, obtaining some new insight even into the classical expansions of Engel and Sylvester and Cantor's product representation.

THEOREM 3: *Assume that* $0 < c_j \leqq 1, j = 2, 3, \cdots, t$ *are such that for every* $n \geqq 2$, *there is an integer* $k_j = k_j(n)$ *satisfying*

$$(8) \qquad \alpha_{j+1}(k_j) = c_{j+1}\{\alpha_j(n-1)-\alpha_j(n)\}/\gamma_j(n), \quad 1 \leqq j \leqq t-1.$$

Let further $r \geqq 1$ be an integer and put $c_1 = \alpha_1(r-1)$. Then the events $\{z_j \leqq c_j\}, j = 1, 2, \cdots t$ are independent, i.e.,

$$\lambda(z_1 \leqq c_1, z_2 \leqq c_2, \cdots, z_t \leqq c_t) = c_1 c_2 \cdots c_t.$$

Before giving its proof, let us clarify the statement of Theorem 3. Since the z's are dependent random variables, the events $\{z_j \leqq u_j\}$ with arbitrary real numbers $0 < u_j \leqq 1$ are dependent as well. Our aim in Theorem 3 was to show that for some expansions, i.e., for certain α's and γ's, we can find (non-continuous) sequences c_j for u_j such that the events above are independent. Formula (8) explicitly gives these sequences $\{c_j\}$, when they exist. Since the c_j are not arbitrary, the events $\{z_j \leqq c_j\}$ cannot describe completely the behavior of the z_j, but they may provide interesting discrete approximations. Immediately following the proof, we shall give several examples for determining the c_j and applications will also be presented.

PROOF: By the algorithm (1) and (2) and by the assumption (8),

$$(9) \qquad z_j \leqq c_j \text{ if, and only if, } d_j > k_{j-1}(d_{j-1}), j \geqq 2, \text{ and } d_1 > r.$$

Therefore

$$\begin{aligned}\lambda(z_1 \leqq c_1, \cdots, z_{t+1} \leqq c_{t+1}) &= \lambda(d_1 > r, d_2 > k_1(d_1), \cdots, \\ &\qquad d_t > k_{t-1}(d_{t-1}), z_{t+1} \leqq c_{t+1}) \\ &= \sum{}' \lambda(d_1 = j_1, \cdots, d_t = j_t, z_{t+1} \leqq c_{t+1})\end{aligned}$$

where the summation $\sum'$ is over all t-vectors $(j_1, \cdots, j_t)$ for which $j_1 > r, j_2 > k_1(j_1), \cdots, j_t > k_{t-1}(j_{t-1})$. Thus by Theorem 2,

$$\begin{aligned}\lambda(z_1 \leqq c_1, \cdots, z_{t+1} \leqq c_{t+1}) &= c_{t+1} \sum{}' \lambda(d_1 = j_1, \cdots, d_t = j_t) \\ &= c_{t+1} \lambda(z_1 \leqq c_1, \cdots, z_t \leqq c_t).\end{aligned}$$

By induction over t this yields that

$$\lambda(z_1 \leqq c_1, \cdots, z_{t+1} \leqq c_{t+1}) = c_1 \cdots c_{t+1}.$$

The fact that this equation does mean the independence of the events $\{z_k < c_k\}$ follows from Theorem 2. The proof is complete.

In the remainder of the paper, we discuss the conclusion of Theorem 3 and deduce some of its consequences for the special cases of Examples 1–3. We first show that it generalizes some earlier results of the present author. Note that for the Oppenheim expansion of Example 2, (8) is satisfied with $c_{j+1} = 1/r_j$, where r_j is an arbitrary positive integer, whenever the $h_j(n)$ are integer valued. Hence Theorem 3 implies the following

COROLLARY 1: *With the notations of Example* 2, *we define the positive integers* $T_j = T_j(x)$ *as*

$$T_j < d_j/h_{j-1}(d_{j-1}) \leqq T_j+1, \ j \geqq 1,$$

where we put $h_0(j) = 1$. *Then, if the* $h_j(n)$ *are integer valued,* $T_1, T_2, \cdots$ *are stochastically independent and for* $s = 2, 3, \cdots$

$$\lambda(T_j+1 = s) = 1/s(s-1).$$

PROOF: By (9) and by the definition of the T's, $z_j \leqq 1/r_j$ if, and only if, $T_j+1 > r_j$, where the r_j are arbitrary positive integers. Thus Theorems 2 and 3 immediately yield our corollary.

Though this Corollary was stated in my recent paper [6], we reformulated it here to show the strength of Theorem 3, and to state one of its consequences not mentioned in [6]. First note that the T_j are distributed as the denominators in the Lüroth expansion, hence everything known for the Lüroth denominators can be restated for the T's, see Jager and de Vroedt [7], Salát [10], Galambos [1] and Vervaat [12]. In particular, by the result on p. 116 of Vervaat [12], we have that, as $N \to +\infty$,

$$\lim \lambda(N^{-1} \sum_{j=1}^{N} T_j - \log N < x) = F(x)$$

exists and is an absolutely continuous proper distribution function. From this limit relation it follows that

$$\sum_{j=1}^{N} T_j/N \log N \to 1 \tag{10}$$

in probability. As for the Lüroth denominators it then follows that the limit relation (10) cannot be strengthened to hold for almost all x. All these interesting properties are believed to be new even for the classical expansions, except that of Lüroth.

Turning to Example 3, we consider the case $1 < g < 2$ and when g is the solution of an equation $g^{a+1}-g^a = 1$ for some integer $a \geqq 1$. Theorem 3 is again applicable with $c_j = g^{-m}$, where m is an arbitrary integer. Indeed, from (8) we get that for the above choice of c_j, $k_j(n) = n+a+m$. Through the relation (9) we therefore get that the variables $T_j = d_j-d_{j-1}$ with $d_0 = 0$, are stochastically independent, a result of Galambos [2] which has interesting statistical applications [3]. We remark here that this earlier result of the author is implicitly reobtained in Vervaat [12], through example 1.2 on p. 104 and the discussion on p. 116. Vervaat's work, however, does not cover Example 3 for any other g. Theorem 3 is applicable to the general case of Example 3, by choosing $c_j = g^{-m}/(g-1)$ where m is a positive integer such that $c_j \leqq 1$. For these m, $k_j(n)$ has the same form as before and through (9) we have the joint distribution of $T_j = d_j-d_{j-1}$ if this difference is at least M defined by

$$g^{-M} \leqq g-1 < g^{-M+1}$$

and $T_j = 0$ otherwise. This result again appears to be new.

To several special cases of the Balkema-Oppenheim expansion as well, Theorem 3 is applicable. One can go through the extensive list of examples on p. 104–109 of Vervaat [12], for instance. I wish to point out that Theorem 3 actually is applicable in connection with any Balkema-Oppenheim expansion when $h(n)$ is monotonic as follows. Consider an arbitrary Balkema-Oppenheim algorithm and apply this in (1) and (2) for $j = 1$. Take a sequence $0 < c_j \leqq 1$, and define $\alpha_2(n)$ by (8) as follows. We choose a monotonic function $k(n)$ (in most cases $h(n)$ itself is possible), so that the right hand side of (8) should define a monotonic function $\alpha_2(.)$ at values of $k(n)$. For any integer m not taken by $k(n)$, $\alpha_2(m)$ is defined arbitrarily. We now complete the definition of the algorithm by taking $\gamma_2(n)$ corresponding to a Balkema-Oppenheim algorithm with $k(n)$ as the new map h in the second step. We now proceed to define each successive step in this same manner. This results in an algorithm each step of which is a Balkema-Oppenheim algorithm but possibly with varying α's and h's. Our results are then in principle applicable to these cases.

REFERENCES

[1] J. Galambos: The ergodic properties of the denominators in the Oppenheim expansion of real numbers into infinite series of rationals. *Quart. J. Math. Oxford Ser., 21* (1970) 177–191.

[2] J. Galambos: A generalization of a theorem of Borel concerning the distribution of digits in dyadic expansions. *Amer. Math. Monthly, 78* (1971) 774–779.

[3] J. Galambos: On a model for a fair distribution of gifts. *J. Appl. Probability, 8* (1971) 681–690.

[4] J. Galambos: Some remarks on the Lüroth expansion. *Czechosl. Math. J.*, *22* (1972) 266–271.

[5] J. Galambos: Probabilistic theorems concerning expansions of real numbers. *Periodica Math. Hungar.*, *3* (1973) 101–113.

[6] J. Galambos: Further ergodic results on the Oppenheim series. *Quart. J. Math. Oxford Ser.*, *25* (1974) (to appear).

[7] H. Jager and C. de Vroedt: Lüroth series and their ergodic properties. *Proc. Nederl. Akad. Wet, Ser. A*, *72* (1969) 31–42.

[8] A. Oppenheim: Representations of real numbers by series of reciprocals of odd integers. *Acta Arith.*, *18* (1971) 115–124.

[9] A. Oppenheim: The representation of real numbers by infinite series of rationals. *Acta Arith.*, *21* (1972) 391–398.

[10] T. Šalát: Zur metrische Theorie der Lürothschen Entwicklungen der reellen Zahlen. *Czechosl. Math. J.*, *18* (1968) 489–522.

[11] F. Schweiger: Metrische Sätze über Oppenheimentwicklungen. *J. Reine Angew. Math.*, *254* (1972) 152–159.

[12] W. Vervaat: Success epochs in Bernoulli trials with applications in number theory. *Math. Centre Tracts, Vol. 42*, (1972) Amsterdam.

(Oblatum: 23–XI–1972)

Department of Mathematics
Temple University
Philadelphia
Pa. 19122, U.S.A.