ERIK ELLENTUCK

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UNCOUNTABLE SUBORDERINGS OF THE ISOLS

by

Erik Ellentuck*

1. Introduction

Let $(P, \leqq)$ be a partially ordered set. We say that $(P, \leqq)$ can be *embedded* in the isols (Λ) if there is a one-one function $f: P \to \Lambda$ such that $x \leqq y$ if and only if $f(x) \leqq f(y)$ for all $x, y \in P$ (by $f(x) \leqq f(y)$ we mean $(\exists z \in \Lambda) f(x)+z = f(y)$). In [3] we showed

THEOREM: *If $(P, \leqq)$ is a partially ordered set then the following conditions are equivalent.* (i) *P is countable,* (ii) *$(P, \leqq)$ can be embedded in the cosimple isols (Λ_Z),* (iii) *$(P, \leqq)$ can be embedded in the cosimple regressive isols (Λ_{RZ}).*

In this paper we shall try to extend the preceding result and characterize those uncountable partial orderings which can be embedded in the isols and in the regressive isols, respectively. We shall find that these classes are quite different. In the regressive case we use methods that are almost entirely from isol theory (cf. [1]). However, for the case of embeddings in the isols we rely on [6]. Since the material in [6] is also contained in [7], and the latter is easily accessible, [7] will be used as our reference source.

2. Embeddings

Let ω be the non-negative integers, and j the pairing with k, l as first, second inverse. For any set α we let $\times^n \alpha$ be the n-fold direct power of α, $|\alpha|$ = the cardinality of α, and $\langle \alpha \rangle$ the recursive equivalence type of α. $\aleph$ will be the cardinality of the continuum. We say that a partially ordered set $(P, \leqq)$ is *regular* if (i) every $x \in P$ has at most countable many predecessors, and the set S consisting of all those $x \in P$ having uncountable many successors satisfies (ii) S is linearly ordered by $\leqq$ and each $x \in S$ has finitely many predecessors, and (iii) if $x \in S$, $y \in P$-S then $x \leqq y$.

THEOREM 1: *A partially ordered set $(P, \leqq)$ can be embedded in the regressive isols if and only if it is regular and has cardinality at most $\aleph$.*

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PROOF: (a) Suppose $(P, \leqq)$ can be embedded in Λ_R. W.l.g. we can assume that $P \subseteq \Lambda_R$. Clearly $|P| \leqq \aleph$. In [1] a unique degree of unsolvability is associated with each $x \in \Lambda_R$. It is the degree of any retraceable set contained in x and it is least among the degrees of all sets contained in x. We denote this degree by $\Delta(x)$. By P17(c) of [1] if $x, y \in \Lambda_R$, x is infinite, and $x \leqq y$ then $\Delta(x) = \Delta(y)$. Now any isol has at most countably many predecessors, afortiori the same is true of $(P, \leqq)$. Let S be as in the definition of regular. Then by P17(c) each $x \in S$ is finite, i.e. $S \subseteq \omega \subseteq \Lambda_R$. The elements of S must be linearly ordered by $\leqq$, if $x \in S$, $y \in P\text{-}S$ then $x \leqq y$, and each $x \in S$ has only finitely many predecessors. Thus $(P, \leqq)$ is regular.

(b) Let $(P, \leqq)$ be a regular partially ordered set having cardinality at most $\aleph$. Say that a function $f: \omega \to \omega$ *dominates* partial recursive functions if for every partial recursive function g we can find an m such that $n \notin \delta g$ or $g(n) < f(n)$, for all $n > m$. By an unpublished result of S. Tennenbaum a degree $\boldsymbol{d}$ contains a function which dominates partial recursive functions if and only if $\boldsymbol{d} \geqq \mathbf{0}'$. Now let f be any such function. W.l.g. we may assume that $f(0) \neq 0$. In [4] it is shown that if we define $t(0) = 0$ and $t(n+1) = j(t(n), f(n))$ then t is a T-retraceable function retraced by k. By theorem 4, chapter 3 of [7] there is a set of $\aleph$ pairwise incomparable degrees, each greater that $\mathbf{0}'$. Let D be a set of $\aleph$ functions each dominating partial recursive functions and having pairwise incomparable degrees. For each $f \in D$ let t_f be the T-retraceable function constructed above and let $\tau_f = \rho t_f$. It is not hard to set that f, t_f and τ_f have the same degree. Let $S \subseteq P$ be as in the definition of regular. S contains at most countably many elements and is linearly ordered by $\leqq$ as an initial segment (perhaps improper) of ω. We will embed S as this segment. Since $x \in S$, $y \in P\text{-}S$ implies $x \leqq y$ it will suffice to embed $P\text{-}S$ in $\Lambda_R - \omega$. For each $A \subseteq P$ let $A' = \{y \in P\text{-}S \mid (\exists x \in A)(y \leqq x \vee x \leqq y)\}$ and let $A^0 = A$, $A^{n+1} = (A^n)'$ and $A^\omega = \cup_{n<\omega} A^n$. Define an equivalence relation $\sim$ on $P\text{-}S$ by $x \sim y$ if $x \in \{y\}^\omega$ and $y \in \{x\}^\omega$. We let E be the set of all equivalence classes. E is a partition of $P\text{-}S$ into at most $\aleph$ sets each containing at most countably many elements. Elements of $P\text{-}S$ belonging to different equivalence classes are incomparable with one another. Thus it will suffice to embed an equivalence class in a set of isols having a fixed degree, different classes corresponding to different degrees. Let $u \in E$ and $f \in D$. We will embed u in a class of regressive isols each having the degree of f. Let t_f and τ_f be as above (in the following drop the subscript f). By a result of [5] there is a reflexive recursive $R \subseteq \times{}^2\omega$ which partially orders ω in such a way that every countable partial ordering can be embedded in (ω, R). Since $(u, \leqq)$ can be embedded in (ω, R) it will suffice to embed (ω, R) in Λ_R. Recall that τ is retraced by k and let $k^*(x)$ be the least n

such that $k^n(x) = k^{n+1}(x)$ where the superscript denotes iteration. Let $\mu^{(n)} = \{x|k^*(x) = n\}$ and for any set $\alpha \subseteq \omega$ let $\alpha^{(n)} = \alpha \cap \mu^{(n)}$. Our embedding will be defined by $h(n) = \cup \{\tau^{(m)}|mRn\}$. To show that $h(n)$ is retraceable note that $h(n)$ is separated from $\tau - h(n)$ by the recursive set $\cup \{\mu^{(m)}|mRn\}$. By P5 of [1], $h(n)$ is a retraceable set and by P17(c) of [1] it has the same degree as τ. Since $h(n) \subseteq \tau$, it is clear that $h(n)$ is immune and consequently $\langle h(n)\rangle \in \Lambda_R$. If xRy then $h(x) \subseteq h(y)$ and $h(x)$ is separated from $h(y) - h(x)$ by the recursive set $\cup \{\mu^{(n)}|nRx\}$. Therefore $\langle h(x)\rangle \leqq \langle h(y)\rangle$ in the isol ordering. By lemma 4 of [3] if α, $\beta \subseteq \tau$ are infinite recursively equivalent sets, then $\alpha \cap \beta \neq \emptyset$ (T-retraceability implies this). If $\sim xRy$ but $\langle h(x)\rangle \leqq \langle h(y)\rangle$ then for some one-one partial recursive function p, $\tau^{(x)} \subseteq \delta p$ and p maps $\tau^{(x)}$ into $\cup \{\tau^{(n)}|\ n \neq x\}$. But this contradicts lemma 4. Thus if $\sim xRy$ then $\sim \langle h(x)\rangle \leqq \langle h(y)\rangle$ and h embeds (ω, R) into Λ_R. q.e.d.

A set A of functions is *independent* if no function in A is recursive in the function theoretic join of finitely many other functions in A. By theorem 4, chapter 3 of [7] we may actually take the set D used in the proof of theorem 1 to be an independent set of functions.

LEMMA 1: *If* $x_i \in \Lambda_R$ *for* $i \leqq n$ *and* $x_0 \leqq x_1 + \cdots + x_n$ *then* $\Delta(x_0) \leqq \Delta(x_1) \cup \cdots \cup \Delta(x_n)$.

PROOF: Let $\tau_i \in x_i$ for $i \leqq n$ where τ_i is retraceable for $1 \leqq i \leqq n$. W.l.g. we may assume that $\tau_i \subseteq \alpha_i$ for $1 \leqq i \leqq n$ where the α_i are recursive and form a partition of ω. Let $\tau = \tau_1 \cup \cdots \cup \tau_n$. We also assume that $\tau_0 \subseteq \tau$ and there are disjoint r.e. sets γ, δ such that $\tau_0 \subseteq \gamma$ and $\tau - \tau_0 \subseteq \delta$. Now consider any $1 \leqq i \leqq n$. $\tau_0 \cap \alpha_i \subseteq \tau_i$ and, $\tau_0 \cap \alpha_i$ and $\tau_i - (\tau_0 \cap \alpha_i)$ are separated by γ, δ. By P15 of [1], $\Delta(\tau_0 \cap \alpha_i) \leqq \Delta(\tau_i)$ and by another application of P15, $\Delta(\tau_0) \leqq \Delta(\tau_1) \cup \cdots \cup \Delta(\tau_n)$. But $\Delta(x_0) \leqq \Delta(\tau_0)$(recall that $\Delta(x_0)$ is the least degree of a set in x_0) and $\Delta(x_i) = \Delta(\tau_i)$ for $1 \leqq i \leqq n$. q.e.d.

THEOREM 2: *If* $(P, \leqq)$ *is a partially ordered set of cardinality at most* $\aleph$ *and every* $x \in P$ *has at most finitely many predecessors then* $(P, \leqq)$ *can be embedded in the isols.*

PROOF. Let D be a set of $\aleph$ independent functions and let $E = \{\langle\tau_f\rangle| f \in D\}$. Let $g : P \to E$ be a one-one function and for $y \in P$ let $h(y) = \sum\{g(x)|x \in P \wedge x \leqq y\}$ where the sum is taken in the isols. If $x, y \in P$ and $x \leqq y$ it is clear that $h(x) \leqq h(y)$. If $\sim x \leqq y$ and $h(x) \leqq h(y)$ then $g(x) \leqq h(y)$ which contradicts lemma 1. q.e.d.

In particular, if we take A to be a set of cardinality $\aleph$ and P to be the set of all finite subsets of A, then $(P, \subseteq)$ can be embedded in the isols by theorem 2 but not in the regressive isols because every element of P has uncountably many successors. Our next goal is to find necessary and sufficient conditions for a partial ordering to be embeddable in the isols. We shall only be able to do this under the assumption of the continuum hypothesis (CH). Without this assumption our conditions will only be sufficient and will not include the embedding of theorem 2.

First, we state theorem 1, chapter 3 of [7]. (*) Let $(P, \leqq)$ be a partially ordered set, and let M and N be disjoint subsets of P such that M has cardinality less than $\aleph$, N is countable, and no member of N is less than any member of M. For each $n \in N$ the set $M_n = \{m \in M | m < n\}$ is countable and any two members of M_n have an upper bound in M_n. Let A be a set of degrees order isomorphic to M. Then there exists a set B of degrees such that $A \cup B$ is order isomorphic to $M \cup N$ by means of an extension of the given order isomorphism between A and M.

THEOREM 3: *The same as* (*) *except* A *and* B *are sets of isols, and we are given that each element of* A *has a representative of degree* $\geqq \mathbf{0}'$ *whose degree ordering is isomorphic to the corresponding isol ordering, and we conclude that the same is true of* $A \cup B$.

PROOF. Our argument is a direct modification of the proof of (*) which the reader is expected to be familiar with. First we state theorem 59 of [2]. If $\langle\alpha\rangle \leqq \langle\beta\rangle$ and $\mathbf{0}' \leqq \Delta(\beta)$ then $\Delta(\alpha) \leqq \Delta(\beta)$. Now let $M = \{m_t | t \in V\}$, $A = \{\langle\alpha_t\rangle | t \in V\}$, and $N = \{n_i | i < \omega\}$. Here we assume that $\mathbf{0}' \leqq \Delta(\alpha_t)$ and the degree ordering of the α_t is isomorphic to the isol ordering of the $\langle\alpha_t\rangle$. Also assume that the order isomorphism between A and M is such that if $s, t \in V$ then $\langle\alpha_s\rangle \leqq \langle\alpha_t\rangle$ if and only if $m_s \leqq m_t$. Let a_t be the characteristic function of α_t. Then in [7] a sequence of functipns $\{b_i | i < \omega\}$ is constructed so as to satisfy

(R1) b_j is recursive in b_i if $n_j \leqq n_i$,

(R2) a_t is recursive in b_i if $m_t \leqq n_i$,

(R3) b_k is not recursive in b_i if $\sim n_k \leqq n_i$,

(R4) a_t is not recursive in b_i if $\sim m_t \leqq n_i$,

(R5) b_i is not recursive in a_t if $t \in V$.

The construction of the b_i amounts to a proof of (*). We modify this construction by requiring that each b_i be a characteristic function, adding extra steps so as to insure $\mathbf{0}' \leqq \Delta(b_i)$, and if $\beta_i = \{n | b_i(n) = 1\}$ then β_i is immune. It is not hard to see that this does not interfere with the construction in [7]. Let $\{p_i | i < \omega\}$ be an enumeration of the primes in increasing order. (R1) is effected by finding a number r such that $b_j(x) = b_i(p_j^{r+x})$ for all x, i.e., $x \in \beta_j$ if and only if $p_j^{r+x} \in \beta_i$. By the obvious separation this gives $\langle\beta_j\rangle \leqq \langle\beta_i\rangle$. For each $n_i \in N$ let $M_{n_i} = \{m_{ij} | j < \omega\}$, cf. (*) for the definition of M_n, and let $\langle\alpha_{ij}\rangle$ be the corresponding element of A under the order isomorphism between A and M where α_{ij} is one of the α_t. Let a_{ij} be the characteristic function of α_{ij}. Then (R2) is effected by finding a number r such that $a_{ij}(x) = b_i(6p_j^{r+x})$ for all x, i.e., $x \in \alpha_{ij}$ if and only if $6p_j^{r+x} \in \beta_i$. Again by the obvious separation this gives $\langle\alpha_{ij}\rangle \leqq \langle\beta_i\rangle$. (R3) says that b_k is not recursive in b_i. Since $\mathbf{0}' \leqq \Delta(b_j)$ for all j we may use theorem 59 to conclude that $\sim \langle\beta_k\rangle \leqq \langle\beta_i\rangle$. (R4) and (R5) are treated in the same way. q.e.d.

The following definition is from [7]. A partially ordered set $(P, \leqq)$ is *completely normal* if there exists an ordinal α and a collection $\{B_\gamma | \gamma < \alpha\}$ of subsets of P such that $P = \cup \{B_\gamma | \gamma < \alpha\}$ and such that for each ordinal $\gamma < \alpha$: (1) $\cup \{B_\delta | \delta < \gamma\}$ has cardinality less than $\aleph$, (2) B_γ is at most countable and disjoint from $\cup \{B_\delta | \delta < \gamma\}$, (3) no member of B_γ is less than any member of $\cup \{B_\delta | \delta < \gamma\}$, (4) for each $n \in B_\gamma$ the set $L_n^\gamma = \{x | x < n \wedge x \in \cup \{B_\delta | \delta < \gamma\}\}$ is at most countable and every two elements of L_n^γ have an upper bound in L_n^γ.

A partially ordered set $(P, \leqq)$ is *normal* if there exists a completely normal partially ordered set $(Q, \leqq')$ in which $(P, \leqq)$ can be embedded. Notice that if $(P, \leqq)$ is normal that each $x \in P$ has at most countably many predecessors and $|P| \leqq \aleph$.

THEOREM 4: *If* $(P, \leqq)$ *is a partially ordered set then normality is a sufficient condition for* $(P, \leqq)$ *to be embeddable in the isols. If* CH *then it is also a necessary condition. Moreover, it is a necessary condition only if* CH.

PROOF: (a) By iterated application of theorem 3 it follows that every completely normal partially ordered set can be embedded in the isols.

(b) We show that if CH then $(\Lambda, \leqq)$ is normal and consequently so must any $(P, \leqq)$ embeddable in $(\Lambda, \leqq)$. Lemma 2, chapter 3 of [7] says that if $(P, \leqq)$ is a partially ordered set of cardinality $\leqq \aleph$, then $(P, \leqq)$ is normal if and only if each $x \in P$ has at most countably many predecessors. But $(\Lambda, \leqq)$ certainly has the latter property.

(c) Lemma 5, chapter 3 of [7] says that if $(P, \leqq)$ is a partially ordered set of cardinality $\aleph$ such that any two elements of P have an upper bound in P and any $x \in P$ has at most countably many predecessors, then $(P, \leqq)$ is normal if and only if CH. The example given immediately after theorem 2 meets these conditions, is embeddable in $(\Lambda, \leqq)$, and is normal only if CH. q.e.d.

We conclude our paper by giving some sufficient conditions for embedding a partial ordering in $(\Lambda, \leqq)$.

COROLLARY 1: *If* $(P, \leqq)$ *is a partially ordered set of cardinality* $\leqq \aleph_1$

then $(P, \leqq)$ is embeddable in the isols if and only if each $x \in P$ has at most a countable number of predecessors.

COROLLARY 2: *If $(P, \leqq)$ is a partially ordered set of cardinality $\leqq \aleph$ such that each $x \in P$ has at most $\aleph_1$ successors then $(P, \leqq)$ is embeddable in the isols if and only if each $x \in P$ has at most countably many predecessors.*

See corollary 2, theorem 3, chapter 3 of [7] for proofs.

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Rutgers, The State University
New Brunswick, New Jersey
and
The Institute for Advanced Study
Princeton, New Jersey