COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 27, nº 1 (1973), p. 49-61

<http://www.numdam.org/item?id=CM_1973__27_1_49_0>

COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 1, 1973, pag. 49–61
Noordhoff International Publishing
Printed in the Netherlands

ON PINCHED MANIFOLDS WITH FUNDAMENTAL GROUP Z_2

Karsten Grove and Hermann Karcher [1]

Introduction

A Riemannian manifold M of dimension $n \geqq 2$ is said to be δ-pinched if the sectional curvature of M, $K : G_2(TM) \to \mathrm{R}$ takes values only in the interval $[\delta \cdot A, A] \subset \mathrm{R}_+$ for some $A > 0$ and $\delta > 0$. In this paper we shall study compact δ-pinched manifolds M with $\pi_1(M) = Z_2$.

It is well known that M is homeomorphic to S^n if M is simply-connected and $\delta > \frac{1}{4}$, see Klingenberg [10], M is even diffeomorphic to S^n when $\delta > 0{,}80$, see Ruh [13] and Shiohama, Sugimoto and Karcher [15]. From this we have especially that any δ-pinched manifold, M with $\pi_1(M) = Z_2$ is homotopy equivalent to the real projective space, RP^n when $\delta > \frac{1}{4}$. However within the class of homotopy projective spaces there are several manifolds of different topological and differentiable type, see e.g. Lopéz de Medrano [12].

The main theorem of this paper is the following

THEOREM: *If M is a connected, complete δ-pinched Riemannian manifold with $\pi_1(M) = Z_2$ and $\delta \geqq 0{,}70$, then M^n is diffeomorphic to the real projective space*, $\mathrm{R}P^n$.

Since the fundamental group of an even dimensional compact manifold with positive curvature by Synge's theorem is either $\{0\}$ or Z_2 our theorem gives a classification of δ-pinched even dimensional manifolds with $\delta > 0{,}80$.

The idea of the proof is first to desuspend the involution on the covering space, $\tilde{M} \equiv S^n$ in such a way that we get a specific diffeomorphism from S^{n-1} to the desuspended submanifold of $\tilde{M}$, – as in the classical proof of the sphere theorem, Section 1. By restricting δ further we obtain, using the diffeotopy theorem of [15], that the induced involution on S^{n-1} is conjugate to the antipodal map of S^{n-1} and that the conjugation diffeomorphism is diffeotopic to the identity map of S^{n-1}, thereby giving us the desired result.

[1] This work was done under the program 'Sonderforschungsbereich Theoretische Mathematik' (SFB 40) at the University of Bonn.

In Section 3 we generalize the diffeotopy theorem of [15] and the involution conjugation theorem to arbitrary manifolds.

We like to mention that the problem was originally brought up by Shiohama in a discussion with one of the authors. He has independently obtained a similar solution.

1. Curvature and desuspension of the involution

Let us assume now and for the rest of the paper that M is a connected, complete, n-dimensional δ-pinched Riemannian manifold with $\delta > \frac{1}{4}$ and $\pi_1(M) = Z_2$. Let us also w.l.o.g. assume that the metric is normalized i.e. $\delta \leqq K_\sigma \leqq 1$ for all two-planes, $\sigma \in G_2(TM)$.

From the sphere theorem we have that the universal covering space, $\tilde{M}$ of M is homeomorphic to S^n. Let $T : \tilde{M} \to \tilde{M}$ be the action of the non-trivial element in $\pi_1(M)$ on $\tilde{M}$. Then T is a fixed point free isometry on $\tilde{M}$ with $T^2 = 1_{\tilde{M}}$ and $M \equiv \tilde{M}/T(x) = x$.

For an arbitrary $p \in \tilde{M}$ put $E(p) = \{x \in \tilde{M} | \mathrm{d}(p, x) = \mathrm{d}(T(p), x)\}$ where $\mathrm{d} : \tilde{M} \times \tilde{M} \to \boldsymbol{R}$ is the distance function on $\tilde{M}$. It is obvious that $E(p)$ is an invariant subset of $\tilde{M}$ under T since T is an isometry and $T^2 = 1_{\tilde{M}}$. If now $E(p)$ is a sphere we have desuspended $T : \tilde{M} \to \tilde{M}$. First we note that there are at least two shortest geodesics from p to $T(p)$ since the midpoint of a unique segment would be a fixed point for T. In particular $T(p)$ is in the cut-locus $C(p)$ of p for any $p \in \tilde{M}$. Therefore since $\delta > \frac{1}{4}$ we get from the estimate on the cut-locus distance $\mathrm{d}(p, C(p))$, see Klingenberg [10], that $\mathrm{d}(p, T(p)) \geqq \pi$ for all $p \in \tilde{M}$. For any point $x \in \tilde{M}$ we then have using Toponogov's triangle comparison theorem, see e.g. Gromoll, Klingenberg, Meyer [5], p. 184, that $\mathrm{d}(x, p) + \mathrm{d}(x, T(p)) \leqq 2\pi \cdot \delta^{-\frac{1}{2}} - \pi$ for all $p \in \tilde{M}$. This together with the cut-locus estimate again tells us that $E(p)$ is a T-invariant submanifold of $\tilde{M}$ diffeomorphic to S^{n-1} when $\delta > \frac{4}{9}$ i.e. $T : \tilde{M} \to \tilde{M}$ desuspends when $\delta > \frac{4}{9}$. However we shall now see that by choosing $p_0 \in \tilde{M}$ more carefully, $E(p_0)$ is a sphere even when $\delta > \frac{1}{4}$.

The displacement function for T, $\mathrm{f}_T : \tilde{M} \to \boldsymbol{R}$ defined by $\mathrm{f}_T(p) = \mathrm{d}(p, T(p))$ for all $p \in \tilde{M}$ is continuous. For any minimum point $p_0 \in \tilde{M}$ of f_T we have that a minimizing geodesic from p_0 to $T(p_0)$ and its T-image determine a simple closed T-invariant geodesic (for a general theory of isometry- invariant geodesics see Grove [6], [7]). Let $\gamma : [0, 2\mathrm{d}(p_0, T(p_0))] \to \tilde{M}$ be such a geodesic i.e. $\gamma(0) = p_0$, $\gamma(\mathrm{d}(p_0, T(p_0))) = T(p_0)$ and $\gamma[\mathrm{d}(p_0, T(p_0)), 2\mathrm{d}(p_0, T(p_0))] = T(\gamma[0, \mathrm{d}(p_0, T(p_0))])$. Let Γ denote the image of γ in $\tilde{M}$.

For $p_0 \in \tilde{M}$ a minimum point of f_T we have,

LEMMA (1.1): *For any* $x \in \tilde{M}$, $\mathrm{d}(x, p_0) \leqq \frac{1}{2}\pi\delta^{-\frac{1}{2}}$ *or* $\mathrm{d}(x, T(p_0)) \leqq \frac{1}{2}\pi\delta^{-\frac{1}{2}}$.

PROOF: Since $d(p_0, T(p_0)) \leqq \text{diam}\,(\tilde{M}) \leqq \pi \cdot \delta^{-\frac{1}{2}}$ by the theorem of Myers, see G.K.M. [5] p. 212, the claim is obvious for $x \in \Gamma$, so assume that $x \notin \Gamma$. Let $q \in \Gamma$ be such that $d(x, \Gamma) = d(x, q)$, it then follows using the second variation formula of Berger [1], that $d(x, q) \leqq \frac{1}{2}\pi \cdot \delta^{-\frac{1}{2}}$. Assume $d(p_0, q) \leqq \frac{1}{2}\pi\delta^{-\frac{1}{2}}$ (otherwise $d(T(p_0), q) \leqq \frac{1}{2}\pi\delta^{-\frac{1}{2}}$) and use 'Toponogov' on a triangle with vertices (p_0, q, x) to get $d(x, p_0) \leqq \frac{1}{2} \cdot \pi \cdot \delta^{-\frac{1}{2}}$ (or $d(x, T(p_0)) \leqq \frac{1}{2}\pi\delta^{-\frac{1}{2}}$); note $\sphericalangle\, p_0 q x = 90°$.

REMARK: Lemma 1.1 holds also for a maximum point of f_T, Shiohama [14], but the present proof is simpler.

From Lemma 1.1 and the cut-locus estimate we have that the distance functions, $\phi_1 = d(p_0, \cdot)$ and $\phi_2 = d(T(p_0), \cdot)$ are differentiable in a neighbourhood of $E(p_0) = \Phi^{-1}(0)$ with $\Phi = \phi_1 - \phi_2$. Since $\nabla\phi_1(x)$ is the velocity vector of the unique minimizing normal geodesic from p_0 to x and similar for $T(p_0)$ it follows that $\nabla\Phi_{|E(p_0)} \neq 0$ so $E(p_0)$ is a submanifold of $\tilde{M}$. Furthermore each normal geodesic from p_0 of length less that π intersects $E(p_0)$ transversely exactly once, thus $E(p_0)$ is diffeomorphic to S^{n-1} i.e. we have proved

THEOREM (1.2): *A fixed point free isometric involution on a δ-pinched sphere ($\delta > \frac{1}{4}$) desuspends.*

REMARK: For any fixed point free involution T on a homotopy-sphere Σ^n there is an associated invariant, $\sigma(T, \Sigma^n)$ the Browder-Livesay invariant, such that for $n \geqq 6$ $\sigma(T, \Sigma^n) = 0$ if and only if $T : \Sigma^n \to \Sigma^n$ desuspends, see e.g. Lopez de Medrano [12]. Here we have proved directly that $T : \tilde{M} \to \tilde{M}$ desuspends if $\delta > \frac{1}{4}$ and thus $\sigma(T, \tilde{M}) = 0$, $n \geqq 6$. From this we see that all the manifolds Σ^n/T with $\sigma(T, \Sigma^n) \neq 0$ do not admit a Riemannian metric whose sectional curvature is δ-pinched with $\delta > \frac{1}{4}$.

By means of the exponential map of $\tilde{M}$ at p_0 we get a diffeomorphism of the standard sphere S^{n-1} to $E(p_0)$ i.e. the involution $T_| : E(p_0) \to E(p_0)$ induces an involution $\bar{T}$ on S^{n-1} so that $E(p_0)/T(x) = x$ is diffeomorphic to $S^{n-1}/\bar{T}(u) = u$. If $\bar{T}$ is conjugate to the antipodal map of S^{n-1} we obtain that M is homeomorphic to RP^n, since any homeomorphism on the sphere extends to the disc. This is not true for diffeomorphisms, if $\Gamma_n = \text{Diff}\,(S^{n-1})/\text{Diff}\,(D^n) \neq 0$; so we need here that the conjugation map is isotopic to the identity map $1_{S^{n-1}}$. However we know that $\Gamma_1 = \cdots = \Gamma_6 = 0$, Cerf [3] and Kervaire and Milnor [9], so in dimensions less than seven we can proceed as follows.

PROPOSITION (1.3): *The image of the sectional curvature, $\tilde{K} : G_2(TE(p_0)) \to R$ of $E(p_0)$ lies in the interval*

$$\left[\delta - \frac{F(\delta)}{4G(\delta)},\ 1 + \frac{F(\delta)}{4G(\delta)}\right],$$

where

$F(\delta) = \delta^{\frac{1}{2}} \cot(\frac{1}{2}\pi\delta^{\frac{1}{2}}) - \cot(\frac{1}{2}\pi\delta^{\frac{1}{2}})$ *and* $G(\delta) = 1 - \cot^2(\frac{1}{2}\pi\delta^{\frac{1}{2}})$.

PROOF: Since $E(p_0) = \Phi^{-1}(0) = (\phi_1 - \phi_2)^{-1}(0)$ we have for the sectional curvature of a two-plane $\sigma \in G_2(TE(p_0))$ spaned by two orthonormal vectors $\{v, w\}$ at $x \in E(p_0)$,

$$\tilde{K}_\sigma = K_\sigma + \frac{1}{||\nabla\phi||_x^2} \det \begin{Bmatrix} \langle \nabla_v \nabla\phi, v\rangle & \langle \nabla_v \nabla\phi, w\rangle \\ \langle \nabla_v \nabla\phi, w\rangle & \langle \nabla_w \nabla\phi, w\rangle \end{Bmatrix},$$

(this is a special case of a formula due to P. Dombrowski, see e.g. G.K.M. [5] p. 109). For K_σ we have $\delta \leqq K_\sigma \leqq 1$ and the estimate for $||\nabla\Phi||_x^2$ is obtained by estimating the angle at x in a geodesic triangle with vertices $(p_0, x, T(p_0))$, quite similar to Gromoll [4] pp. 357, 364. The estimate for the other terms are also done similar to Gromoll [4] p. 364, except for a sign mistake there.

From proposition 1.3 we see that $E(p_0)$ is $\varepsilon(\delta)$-pinched with

$$\varepsilon(\delta) = \left(\delta - \frac{F(\delta)}{4G(\delta)}\right) \Big/ \left(1 + \frac{F(\delta)}{4G(\delta)}\right)$$

when δ is sufficiently big. Now $\varepsilon(\delta) \to 1$ as $\delta \to 1$ so there is a δ_2 such that $\varepsilon(\delta_2) = \delta_1 = \frac{1}{4}$, i.e. with M δ_2-pinched, $E(p_0)$ is $\frac{1}{4}$ - pinched and $T : E(p_0) \to E(p_0)$ desuspends. By induction we get a sequence with $\varepsilon(\delta_{k+1}) = \delta_k$ such that if M is δ-pinched with $\delta > \delta_k$ then $T : \tilde{M} \to \tilde{M}$ desuspends k times. This together with the remarks before Proposition 1.3 and the fact that any fixed point free involution is conjugate to the antipodal map in dimensions less than four (in the topological category proved by Livesay [11]), gives us

THEOREM (1.4): *Let* $M^n (n \leqq 6)$ *be a connected, compact* δ*-pinched manifold with* $\pi_1(M) = Z_2$. *Then* M *is diffeomorphic to* $\mathrm{R}P^n$ *if* $\delta > \delta_{n-3}$. *We note that*

$$\delta_1 = \tfrac{1}{4},\ \delta_2 \leqq 0,56 \text{ and } \delta_3 \leqq 0,69.$$

In the next paragraph we shall study the involution $\bar{T} : S^{n-1} \to S^{n-1}$ in detail. We shall see that independent of n, $\bar{T}$ is conjugate to the antipodal map $(-I) : S^{n-1} \to S^{n-1}$ and the obtained conjugation diffeomorphism isotopic to the identity map of S^{n-1} as soon as $\delta \geqq 0.70$. This will prove our main theorem.

2. Conjugation of the desuspended involution

Let $S_p^{n-1} \subset T_p\tilde{M}$ be the unit sphere in the tangent-space of $\tilde{M}$ at $p \in \tilde{M}$. The diffeomorphism

$$H_{po} : S_{po}^{n-1} \to E(p_0)$$

mentioned in Section 1 sends a unit vector $\mathrm{u} \in S_{po}^{n-1}$ to the unique intersection of $E(p_0)$ with the normal geodesic of length less than π determined by $\mathrm{u} \in S_{po}^{n-1}$. The involution

$$\bar{T} : S_{po}^{n-1} \to S_{po}^{n-1}$$

is then defined by

$$\bar{T} = H_{po}^{-1} \circ T_{|E(po)} \circ H_{po}.$$

Let us now give another expression for $\bar{T}$. Analogue to the map H_{po} we have the diffeomorphism

$$H_{T(po)} : S_{T(po)}^{n-1} \to E(p_0).$$

Put

$$h = H_{T(po)}^{-1} \circ H_{po}$$

and denote by $T_{*T(po)}$ the differential of T at $T(p_0)$. Then we have $\bar{T} = T_{*T(po)} \circ h$.

We want the involution $\bar{T}$ to be conjugate to the antipodal map

$$(-I) : S_{po}^{n-1} \to S_{po}^{n-1}$$

i.e. to find a diffeomorphism

$$\mu : S_{po}^{n-1} \to S_{po}^{n-1}$$

such that $(-I) \circ \mu = \mu \circ \bar{T}$.

Now for each $\mathrm{u} \in S_{po}^{n-1}$ let $\mu(\mathrm{u})$ be the mid point of the unique minimizing geodesic on S_{po}^{n-1} joining u and $(-I) \circ \bar{T}(\mathrm{u})$ i.e. $\mu(\mathrm{u}) = \exp_{\mathrm{u}} (\frac{1}{2} \exp_{\mathrm{u}}^{-1}(-\bar{T}(\mathrm{u})))$. Note that S_{po}^{n-1} carries the standard metric of constant curvature 1. Since $(-I) : S^{n-1} \to S^{n-1}$ is an isometry it follows that $(-I) \circ \mu = \mu \circ \bar{T}$. Thus to prove that $\bar{T}$ is conjugate to $(-I)$ we shall prove that μ is a diffeomorphism. Setting $\mu_t = \exp_{\mathrm{u}} (t \exp_{\mathrm{u}}^{-1} (-\bar{T}(\mathrm{u})))$ we have that $\mu = \mu_{\frac{1}{2}}$ and furthermore from the expression of μ_t we see, that we are in position to apply the diffeotopy theorem of [15] as soon as we can prove that $(-I) \circ \bar{T}$ is C^1-close to the identity map I of S_{po}^{n-1}. At the same time we will get that μ is isotopic to the identity map. The last property enables us to extend the Z_2-equivariant diffeomorphism

$$\mu \circ H_{po}^{-1} : E(p_0) \to S^{n-1}$$

to a Z_2-equivariant diffeomorphism from $\tilde{M}$ to S^n as follows: First observe that the differential T_* of T induces an involution of the normal bundle $\nu : NE(p_0) \to E(p_0)$ of the submanifold $E(p_0)$ in $\tilde{M}$. In a trivialization $NE(p_0) \cong E(p_0) \times R$ this involution is simply given by $(p, t) \to (T(p), -t)$. Similarly we have on the normal bundle of S^{n-1} in S^n, $NS^{n-1} \cong S^{n-1} \times R$, the involution $(u, t) \to (-u, -t)$. Extend the given

$$\mu \circ H_{p_0}^{-1} : E(p_0) \to S^{n-1}$$

first to a Z_2-equivariant bundle map

$$E(p_0) \times R \cong NE(p_0) \to NS^{n-1} \cong S^{n-1} \times R \text{ by } (p, t) \to (\mu \circ H_{p_0}^{-1}(p), t).$$

The restriction of this bundle map to suitably small metric ε-disc-bundles of $NE(p_0)$ and NS^{n-1} defines via the exponential maps of $\tilde{M}$ and S^n respectively a trivial extension of $\mu \circ H_{p_0}^{-1}$ to a Z_2-equivariant diffeomorphism F between tubular neighbourhoods of $E(p_0)$ in $\tilde{M}$ and of S^{n-1} in S^n. The boundary of each of these tubular neighbourhoods has two components each diffeomorphic to S^{n-1}. On $\tilde{M}$ these boundary spheres are transversally intersected by the shortest geodesics from p_0 and $T(p_0)$ for small ε (on S^n the geodesics from the poles n, s intersect orthogonally). Therefore we can compose the restriction of F to the 'upper' part of the tubular neighbourhood with exponential maps, to obtain a diffeomorphism $\exp_n^{-1} \circ F \circ \exp_{p_0}$ of 'ring' domains ($\cong S^{n-1} \times [0, \varepsilon]$). Now use the diffeotopy from $\mu : S^{n-1} \to S^{n-1}$ to the identity to extend this map over the hole in the 'ring' to a diffeomorphism of balls. Composition with $\exp_{p_0}^{-1}$ and $\exp_n$ gives the extension $\bar{F}$ of F over the 'upper hemissphere' of $\tilde{M}$, and $(-I) \circ \bar{F} \circ T$ extends this equivariantly to a diffeomorphism $\bar{F} : \tilde{M} \to S^n$, i.e. M is diffeomorphic to RP^n.

We prove next that $(-I) \circ \bar{T} : S^{n-1} \to S^{n-1}$ satisfies the assumptions of the diffeotopy theorem of [15, p. 16] as soon as δ is big enough.

Put $\beta = \sphericalangle(u, (-I) \circ \bar{T}(u))$ in $R^n = T_{p_0}\tilde{M}$ i.e. β is the distance on $S_{p_0}^{n-1}$ from u to $-\bar{T}(u)$. Furthermore for $A \in T_u S_{p_0}^{n-1}$ put $\Phi = \sphericalangle(A, ((-I) \circ \bar{T})_{*u}(A))$ in $R^n = T_{p_0}\tilde{M}$ after canonical identification.

We shall estimate β and Φ in terms of δ.

From the expression $\bar{T} = H_{p_0}^{-1} \circ T_{|E(p_0)} \circ H_{p_0}$ we see that $\alpha = \sphericalangle(u, \bar{T}(u))$ is the angle at p_0 in a geodesic triangle on $\tilde{M}$ with vertices $(p_0, H_{p_0}(u), T(H_{p_0}(u))$. Now since $d(H_{p_0}(u), T(H_{p_0}(u))) \geqq \pi$ as we have seen and $d(p_0, H_{p_0}(u)) = d(p_0, T(H_{p_0}(u))) \leqq \frac{1}{2}\pi\delta^{-\frac{1}{2}}$ by lemma 1.1 we get using Toponogov's triangle theorem, that $\cos \alpha \leqq \cos(\pi\delta^{\frac{1}{2}})$ i.e. $\alpha \geqq \pi\delta^{\frac{1}{2}}$ and therefore

LEMMA (2.1): $\beta = \sphericalangle(u, (-I) \circ \bar{T}(u)) \leqq \pi(1-\delta^{\frac{1}{2}}) < \pi/2$.

To estimate $\Phi = \measuredangle (A, ((-I)\circ \bar{T})_{*u}(A))$ we shall use the expression $\bar{T} = T_{*T(p_0)}\circ h$.

For $u \in S_p^{n-1} \subset T_p\tilde{M}$ denote by $\mathscr{T}_u : T_u(T_p\tilde{M}) \to T_p\tilde{M}$ the canonical identification of $T_u(T_p\tilde{M})$ and $T_p\tilde{M}$.

Let $A \in T_u S_{p_0}^{n-1}$ be a unit tangent vector of $S_{p_0}^{n-1}$ at u and let $h_*(A) \in T_{h(u)}S_{T(p_0)}^{n-1}$ be the differential of h applied to A. Put $X = \mathscr{T}_{h(u)}(h_*(A))$. Since now T_*: $T_{T(p_0)}\tilde{M} \to T_{p_0}\tilde{M}$ is a linear isometry we get that $T_*(X) = T_*(\mathscr{T}_{h(u)}(h_*(A))) = \mathscr{T}_{T_{*T(p_0)}(h(u))}((T_{*T(p_0)}\circ h)_{*u}(A)) = \mathscr{T}_{\bar{T}(u)}(\bar{T}_{*u}(A))$ i.e. $\Phi = \measuredangle(-T_*(X), \mathscr{T}_u(A)) = \measuredangle(-T_*(X), r)$, when we put $r = \mathscr{T}_u(A)$. With $w = h^{-1}(\|X\|^{-1}X)$ we also have $\Phi = \measuredangle(-T_*(h(w)), r) \leqq \beta+\gamma$, where $\beta = \measuredangle(-T_*(h(w)), w)$ and $\gamma = \measuredangle(w, r)$ i.e. $\Phi \leqq \pi(1-\delta^{\frac{1}{2}})+\gamma$ by Lemma 2.1.

Note that $\gamma = \measuredangle(w, r)$ only depends on h and not on T (except that h depends on T by the choice of p_0 and $T(p_0)$). γ is the angle at p_0 in the geodesic triangle on $\tilde{M}$ with vertices (p_0, q, m), where $q = \exp_{p_0}(\frac{1}{2}\pi r)$ and $m = H_{p_0}(w)$. Having eliminated T as remarked this problem is the same as a problem in [15] (including notation), so we can just take the formulas from there. Thus we have [15, 9.8] that

$$\gamma \leqq A(\delta)+B(\delta), \tag{2.2}$$

where

$$B(\delta) = \pi\cdot\delta^{-\frac{1}{2}}(\cos(\tfrac{1}{2}\pi\delta^{\frac{1}{2}})+\tfrac{1}{2}(1-\delta^{\frac{1}{2}})), \quad [15, 7.1]. \tag{2.3}$$

and $A(\delta)$ is given by

$$\cos(\delta^{\frac{1}{2}}A(\delta)) = 1-2\sin^2(\tfrac{1}{2}\pi\delta^{\frac{1}{2}})\cdot\{1-\sin^2(\tfrac{1}{2}\pi\delta^{\frac{1}{2}})\cdot\cos^2(9.6.1)\}. \tag{2.4}$$

This formula is the same as [15, 9.7.3] combined with [15, 9.6.3], also (9.6.1) refers to [15].

We could now use the estimates on $A(\delta)$ and $B(\delta)$ from [15] and thereby get a δ_0 such that $\delta > \delta_0$ implies that μ is a diffeomorphism isotopic to $1_{S^{n-1}}$ and thus M diffeomorphic to RP^n.

We shall here take the opportunity to prove a lemma on slowly rotating Jacobi fields which simplifies and improves estimate 9.6.1 of [15], and which might be of independent interest.

Lemma (2.5): *Let c be a normal geodesic in a Riemannian manifold whose sectional curvature K satisfies $0 \leqq \delta \leqq K \leqq \Delta$. Let J be a Jacobi-field normal to c and P a parallel field along c parallel to J (or J') at $t = 0$. With $\theta(t) = \measuredangle(J(t), P(t))$ we have,*

(1) If $J(0) = 0$:

$$\sin\,\theta(t) \leqq \left(\frac{\Delta}{\delta}\right)^{\frac{1}{2}}\left\{\frac{\sin(\delta^{\frac{1}{2}}\cdot t)}{\sin(\Delta^{\frac{1}{2}}t)}\right\} - \left(\frac{2\Delta}{\Delta+\delta}\right)^{\frac{1}{2}}\left\{\frac{\sin((\frac{1}{2}(\Delta+\delta))^{\frac{1}{2}}\cdot t)}{\sin(\Delta^{\frac{1}{2}}t)}\right\}$$

(2) If $J'(0) = 0$:

$$\sin\theta(t) \leqq \frac{\cos(\delta^{\frac{1}{2}}t) - \cos((\frac{1}{2}(\Delta+\delta))^{\frac{1}{2}}\cdot t)}{\cos(\Delta^{\frac{1}{2}}t)}$$

In (1) we have at $t = \frac{1}{2}\pi\delta^{-\frac{1}{2}}(\Delta = 1)$:

$\delta =$	0,65	0,70	0,75	0,80
$\theta \leqq$	10°,1	7°,6	5°,8	4°,8

PROOF: Let P be the parallel unit field along c with $P(0)$ parallel to $J'(0)$ (or $J(0)$) and let Q be the parallel unit field orthogonal to P and at a given t_0 satisfying $\|J(t_0)\|^{-1}J(t_0) = \cos\theta(t_0)\cdot P + \sin\theta(t_0)\cdot Q$. From the Jacobi-equation we have

$$\langle J, Q\rangle'' + \langle R(J, c)c, Q\rangle = 0.$$

Since J and Q are orthogonal to $\dot{c}$ we have (note the symmetry in J and Q), that

$$|\langle R(J, \dot{c})\dot{c}, Q\rangle - \tfrac{1}{2}(\Delta+\delta)\langle J, Q\rangle| \leqq \tfrac{1}{2}(\Delta-\delta)\|J\|.$$

Therefore

$$(*)\quad -\tfrac{1}{2}(\Delta-\delta)\|J\| \leqq \langle J, Q\rangle'' + \tfrac{1}{2}(\Delta+\delta)\langle J, Q\rangle \leqq \tfrac{1}{2}(\Delta-\delta)\|J\|.$$

The Rauch-Berger-comparison theorems, see G.K.M. [5] and Berger [2], give explicit bounds for $\|J\|$:

(a) if $J(0) = 0$, $\|J'(0)\|\cdot\Delta^{-\frac{1}{2}}\sin(\Delta^{\frac{1}{2}}t) \leqq \|J(t)\| \leqq \leqq \|J'(0)\|\delta^{-\frac{1}{2}}\sin(\delta^{\frac{1}{2}}t)$

(b) if $J'(0) = 0$, $\|J(0)\|\cos(\Delta^{\frac{1}{2}}t) \leqq \|J(t)\| \leqq \|J(0)\|\cdot\cos\delta^{\frac{1}{2}}t$.

Now, if we have a differential inequality, $a'' + k\cdot a \leqq f$ $(k > 0)$ and the corresponding equality $A'' + k\cdot A = f$, where the functions a and A have the same initial conditions $a(0) = A(0)$ and $a'(0) = A'(0)$, then $a(t) \leqq A(t)$ for $t \leqq \pi\cdot k^{-\frac{1}{2}}$. To prove this statement put $b = a - A$, hence $b'' + k\cdot b \leqq 0$, $b(0) = b'(0) = 0$. For $s(t) = \sin(k^{\frac{1}{2}}t)$ we have $s'' + ks = 0$, $s(0) = 0$ and $s'(0) = 0$. Therefore

$$\lim_{t\to 0}\frac{b(t)}{s(t)} = 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\frac{b}{s}(t) = \frac{s\cdot b' - bs'}{s^2}(t) = \int_0^t \frac{sb'' - bs''}{s^2} \leqq 0,$$

hence $b/s \leqq 0$ and thus $b \leqq 0$ for $t \leqq \pi\cdot k^{-\frac{1}{2}}$.

In the case $J(0) = 0$ we take $a = \langle J, Q\rangle$, hence $a(0) = 0$ $a'(0) = 0$ $(Q \perp P\|J'(0))$ and $f = \|J'(0)\|\frac{1}{2}(\Delta-\delta)\cdot\delta^{-\frac{1}{2}}\sin(\delta^{\frac{1}{2}}t)$. Using the above statement together with (a) and $(*)$ we get

$$(1')\quad a(t) \leqq A(t) = J'(0)\{\delta^{-\frac{1}{2}}\sin(\delta^{\frac{1}{2}}t) - (\tfrac{1}{2}(\Delta+\delta))^{-\frac{1}{2}}\cdot\sin((\tfrac{1}{2}(\Delta+\delta))^{\frac{1}{2}}t)\}.$$

In the case $J'(0) = 0$ take $a = \langle J, Q \rangle$, hence $a(0) = 0$ ($Q \perp P||J(0)$), $a'(0) = 0$ and $f = ||J(0)|| \cos(\delta^{\frac{1}{2}} t)$. As before using (b) and (*) we get

$$(2') \quad a(t) \leqq A(t) = ||J(0)|| \cdot (\cos(\delta^{\frac{1}{2}} t) - \cos([\tfrac{1}{2}(\Delta + \delta)]^{\frac{1}{2}} t)).$$

In both cases $(-a)$ satisfies the same differential inequality so the lemma is proved since

$$\sin \theta(t_0) = \frac{|\langle Q, J \rangle|}{||J||}(t_0) \leqq \frac{A(t_0)}{J(t_0)}.$$

REMARK: If the curvatures are not necessarily positive, then the trigonometric functions have to be replaced by the corresponding linear ones ($\delta = 0$) or hyperbolic ones ($\delta < 0$).

Instead of $\cos^2$ (9.6.1) in (2.4) we can now write $1 - \sin^2(\theta(\frac{1}{2}\pi\delta^{-\frac{1}{2}}))$ i.e. $A(\delta)$ is determined by

$$(2.6) \quad \cos(\delta^{\frac{1}{2}} A(\delta)) = 1 - 2\sin^2(\tfrac{1}{2}\pi\delta^{\frac{1}{2}}) \cdot \{1 - \sin^2(\tfrac{1}{2}\pi\delta^{\frac{1}{2}}) \cdot (1 - \sin^2(\theta(\tfrac{1}{2}\pi\delta^{-\frac{1}{2}})))\},$$

where θ is given by Lemma 2.5 with $\Delta = 1$.

Lemma 2.1, $\Phi \leqq \beta + \gamma$, (2.2), (2.3) and (2.6) gives us that $\beta \leqq 27°,6$ and $\Phi \leqq 133°,7$ when $\delta \geqq 0,70$ i.e. from the diffeotopy theorem [15, p. 16] we get that μ_t is a diffeomorphism for all $t \in [0, 1]$, especially $\mu_{\frac{1}{2}} = \mu$ is a diffeomorphism isotopic to the identity map of S^{n-1}. As mentioned earlier this proves our

MAIN THEOREM: *Let M be a connected, compact n-dimensional ($n \geqq 2$), δ-pinched Riemannian manifold with $\pi_1(M) = Z_2$. Then M is diffeomorphic to the real projective space RP^n, when $\delta \geqq 0,70$.*

REMARK: Our proof can only give diffeomorphism. This raises the question if a homeomorphism result can be obtained with a smaller δ. – As mentioned in the introduction our theorem together with the diffeentiable sphere-pinching theorem [13] and [15] gives a classification up to diffeomorphism of even dimensional δ-pinched manifolds with $\delta >$ 0,80. Note that the pinching-constant in the S^n-case is bigger than in the RP^n-case!

3. A diffeotopy theorem

In this paragraph we shall give another application of Lemma 2.5.

Let M be a connected, compact Riemannian manifold. For the sectional curvature, K of M we then have $\min K = \delta \leqq K \leqq \Delta = \max K$.

A diffeotopy between f and $g \in \text{Diff}(M)$ results in a diffeotopy between 1_M and $f^{-1} \circ g$. Therefore we restrict our attention to that situation.

Let $\mathrm{D}(p) = \mathrm{d}(p, C(p))$ be the cut-locus distance of $p \in M$. D is a continuous function. Note that if $\delta > 0$, if M is simply connected and if dim (M) even or $\delta/\Delta > \frac{1}{4}$ we have the bound $\mathrm{D}(p) \geqq \Delta^{-\frac{1}{2}} \cdot \pi$. Assume now that $\mathrm{d}(p, f(p)) < \mathrm{D}(p)$ for all $p \in M$, then

$$F : M \times \mathrm{I} \to M$$

defined by $F(p, t) = \exp_p (t \cdot \exp_p^{-1}(f(p)))$ is a well-defined homotopy with $F_0 = 1_M$ and $F_1 = f$. We shall give conditions in terms of curvature that this homotopy is a diffeotopy. The conditions will be such that fairly large balls in the C^1-metric of Diff (M) are contractible.

Let $A \in T_pM$ be a unit tangent vector at $p \in M$. Denote by $\tau_p : T_pM \to T_{f(p)}(M)$ the parallel translation along the unique minimizing geodesic $(f(p) \notin C(p))$ from p to $f(p)$. Then we have,

THEOREM (3.1): *Let $f \in$ Diff (M) and put $\beta(p) = \mathrm{d}(p, f(p))$. Then there is a number* $\mathrm{C}(\Delta, \delta, \beta)$, *which can be estimated explicitly, such that if*

(1) $\beta(p) < \mathrm{D}(p)$ *for all* $p \in \mathrm{M}$ *and further* $\beta < \Delta^{-\frac{1}{2}}\pi$(if $\Delta > 0$)

(2) $\sphericalangle\, (f_{*p}(A), \tau_p(A)) < C(\delta, \Delta, \beta)$ *for all* $A \in S_p \subset T_pM$.

then f is diffeotopic to 1_M.

PROOF: Since M is compact and we have already a homotopy F, it is sufficient to prove that each F_t has maximal rank at each point, since then $\{F_t\}$ is a family of differentiable covering maps and $F_0 = 1_M$ implies that each of the coverings F_t is a diffeomorphism.

From $\beta = \mathrm{d}(p, f(p)) < \mathrm{D}(p)$ we have that the shortest connection from p to $f(p)$ has no conjugate points. The reason for $\beta < \Delta^{-\frac{1}{2}}\pi$ if $\Delta > 0$ is that we are going to use comparison theorems.

Now for a unit tangent vector $A \in T_pM$ consider the geodesic, γ_A defined by $\gamma_A(s) = \exp_p\,(s \cdot A)$. Clearly $\gamma_A'(0) = A$ and $(F_t)_{*p}(A) = \mathrm{d}/\mathrm{d}s(F_t \circ \gamma_A)_{|s=0}$. Since all the curves c_s defined by $c_s(t) = F(\gamma_A(s), t)$ are geodesics it follows that $J(t) = (F_t)_{*p}(A)$ is a Jacoby field along c_0 with $J(0) = A$ and $J(1) = f_{*p}(A)$. Thus F_t has maximal rank at p if none of these Jacoby fields ever vanishes. Assuming the contrary i.e. that there is a t_0 and an A such that $J(t_0) = 0$, we shall derive an estimate $\sphericalangle\, (f_{*p}(A), \tau_p(A)) \geqq \mathrm{C}(\delta, \Delta, \beta)$ and thereby finish the proof of the theorem.

We write J as sum of its tangential and normal component $J = J_{\mathrm{T}} + J_{\mathrm{N}}$. Then

$$J_T(t) = a(t-t_0)\dot{c}_0(t), \ ||\dot{c}_0|| = \beta$$

and

$$||J'_N(t_0)|| \cdot \Delta^{-\frac{1}{2}}\beta^{-1} \cdot \sin(\Delta^{\frac{1}{2}}\beta(t-t_0)) \leqq ||J_N(t)|| \leqq ||J'_N(t_0)|| \cdot \delta^{-\frac{1}{2}}\beta^{-1} \sin(\delta^{\frac{1}{2}}\beta(t-t_0))$$

(If e.g. $\delta < 0$, then the last expression changes to $(-\delta)^{-\frac{1}{2}}\beta \sinh((-\delta)^{\frac{1}{2}} \beta(t-t_0))$.)

We have $A = J_T(0)+J_N(0)$, $f_{*p}(A) = J_T(1)+J_N(1)$ and

$$\tau_p(A) = -a \cdot t_0 \cdot \dot{c}_0(1)+Q(1),$$

where Q is a parallel field with $Q(0) = J_N(0)$.

Define angles $\alpha_i = \sphericalangle (J(i), \dot{c}(i))$ i.e. $\tan \alpha_i = ||J_T(i)||^{-1} \cdot ||J_N(i)||$, $i = 0, 1$ and put

$v = ||J_T(1)||^{-1} J_T(1) \cos \alpha_1 - ||Q(1)||^{-1} Q(1) \cdot \sin \alpha_1$ (hence $\sphericalangle (f_{*p}(A), \dot{c}(1)) = \sphericalangle (v, \dot{c}(1)) = \alpha_1$). Clearly $\sphericalangle (v, f_{*p}(A)) \leqq \sphericalangle (-Q(1), J_N(1)) \leqq \theta(t_0 \cdot \beta)+\theta((1-t_0)\beta)$ by lemma 2.5, so we get

$$(*) \quad \sphericalangle (\tau_p A, f_{*p}(A)) \geqq \sphericalangle (\tau_p A, v) - \sphericalangle (v, f_{*p}(A)) \geqq \pi - \sphericalangle (\tau_p A, -v) - (\theta(t_0 \cdot \beta)+\theta((1-t_0)\cdot\beta).$$

Finally

$$\begin{aligned} |\tan(\sphericalangle (\tau_p A, -v))| &= |\tan(\alpha_1-\alpha_0)| = |(1+\tan\alpha_1 \tan\alpha_0)^{-1} \cdot (\tan\alpha_1 - \tan\alpha_0)| \\ &\leqq \tfrac{1}{2}|(\tan\alpha_0^{-1}\tan\alpha_1)^{\frac{1}{2}} - (\tan\alpha_1^{-1}\tan\alpha_0)^{\frac{1}{2}}| \\ &\leqq \max_{0\leqq t\leqq 1} \frac{1}{2}\left| \left\{\frac{\sin(\delta^{\frac{1}{2}}\beta(1-t))}{\delta^{\frac{1}{2}}\cdot\beta(1-t)} \cdot \frac{\Delta^{\frac{1}{2}}\beta t}{\sin(\Delta^{\frac{1}{2}}\beta t)}\right\}^{\frac{1}{2}} - \left\{\frac{\sin(\Delta^{\frac{1}{2}}\beta t)}{\Delta^{\frac{1}{2}}\beta t} \cdot \frac{\delta^{\frac{1}{2}}\beta(1-t)}{\sin(\delta^{\frac{1}{2}}\beta(1-t))}\right\}^{\frac{1}{2}} \right| \\ &\leqq \frac{1}{2}\left| \left\{\frac{\delta^{\frac{1}{2}}\beta}{\sin(\delta^{\frac{1}{2}}\beta)}\right\}^{\frac{1}{2}} - \left\{\frac{\sin(\delta^{\frac{1}{2}}\beta)}{\delta^{\frac{1}{2}}\beta}\right\}^{\frac{1}{2}} \right| \end{aligned}$$

supplies an explicit upper bound for $\sphericalangle (\tau_p A, -v)$. If we use this and lemma 2.5 in (*) we get

$$\sphericalangle (f_{*p}(A), \tau_p(A) \geqq C(\delta, \Delta, \beta).$$

This completes the proof.

COROLLARY (3.2): *Let $T : M \to M$ be an arbitrary isometric involution on M and let $I : M \to M$ be another involution. If $T^{-1} \circ I$ satisfies the assumptions on f in Theorem* 3.1, *then I is conjugate to T.*

PROOF: $\mu(p) = \exp_p(\frac{1}{2}\exp_p^{-1}(T^{-1}\circ I(p)))$ conjugates I and T.

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(Oblatum 14–XII–1972)

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Added in proof

The following modifications improve the main theorem to:
M is diffeomorphic to $\mathbb{R}P^n$ *if* $\delta \geqq 0.6$ *and homeomorphic to* $\mathbb{R}P^n$ *if* $\delta \geqq 5.56$

For an arbitrary $n \in E(p_0)$ consider the isometry $\mathcal{O} : T_{p_0}\tilde{M} \to T_{p_0}\tilde{M}$ defined as the composition of the following maps:

s = reflection at the hyperplane perpendicular to the initial direction u of the segment $p_0 n$,
τ_1 = parallel translation along $p_0 n$,
D_1 = simple rotation in $T_n\tilde{M}$ of the tangent vector of $p_0 n$ to the tangent vector of $nT(p_0)$,
τ_2 = parallel translation along $nT(p_0)$,
$-T_{*T(p_0)}$ = $(-\text{ id}) \circ$ (differential of the involution),

D_2 = simple rotation in $T_{p_0}\tilde{M}$ of u to $-\overline{T}(\mathrm{u})$, this corresponds – after canonical identification – to the Levi-Civita translation along the great circle arc from u to $-T(\mathrm{u})$ on the unit sphere of $T_{p_0}\tilde{M}$.

Although the description of the isometry $\mathcal{O}$ looks complicated, it is easy to prove with the arguments in this paper that:

(a) $\forall\ A \in T_{p_0}\tilde{M}\ \sphericalangle\ (A, \mathcal{O}A) \leqq \pi\delta^{-\frac{1}{2}} \cos\left(\frac{1}{2}\pi\delta^{\frac{1}{2}}\right)+\pi(\delta^{-\frac{1}{2}}-1)\ \sin\left(\frac{1}{2}\pi\delta^{\frac{1}{2}}\right)$

If A is a tangent vector at u of the unit sphere in $T_{p_0}M$ then we have also with the function Θ of Lemma 2.5 (1):

(b) $\sphericalangle\ (h_*(A), \tau^2 \circ D_1 \circ \tau_1 \circ s(A)) \leqq 2\Theta\left(\frac{1}{2}\pi\delta^{-\frac{1}{2}}\right)$.

(a) and (b) provide a simpler and slightly better estimate of Φ than (2.2), (2.3) and (2.6). Secondly we use (3.17) of [8] to improve the diffeotopi theorem of [15] and get with (a) and (b) the result: $\mu_{\frac{1}{2}}$ (see § 2) is a diffeomorphism if $\delta \geqq 0.56$ and each μ_t is a diffeomorphism if $\delta \geqq 0.6$.