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DYNAMICAL REPRESENTATIONS IN NILMANIFOLDS

by

William Parry

The results presented here are a continuation of [1] and are best viewed as a generalisation of the classic work of Von-Neumann [2], Halmos and Von-Neumann [3] and Abramov [4]. In [1] we classified ergodic unipotent affine transformations of nilmanifolds; for it turned out that metric (measure theoretic) isomorphisms and homomorphisms between two such systems are necessarily (a.e.) affine and therefore algebraic. Having clarified the relationships between ergodic unipotent affine transformations of nilmanifolds, we propose to use these systems as models for general metric dynamical systems. We do this in analogy with the work of [2], [3] concerning discrete spectra which is based on the simple model of a translation of a torus, or even of a circle. Abramov's theory is likewise based on the model of a unipotent affine transformation of a torus. (It is not usually formulated this way; for an account of this point of view c.f. [5].) The theory of entropy can be formulated in a similar way as a theory based on the Bernoulli transformations as models, but in this case considerable effort is required to show this for it depends on Sinai's result [6] concerning Bernoulli factors of transformations with positive entropy and its further elaboration requires Ornstein's results [7], [8], [9] on isomorphy, factors and limits of Bernoulli transformations.

Of course the best example of a structure theory based on models is that of group representations. We shall be concerned with metric dynamical systems and representations of them as unipotent affine transformations on nilmanifolds. It is in this sense that we speak of dynamical representations in nilmanifolds. Our main results state that the class of ergodic unipotent affine transformations on nilmanifolds is closed under reasonable finite operations:

If T is a totally ergodic dynamical sytem on a Lebesgue space with sufficiently many representations in nilmanifolds, then T is the projective limit of a sequence of unipotent affines.

If a finite number of unipotent affines on nilmanifolds are sufficient, then T is a unipotent affine on a nilmanifold.

If T_1, T_2 are metrically isomorphic ergodic unipotent affines on compact homogeneous spaces of locally compact connected nilpotent groups, then each such isomorphism is essentially affine.

Every metric factor of an ergodic unipotent affine transformation on a nilmanifold is also a unipotent affine transformation on a nilmanifold.

It should be stated that our principal results remain valid if unipotent affines are replaced by a nilflow. Indeed the proofs go over virtually word for word. Specifically if 'unipotent affine' is replaced by 'nilflow' in (2.2), (3.1), (3.3), (5.1) these statements remain valid. We should also mention that the same theorems (with unipotent affine or with nilflow) remain valid in the topological category, i.e. if (single or flows of) ergodic measure preserving transformations are replaced by (single or flows of) topologically transitive homeomorphisms of a compact connected metric space and if metric homomorphisms are replaced by continuous surjective maps.

1. Preliminaries

Let N be a connected separable locally compact nilpotent group, and let F be a closed subgroup such that N/F is compact. N/F is called a *nilspace*. If N is a Lie group then N/F is called a *nilmanifold*. In the latter case F_0, the connected component of the identity, is a normal subgroup of N and F/F_0 is discrete in N/F_0 so that every nilmanifold is the quotient of a Lie group by a uniform discrete group. [10].

If N/F is a nilspace there is a unique normalised *Haar* measure m (defined on the Borel sets) which is invariant under left translations. Measure theoretic statements will always refer to this measure.

If N_1/F_1, N_2/F_2 are two nilspaces and A is a continuous endomorphism of N_1 onto N_2 such that $AF_1 \subset F_2$ and if $a \in N_1$ then $T(xF_1) = aA(x)F_2$ is called an *affine* transformation, sometimes abbreviated to $T = aA$. Affine transformations preserve Haar measure. If A is surjective with $AF_1 = F_2$ then T is an *invertible* affine. An affine transformation $T = aA$ of a nilspace N/F onto itself is called *unipotent* if ${}_n\bigcap B^n(N) = e$ where $Bx = x^{-1}A(x)$.

Let N be a connected separable locally compact nilpotent group and let K be a compact normal subgroup of N such that N/K is a Lie group. Let A be a unipotent automorphism of N. *Then for some integer n, $A(K \cap A^{-1}K \cap \cdots \cap A^{-(n-1)}K = K \cap A^{-1}K \cap \cdots \cap A^{-(n-1)}K$.* For if $K_- = K \cap A^{-1}K \cap A^{-2}K \cap \cdots$ then $AK_- \subset K_-$. However, $B^nK_- \downarrow e$ and therefore $B^n(\kappa)AK_- \to AK_-$ when $\kappa \in K_-$. But $B^n(\kappa)AK_- = \kappa^{(-1)^n}AK_-$ so that $\kappa \in AK_-$. In other words $AK_- = K_-$ and $AK \cap K_- = K_-$ i.e. $K \supset A^{-1}K \cap A^{-2}K \cap \cdots$ Since N/K is a Lie group and therefore has no small subgroups, there exists a neighbourhood U of K with the property that if H is a closed subgroup with $U \supset H \supset K$, then $K \supset H$. For some n, $K.(A^{-1}K \cap A^{-2}K \cap \cdots \cap A^{-n}K) \subset U$, hence

$$K \supset A^{-1}K \cap \cdots \cap A^{-n}K,$$

which proves our assertion.

As an easy consequence of the above we have:

(1.2) *If $T = aA$ is a unipotent affine transformation of the nilspace N/F, then there exists a sequence of compact normal subgroups $K_n \downarrow e$ with $AK_n = K_n$ and N/K_n a Lie group. T is a projective limit of the unipotent affines induced on the nilmanifolds N/K_n. $F \simeq N/K_n/(K_nF/K_n)$.*

In the last chapter we shall need the following, no doubt standard, result:

(1.3) *If N is a connected separable nilpotent locally compact group and K is a compact subgroup then K is central.*

The result follows from the corresponding fact for Lie groups. When N is Lie, $N = M/F$ where M is the nilpotent universal covering group of N and F is discrete and central. We may suppose that $K = H/F$ is a maximal compact subgroup of N and therefore connected. It follows that H is connected. F is uniform discrete in some central connected subgroup L and therefore $H \cdot L/F = H/F$ since H/F is maximal compact. Consequently $H \supset L$ and $H/F \supset L/F$ are torii with the same fundamental group F i.e. $H = L$, H is central and K is central.

We shall need the following theorem proved in [1]:

(1.4) *If T_1, T_2 are ergodic unipotent affine transformations of the nilmanifolds X_1, X_2 and if F is a measure preserving transformation of X_1 onto almost all X_2 such that $F T_1 = T_2 F$ a.e. then there exists an affine transformation F' of X_1 onto X_2 such that $F = F'$ a.e.*

Throughout the paper equations and inequalities between measure theoretic objects will be understood to hold up to sets of measure zero.

If T_i are measure preserving transformations of Lebesgue spaces $(X_i, \mathscr{B}_i, m_i)$ $i = 1,2$ and if F is a measure preserving transformation of X_1 onto almost all (a · a) X_2 where $F T_1 = T_2 F$ a.e., we shall say that F is a *representation* of the system (X_1, T_1) on (X_2, T_2) (or in X_2) and write $(X_1, T_1) \xrightarrow{F} (X_2, T_2)$ or $T_1 \xrightarrow{F} T_2$. The representation F is called *invertible* (*or a metric isomorphism*) if there is a representation $T_2 \xrightarrow{F'} T_1$, such that $F' \circ F$ is the identity a.e. *Whenever we speak of representations in nilmanifolds in this paper we shall mean representations as unipotent affine transformations on nilmanifolds.*

In the following we assume $F^{-1}\mathscr{B}' = (F')^{-1}\mathscr{B}'$.

As a consequence of (1.4) we have:

(1.5) *If F and F' are representations of (X, T) on the same unipotent affine transformation T' of the nilmanifold X', where T is ergodic, then there exists an invertible affine transformation φ of X' such that $F' = \varphi \circ F$ a.e. and $\varphi T' = T'\varphi$.*

This statement is analogous to the well known result that eigenfunctions with the same eigenvalue are constant multiples of one another. For the proof we note that the map $x' \to F' \circ F^{-1} x'$ is a well defined measure preserving transformation which commutes with T'. By (1.4) there exists an affine transformation φ of X' with $F' \circ F^{-1} = \varphi$ a.e. i.e. $F' = \varphi \circ F$ a.e. $\varphi^{-1} = F \circ (F')^{-1}$ is clearly well defined and also affine.

We shall find the following proposition useful in supplementing measure theoretic structures with topological structures. (For a similar proposition c.f. Appendix to part 1 of [11].)

PROPOSITION 1.6. *Let T be a measure preserving transformation of the Lesbesgue space $(X, \mathscr{B}, m)$, and let T_i be measure preserving homeomorphisms of $(X_i, \mathscr{B}_i, m_i)$ $i = 1, 2$ where each X_i is a compact metric space, $\mathscr{B}_i$ are the σ-algebras of Borel sets and m_i are normalised measures.*

Let F_i be measure preserving maps of X onto a.a. X_i such that $\mathscr{B} = F_1^{-1}\mathscr{B}_1 \vee F^{-1}\mathscr{B}_2$. Then there exists a measure preserving homeomorphism T' of $(X', \mathscr{B}', m')$ (where X' is a compact metric space) and there exists an invertible measure preserving transformation F of $(X, \mathscr{B}, m)$ onto $(X', \mathscr{B}', m')$ and continuous measure preserving maps F_i' of $(X', \mathscr{B}', m')$ onto $(X_i, \mathscr{B}_i, m_i)$ such that

$$\begin{tikzcd} (X, T) \arrow[r, "F"] \arrow[d, "F_i"'] & (X', T') \arrow[dl, "F'_i"] \\ (X_i, T_i) & \end{tikzcd}$$

Moreover, F_1', F_2' separate points of X'.

PROOF. Let $\mathscr{A} = \{\sum_{m=1}^{N} (f_1^m \circ F_1)(f_2^m \circ F_2) : f_i^m \in C(X_i)\}$
$$\subset L_\infty(X).$$

Since $C(X_1)$, $C(X_2)$ are separable algebras so is $\mathscr{A}$ in the topology of $L_\infty(X)$. Thus the closure $\bar{\mathscr{A}}$ is a separable commutative T invariant C^* sub-algebra of $L_\infty(X)$, and there exists a compact metric space X' with $C(X')$ isometrically isomorphic to $\bar{\mathscr{A}}$, by an isometry U. If U_T is the isometric automorphism of $\bar{\mathscr{A}}$ induced by T, then there exists an isometric automorphism $U_{T'}$ of $C(X')$, where T' is a homeomorphism of X', such that

$$(C(X'), U_{T'}) \xrightarrow{U} (\bar{\mathscr{A}}, U_T).$$

Moreover, for each $i = 1, 2$, $U_{F_1} f_i = f_i \circ F_i$ defines an isometric isomorphism of $C(X_i)$ into $\bar{\mathscr{A}}$, and therefore we have isometric isomorphisms $U_{F_1'}$ such that

$$\begin{tikzcd} (C(X'), U_{T'}) \arrow[r, "U"] & (\bar{\mathscr{A}}, U_T) \\ (C(X_i), U_{T_i}) \arrow[u, "U_{F'_i}"] \arrow[ur, "U_{F_i}"'] & \end{tikzcd}$$

where F'_i is a continuous map of X' onto X_i. The invertible isometry U induces an invertible set mapping of $\mathscr{B}'$ to $\mathscr{B}$, where $\mathscr{B}'$ is the Borel σ-algebra of X' which will pull back the measure m to a measure m' on $\mathscr{B}'$. Since $(X, \mathscr{B}, m)$, $(X', \mathscr{B}', m')$ are Lebesgue spaces, this set mapping is induced by an invertible measure preserving transformation F such that $(X, T) \xrightarrow{F} (X', T')$. Thus we have

$$\begin{tikzcd}
(X, T) \arrow[r, "F"] \arrow[dr, "F_i"'] & (X', T') \arrow[d, "F'_i"] \\
& (X_i, T_i)
\end{tikzcd}$$

$$\begin{aligned}
\text{and } F_1'^{-1}\mathscr{B}_1 \vee F_2'^{-1}\mathscr{B}_2 &= F(F_1^{-1}\mathscr{B}_1 \vee F_2^{-1}\mathscr{B}_2) \\
&= F(\mathscr{B}) \\
&= \mathscr{B}'.
\end{aligned}$$

Since $U'_{F_i}\, C(X_i)$, $i = 1, 2$, generate $C(X')$, F'_1, F'_2 separate points of (X').

2. Comparison of representations

If T is a measure preserving transformation of a measure space $(X, \mathscr{B}, m)$ then T is called *ergodic* if $B \in \mathscr{B}$, $T^{-1}B = B$ implies $mB = 0$ or $mB^c = 0$ and is called *totally ergodic* if T^n is ergodic for $n = 1, 2, \cdots$

If T is a homeomorphism of a compact metric space X then T is called *minimal* if $K \subset X$, K closed and $TK = K$ implies $K = \emptyset$ or $K = X$ and is called *totally minimal* if T^n is minimal for $n = 1, 2, \cdots$ T is called *distal* if for each $x, y \in X$, $x \neq y$, there exists $\varepsilon > 0$ with $d\,(T^n x, T^n y) > \varepsilon$ for all n. If T is distal with a dense orbit, then T is minimal i.e. every orbit is dense [12].

THEOREM 2.1. *Let T be a totally ergodic measure preserving homeomorphism of the compact metric space $(X, \mathscr{B}, m)$ where $\mathscr{B}$ is the σ-algebra of Borel sets and m is a normalised measure which is positive on non-empty open sets. Let (X_i, T_i) $i = 1, 2$ be unipotent affine transformations of nilmanifolds $X_i = N_i/D_i$ where D_i is discrete and $T_1\,(gx_1) = gT_1x_1$ for $g \in G$ a compact central subgroup of N_1. If we have the commutative diagram*

$$\begin{tikzcd}
(X, T) \arrow[r, "f_2"] \arrow[d, "f_1"'] & (X_2, T_2) \arrow[d, "P"] \\
(X_1 T_1) \arrow[r, "P_G"'] & (X_1/G, T_1/G)
\end{tikzcd}$$

where f_1, f_2 are continuous measure preserving maps onto X_1, X_2, which separate points of X and if P is affine and P_G is the natural map of X_1 onto X_1/G then (X, T) can be given the structure of a unipotent transformation of a nilmanifold, whereupon f_1, f_2 become affine.

PROOF. We first show that T is distal from which, together with the existence of a dense orbit (T is ergodic and m is positive on non-empty open sets), we conclude that T is minimal. If $T^{m_n}x \to z \leftarrow T^{m_n}y$ then $T_i^{m_n} f_i(x) \to f_i(z) \leftarrow T_i^{m_n} f_i(y)$. Since T_1, T_2 are distal c.f [13] we see that $f_i(x) = f_i(y)$ $i = 1, 2$ i.e. $x = y$.

Let $\varepsilon, \varepsilon_1, \varepsilon_2$ denote the partitions of X, X_1, X_2 into points and let $\alpha_1 = f_1^{-1}\,\varepsilon_1$, $\alpha_2 = f_2^{-1}\,\varepsilon_2$ so that $\varepsilon = \alpha_1 \vee \alpha_2$, $T\alpha_1 = \alpha_1$, $T\alpha_2 = \alpha_2$.
If $A_i \in \alpha_i$, $A_i = f_i^{-1}x_i$ define for $n \in N_i$
$nA_i = f_i^{-1}nx_i$. $TnA_i = Tf_i^{-1}nx_i = f_i^{-1}T_i nx_i$
$= f_i^{-1}\tau_i(n)T_i x_i = \tau_i(n)T_i A_i$ where $T_i\ (nx_i) = \tau_i(n)T_i x_i$ and τ_i is a unipotent automorphism of N_i.
Let $\alpha_1/G = \{GA_1 : A_1 \in \alpha_1\}$ then α_1/G is a partition and the commutative diagram in the hypothesis asserts that $\alpha_1/G \leqq \alpha_2$ i.e. if $GA_1 \cap A_2 \neq \emptyset$, $A_1 \in \alpha_1$, $A_2 \in \alpha_2$ then $GA_1 \supset A_2$.

1. If $gA_1 \cap A_2 \neq \emptyset$, $A_1 \cap A_2 \neq \emptyset$ and $A_1' \cap A_2' \neq \emptyset$, $g \in G$, A_i, $A_i' \in \alpha_i$ then $gA_1' \cap A_2' \neq \emptyset$.
(Note that these non-empty sets are necessarily single points.) Since T is minimal there is a sequence such that $T^{m_n}(A_1 \cap A_2) \to A_1' \cap A_2'$ and therefore $\emptyset \neq T^{m_n}(gA_1 \cap A_2) = gT^{m_n} A_1 \cap T^{m_n}A_2 \to gA_1' \cap A_2'$ since $T^{m_n}A_i \to A_i'$.

2. If $A_1 \cap A_2 \neq \emptyset$ then $\{g \in G : gA_1 \cap A_2 \neq \emptyset\} = H$ is a closed subgroup of G. For if $gA_1 \cap A_2 \neq \emptyset$, $hA_1 \cap A_2 \neq \emptyset$ then by 1.

$$ghA_1 \cap A_2 \neq \emptyset.$$

It is clear that H is closed and a semi-group. H is therefore a group.

3. If $A_1 \in \alpha_1$, $A_2 \in \alpha_2$, $HA_1 \cap A_2 \neq \emptyset$ then $HA_1 \supset A_2$. Evidently by definition of H, the condition $HA_1 \cap A_2 \neq \emptyset$ is the same as the condition $A_1 \cap A_2 \neq \emptyset$ and therefore we have $GA_1 \supset A_2$. But

$$GA_1 = HA_1 \cup \bigcup_{g \notin H} gA_1$$

and for $g \notin H$, $gA_1 \cap A_2 = \emptyset$. Therefore $GA_1 \supset A_2$ implies $HA_1 \supset A_2$.

4. The map $X_2 \xrightarrow{P'} X_1/H$ given by $x_2 \to G.f_1 \circ f_2^{-1}x_2$ is well defined (by 3.) and sends T_2 to T_1/H (the affine induced by T_1) and since $T_2, T_1/H$ are unipotent, by (1.4) P' is affine. In other words the conditions of the theorem remain valid with G replaced by H so that:

There is no loss in generality in assuming that $gA_1 \cap A_2 \neq \emptyset$ whenever $A_1 \cap A_2 \neq \emptyset$ i.e. $G = H$.

5. The following conditions on $(m, n) \in N_1 \times N_2$ are equivalent (when we assume, as we may $G = H$):

(i) $mA_1 \cap nA_2 \neq \emptyset$ whenever $A_1 \cap A_2 \neq \emptyset$.

(ii) $mGA_1 \supset nA_2$ when $A_1 \cap A_2 \neq \emptyset$ i.e. $GA_1 \supset A_2$.

(iii) $mGA_1 = nGA_1$ for all A_1.

(i) $\Rightarrow$ (ii). If $A_1 \cap A_2 \neq \emptyset$ then $mA_1 \cap nA_2 \neq \emptyset$ and therefore

$$mGA_1 \supset nA_2.$$

(ii) $\Rightarrow$ (iii). Let $GA_1 \supset A_2$ then $mGA_1 = GmA_1 \supset nA_2$. Hence

$$mGA_1 \supset \bigcup_{A_2 \subset GA_1} nA_2 = nGA_1 \text{ i.e. } mGA_1 = nGA_1.$$

(iii) $\Rightarrow$ (i). If $A_1 \cap A_2 \neq \emptyset$ then $GA_1 \supset A_2$ and $GmA_1 = mGA_1 = nGA_1 \supset nA_2$. Hence $mA_1 \cap nA_2 \neq \emptyset$. $M = \{(m, n) \in N_1 \times N_2$: *either (i), (ii), (iii) hold\} is a closed subgroup of* $N_1 \times N_2$, (since (iii) defines a closed group) *and $M \supset G \times e$. As a closed subgroup of a Lie group, M is a Lie group.*

5 *(i) of course defines an action of M on X, since $\alpha_1 \vee \alpha_2 = \varepsilon$. If $x = A_1 \cap A_2$ define $(m, n)x = mA_1 \cap nA_2$ if $(m, n) \in M$.*

6. *$(m, n)A_1 = mA_1$ and $(m, n) A_2 = nA_2$ if $(m, n) \in M$.*

In fact
$$\begin{aligned}(m, n) A_1 &= \bigcup \{(m, n)A_1 \cap A_2 : A_1 \cap A_2 \neq \emptyset\} \\ &= \bigcup \{mA_1 \cap nA_2 : mA_1 \cap nA_2 \neq \emptyset\} \\ &= \bigcup \{mA_1 \cap nA_2 : \text{all } A_2 \in \alpha_2\} \\ &= mA_1. \text{ Similarly for } (m, n)A_2.\end{aligned}$$

7. *M acts transitively on X and the homomorphisms π_1, π_2 defined by $\pi_1(m, n) = m$ and $\pi_2 (m, n) = n$ are surjective.* Since N_i acts transitively on X_i we need only show that some element of M carries $A_1 \cap A_2 \neq \emptyset$ to $mA_1 \cap nA_2 \neq \emptyset$. Certainly there exists $m' \in N_1$ with $m'GA_1 = nGA_1$ for all $A_1 \in \alpha_1$ i.e. $(m', n) \in M$. Furthermore $nGA_1 \supset nA_2$. Hence $m'gA_1 \cap nA_2 \neq \emptyset$ for some and therefore all $g \in G$. But $m'GA_1 \supset nA_2$ and $mGA_1 \supset nA_2$ implies $m'GA_1 = mGA_1$ i.e.
$m'gA_1 = mA_1$ for some $g \in G$. $(m'g, n) \in M$ and carries $A_1 \cap A_2$ to $mA_1 \cap nA_2$.
The surjectivity of π_1, π_2 follows from the definition of M according to 5 (iii).

8. Since $T((m, n)x) = (\tau_1(m), \tau_2(n))Tx$ where τ_1, τ_2 are unipotent, it follows that T is a unipotent affine. f_1, f_2 are affine since $f_1(m, n)x = mf_1(x)$ and $f_2(m, n)x = nf_2(x)$.

9. The connected component M_0 of the identity of M acts transitively on X and $\pi_1|M_0$, $\pi_2|M_0$ are surjective.

In fact X is partitioned into closed M_0 orbits and since M/M_0 is countable and M is transitive (and measure preserving) X is partitioned into a finite number of M_0 orbits which are permuted by T. Since T is totally ergodic, X must be a single M_0 orbit i.e. M_0 acts transitively. $\pi_1(M_0)$, $\pi_2(M_0)$ are open subgroups of N_1, N_2 and therefore

$$\pi_1(M_0) = N_1, \pi_2(M_0) = N_2.$$

In the following theorem all maps are assumed to be measure preserving. Maps which are not specified as affine are assumed to be onto almost all of the target space whereas affine maps are assumed to be surjective. Moreover parts of a diagram involving such maps commute a.e. $(X, T) \xrightarrow{F_1} (X_1, T_1)$ means $F_1 T = T_1 F_1$ a.e. and F_1 maps X onto X_1 a.e.

THEOREM 2.2. *Let T be a totally ergodic transformation of the Lebesgue space X and suppose*

$$\begin{tikzcd} & (X, T) \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ (X_1, T_1) & & (X_2, T_2) \end{tikzcd}$$

where T_i are unipotent affines on nilmanifolds X_i. Then there exists a unipotent affine transformation T_3 on a nilmanifold X_3 and there exist maps F_3, φ_1, φ_2 where φ_1, φ_2 are affine such that the following diagram commutes:

$$\begin{tikzcd} & (X, T) \arrow[ddl, "F_1"'] \arrow[d, "F_3"] \arrow[ddr, "F_2"] & \\ & (X_3, T_3) \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ (X_1, T_1) & & (X_2, T_2) \end{tikzcd}$$

Moreover among the systems (X_3, T_3) there is an unique 'smallest' (up to an invertible affine) $(\overline{X}_3, \overline{T}_3)$ such that $\overline{F}_3 T_3 = \overline{T}_3 \overline{F}_3$ for some affine $\overline{F}_3$.

PROOF. If (X_3, T_3) is any system (affine or not) for which the above diagram commutes (X_3, T_3) factors through the system $(\overline{X}_3, \overline{T}_3)$ where $(\overline{X}_3, \overline{T}_3)$ is the factor system of (X, T) defined by the sub-σ-algebra $F_1^{-1}\mathscr{B} \vee F_2^{-1}\mathscr{B}_2 \subset \mathscr{B}$. ($\mathscr{B}$, $\mathscr{B}_1$, $\mathscr{B}_2$ are σ-algebras of X, X_1, X_2 respectively). To prove the theorem then, we need only impose a transitive connected nilpotent Lie group action on $\overline{X}_3$ such that $\overline{T}_3$ is a unipotent affine. $(\bar{X}_3, \bar{T}_3)$ will then be affinely related to any other candidate (X_3, T_3) by (1.4).

In other words, by replacing (X, T) by $(\bar{X}_3, \bar{T}_3)$, if necessary, we need to prove:

If T is totally ergodic and (X_i, T_i) $(i = 1, 2)$ are unipotent affines on nilmanifolds and if $F_i T = T_i F_i$ a.e. and $F_1^{-1}\mathscr{B}_1 \vee F_2^{-1}\mathscr{B}_2 = \mathscr{B}$ then (X, T) is metrically isomorphic to a unipotent affine on a nilmanifold. (With this representation, F_1, F_2 become affine by (1.4).)

PROPOSTIION 1.6. allows us to assume that T is a homeomorphism of a compact metric space and F_1, F_2 are continuous separating surjective maps. However, the Borel measure m which is preserved by T may not be positive on some non-empty open set. But the support of m (the complement of the largest open set on which m is zero) is T invariant and compact, and its image by F_i is T_i invariant. Since T_i is minimal this means that F_i maps the support of m onto X_i. Thus we may replace X by the support of m. In other words there is no loss in generality in assuming that m is positive on non-empty open sets.

The proof of the theorem is inductive on the length of the lower central series $N_1 \supset N_1^1 \supset \cdots \supset N_1^k \neq N_1^{k+1} = e$, where $X_1 = N_1/D_1$ (D_1 a uniform discrete subgroup of N_1 and $T_1 = a_1 A_1, A_1 D_1 = D_1$). Evidently $X_1 = N_1/D_1 \simeq (N_1/N_1^k \cap D_1)/(D_1/N_1^k \cap D_1)$ and $N_1^k \cap D_1$ is normal since N_1^k is central. By replacing N_1 by $N_1/N_1^k \cap D_1$ we may suppose that $G = N_1^k$ is a torus. ($N_1^k \cap D_1$ is a uniform discrete subgroup of N_1^k). Moreover $T_1(gx_1) = \tau_1(g)T_1 x_1$ for some unipotent automorphism of G. The unimodular matrix representing τ_1 is a nilpotent matrix plus the identity. Hence $G = G_0 \supset G_1 \supset \cdots \supset G_{l+1} = e$ where G_i/G_{i+1} is a torus $\tau_1 G_i = G_i$ and τ_1 induces the identity on G_i/G_{i+1}. At this stage, the proof is inductive on the length of the series

$$G_0 \supset G_1 \supset \cdots \supset G_l \neq G_{l+1} = e.$$

Since the last stage of the proof is typical, it will suffice to suppose: $K = G_1$ is a torus subgroup of N_1 such that $T_1(gx_1) = gTx_1$ for all $g \in K$, $x_1 \in X_1 = N_1/D_1$. Into the diagram

$$\begin{array}{ccc} (X, T) & \xrightarrow{F_2} & (X_2, T_2) \\ {\scriptstyle F_1}\downarrow & & \\ (X_1, T_1) & \xrightarrow[p_K]{} & (X_1/K, T_1/K) \end{array}$$

the inductive hypothesis allows us to introduce a unipotent affine (X_2', T_2') and affine maps φ_1, φ_2 such that

$$\begin{tikzcd}
(X, T) \arrow[rr, "F_2"] \arrow[dr, "F'_2"] \arrow[dd, "F_1"'] & & (X_2, T_2) \\
& (X'_2, T'_2) \arrow[ur, "\varphi_2"'] \arrow[dr, "\varphi_1"] & \\
(X_1, T_1) \arrow[rr, "p_K"'] & & (X_1/K, T_1/K)
\end{tikzcd}$$

commutes. In particular, we have the commutative diagram:

$$\begin{tikzcd}
(X, T) \arrow[r, "F'_2"] \arrow[d, "F_1"'] & (X'_2, T'_2) \arrow[d, "\varphi_1"] \\
(X_1, T_1) \arrow[r, "p_K"'] & (X_1/K, T_1/K)
\end{tikzcd}$$

where F_1, F'_2 separate points. The hypotheses of Theorem 2.1 are therefore satisfied and (X, T) can be given a unipotent affine structure whereupon F_1, F'_2 become affine i.e. F_1, F_2 become affine. The proof of the theorem is complete.

3. Projective limits

We say that F_α is a *unipotent representation* of (X, T) (a measure preserving transformation T of a Lebesgue space $(X, \mathscr{B}, m)$) in a nilmanifold X_α if F_α is a measure preserving transformation of X onto a.a. X_α and $F_\alpha T = T_\alpha F_\alpha$ a.e. where T_α is a unipotent affine transformation of X_α.

Let $\mathscr{B}$, $\mathscr{B}_\alpha$ be the σ-algebras of X, X_α. (X, T) is said to have *sufficiently many* (unipotent) representations (in nilmanifolds) if $\{F_\alpha^{-1}\mathscr{B}_\alpha : F_\alpha \text{ is a representation of } (X, T)\}$ generates $\mathscr{B}$.

THEOREM 3.1. *If (X, T) is a totally ergodic measure preserving transformation with sufficiently many representations then (X, T) is metrically isomorphic to a projective limit of a sequence of unipotent affine transformations on nilmanifolds.*

PROOF. Let $\mathscr{S}$ be the set of unipotent affine transformations representing (X, T). For each α, β with (X_α, T_α), $(X_\beta, T_\beta) \in \mathscr{S}$ there is an unique (up to an invertible affine) $(X_\gamma, T_\gamma) \in \mathscr{S}$ which minimises all others through which (X_α, T_α), (X_β, T_β) factor. If we select one $(X_\alpha, T_\alpha) \in \mathscr{S}$ in each affine equivalence class, it follows that the operation '$\vee$' defined by $\alpha \vee \beta = \gamma$ is unambiguous. Let $\mathscr{S}'$ denote the set of selections. Putting

$\alpha \geqq \beta$ when $\alpha \vee \beta = \alpha$, $\mathscr{S}'$ becomes a directed family. $\mathscr{S}' = \{(X_\alpha, T_\alpha) : \alpha \in I\}$. We note that $\mathscr{A} = \bigcup_{\alpha \in I} F_\alpha^{-1}\mathscr{B}_\alpha$ is a sub-algebra of $\mathscr{B}$ since if $B_\alpha \in F_\alpha^{-1}\mathscr{B}_\alpha$, $B_\beta \in F_\beta^{-1}\mathscr{B}_\beta$ then $B_\alpha \cup B_\beta \in F_\gamma^{-1}\mathscr{B}_\gamma$ where $\gamma = \alpha \vee \beta$. Moreover $\mathscr{A}$ generates $\mathscr{B}$. There is a dense sequence $\{B_n\}$ in $\mathscr{B}$ since $(X, \mathscr{B}, m)$ is separable. Inductively choose $\alpha_n \geqq \alpha_{n-1} \geqq \cdots > \alpha_1$ so that $B_1, \cdots B_n$ can be approximated to within $1/n$ by sets in $F_{\alpha_n}^{-1}\mathscr{B}_{\alpha_n}$, and abbreviate α_n to n. Clearly $F_n^{-1}\mathscr{B}_n \supset F_{n-1}^{-1}\mathscr{B}_{n-1}$ and $\bigcup_n F_n^{-1}\mathscr{B}_n$ generates $\mathscr{B}$. Moreover we have affine maps φ_n with $(X_n, T_n) \overset{\varphi_n}{\to} (X_{n-1}, T_{n-1})$. The inverse or projective limit, $\lim_{\varphi_n}(X_n, T_n) = (X_\infty, T_\infty)$ is defined by:

$$X_\infty = \{(x_1, x_2, \cdots) : \varphi_n(x_n) = x_{n-1}\} \subset X_1 \times X_2 \times \cdots$$

and $T_\infty(x_1, x_2, \cdots) = (T_1x_1, T_2x_2, \cdots)$ and it is a standard result that (X, T) is metrically isomorphic to (X_∞, T_∞) with the given properties of φ_n.

THEOREM 3.2. *If T is an ergodic unipotent affine transformation of a nilspace then T is a projective limit of unipotent affine transformations of nilmanifolds.*

PROOF. Let $T = aA$ on $X = N/F$ where F is closed and $AF = F$, A unipotent. We have seen that there exist compact normal subgroups $K_n \downarrow e$ with $AK_n = K_n$ and N/K_n Lie. Therefore $N/K_n \cdot F$ are nilmanifolds and T is the projective limit of the induced affines on $N/K_n \cdot F$. Standard considerations (c.f. for example [14]) of inverse limits of groups complete the details.

THEOREM 3.3. *If T_i are ergodic unipotent affine transformations of nilspaces X_i $(i = 1, 2)$ and if F is a measure preserving transformation of X_1 onto a.a. X_2 such that $FT_1 = T_2F$ a.e. then there exists an affine transformation F' of X_1 onto X_2 such that $F'T_1 = T_2F'$ and $F = F'$ a.e.*

PROOF. Let (X_1, T_1) be a projective limit of $(\overline{X}_1^n, \overline{T}_1^n)$ and let (X_2, T_2) be a projective limit of (X_2^n, T_2^n) where $(\overline{X}_1^n, \overline{T}_1^n)$, (X_2^n, T_2^n) are unipotent affine transformations of nilmanifolds. Using Theorem 2.2 and the lattice structure of the family of all representations of (X_1, T_1) we see that (X_1, T_1) is also the projective limit of (X_1^n, T_1^n) where

$$\begin{array}{ccc}
(X_1, T_1) & \xrightarrow{F} & (X_2, T_2) \\
\downarrow{\scriptstyle \pi_1^n} & & \downarrow{\scriptstyle \pi_2^n} \\
(X_1^n, T_1^n) & \xrightarrow{\varphi_n} & (X_2^n, T_2^n) \\
\downarrow & & \downarrow \\
(X_1^{n-1}, T_1^{n-1}) & \xrightarrow{\varphi_{n-1}} & (X_2^{n-1}, T_2^{n-1})
\end{array}$$

is commutative, $\{\varphi_n\}$ are affine and $\{\pi_1^n\}$, $\{\pi_2^n\}$ are affine and separating for X_1, X_2. F is thus seen to be equivalent to a projective limit of affines and is therefore affine.

4. Invariant sub-σ-algebras

This section is devoted to results which will be needed in § 5.

Let T be a measure preserving transformation of the Lebesgue space $(X, \mathscr{B}, m)$ and let $\mathscr{A} \subset \mathscr{B}$ be a T invariant sub-σ-algebra. If $\mathscr{F} = \{f : f$ is measurable, $|f| = 1, fT/f$ is $\mathscr{A}$ measurable$\}$. We define $D(\mathscr{A})$ as the smallest σ-algebra with respect to which the functions of $\mathscr{F}$ are measurable, and we define $D_n(\mathscr{A}) = D(D_{n-1}(\mathscr{A}))$ where $D_0(\mathscr{A}) = D(\mathscr{A})$.

T is said to have *generalised discrete spectrum* [mod $\mathscr{A}$] *of finite type* if for some positive integer n, $D_n(\mathscr{N})[D_n(\mathscr{A})]$ is $\mathscr{B}$, where $\mathscr{N}$ is the trivial σ-algebra of null sets and their complements. Since $D_n(\mathscr{A}) \supset D_n(\mathscr{N})$ we see that generalised discrete spectrum of finite type implies generalised discrete spectrum mod$\mathscr{A}$ of finite type. The qualification 'generalised' is dropped when $n = 1$.

(4.1) We shall need the following [15]:

If T is ergodic with discrete spectrum mod $\mathscr{A}$ then there exists a compact abelian group G of measure preserving transformations (acting measurably) such that $T(gx) = gTx$ for $g \in G$ and $\mathscr{A} = \{B \in \mathscr{B} : gB = B$ for all $g \in G\}$.

If T is a unipotent affine transformation of a nilmanifold X we have seen how there exists a torus group G which acts on X in such a way that X/G is another nilmanifold of lower dimension and $Tgx = gTx$.Using a Borel section of the G action one can construct for each $\gamma \in \hat{G}$ (the character group of G) a Borel function f_γ, $|f_\gamma| = 1$, such that $f_\gamma\,(gx) = \gamma(g)f(x)$. Clearly $f_\gamma\,T$ also enjoys this property. Thus $f_\gamma T/f_\gamma$ is G invariant. Hence (X, T) has discrete spectrum mod$\mathscr{A}$ where $\mathscr{A} = \{B \in \mathscr{B} : gB = B$ for all $g \in G.\}$ By induction we have:

(4.2) *A unipotent affine transformation of a nilmanifold has generalised discrete spectrum of finite type.*

THEOREM (4.3). *If T is an ergodic unipotent affine transformation of a nilmanifold N/D (D discrete) and if G is a compact group of measure preserving transformations T_g (acting measurably and effectively on X) such that $TT_g = T_gT$, then G is a central group of translations i.e. there is an isomorphic copy G' of G in $Z/Z \cap D$ (where Z is the centre of N) such that T_g is translation by an element of G'.*

PROOF. By factoring N by $Z \cap D$ we may assume Z is compact. By (1.4) each T_g is affine and since T_g commutes with T, T_g is unipotent. G is compact so that $T_g^{m_n}$ converges to the identity for some sequence m_n. Therefore the automorphism part of T_g is the identity i.e. each T_g is a translation. Hence we have an isomorphism of G into N. (N acts effectively on N/D since $Z \cap D$ has been factored out). The measurability of the G action implies that $\{T_g : g \in G\}$ acts continuously. We may suppose then that $G \subset N$ and $T(gx) = gTx$. By (1.3) G is central.

We conclude this section with a result which has independent interest. We shall use it in much the same way that Rohlin used and Hahn and the author used the special case of a compact abelian group in [16] and [17] respectively. An analogous result for locally compact abelian groups was proved in [18].

PROPOSITION 4.4. *Let G/H be a compact homogeneous space of a locally compact separable group G, with a G invariant normalised Borel measure m. If $f \in L_\infty(X)$ $(X = G/H)$ and $|f(gx)-f(x)| \leqq \rho(g)$ a.e. for each $g \in G$ where $\rho(g)$ is continuous and $\rho(g) \to 0$ as $g \to e$, then there is a continuous function f' such that $f = f'$ a.e. Consequently if $\mathscr{A}$ is a sub-σ-algebra of the Borel σ-algebra $\mathscr{B}$, invariant under G then there exists a closed subgroup $H' \supset H$ such that $\mathscr{A} = \pi^{-1}\mathscr{B}'$ where $\mathscr{B}'$ is the Borel σ-algebra of G/H' and $\pi : G/H \to G/H'$ is the natural map.*

PROOF. We shall deal with the second statement first. In fact by the first part of the theorem if $f \in C(G/H)$ then for almost all x,
$|E(f/\mathscr{A})\,(gx)-E(f/\mathscr{A})| \leqq \|f(gx)-f(x)\|_\infty \equiv \rho(g)$
so that there exists $f' \in C(G/H)$ with $E(f/\mathscr{A}) = f'$ a.e. Therefore $\mathscr{A} =$ smallest σ-algebra with respect to which $V = \{f \in C(G/H) : f$ is $\mathscr{A}$ measurable$\}$ are measurable. But the smallest σ-algebra with respect to which the functions of V are measurable consists of the Borel sets made up of elements of ζ where ζ is the smallest partition on elements of which V functions are constant. V is G invariant so that ζ is G invariant. Let $Z_0 \in \zeta$ be the element of ζ to which $H \in G/H$ belongs. Obviously $hZ_0 = Z_0$ for all $h \in H$. Define $H' = \{g \in G : gZ_0 = Z_0\} \supset H$. H' is a closed subgroup. Since G acts transitively $Z_0 = H'x_0$ if $x_0 = H \in Z_0$ and every element of ζ has the form $gH'x_0$. But $gH'x_0 = gH'$ is a typical inverse image of π i.e. $\mathscr{A}$ consists of sets made up of inverse images of π. In other words $\mathscr{A} = \pi^{-1}\mathscr{B}'$.

We proceed to the proof of the first statement of the Proposition. Let $f \in L_\infty$ (G/H) satisfy $|f(gx)-f(x)| \leqq \rho(g)$ a.e. for each $g \in G$ where $\rho(g)$ is continuous and $\rho(g) \to 0$ as $g \to e$.

$$\{(g, x) : |f(gx)-f(x)| > \rho(g)\} \subset G \times X$$

is measurable and by Fubini's theorem is null. Therefore there exists a null set $N \subset X$ such that if $x \notin N$ then $|f(gx)-f(x)| \leqq \rho(g)$ for all $g \notin N_x$ where $N_x \subset G$ is a null set depending on x. $U_\varepsilon = \{g : \rho(g) < \varepsilon_{/2}\}$

is an open neighbourhood of e, and $U_\varepsilon x$ is a neighbourhood of x. Since N is null, $X-N$ is dense, and $\{U_\varepsilon x : x \in X-N\}$ is a covering of X, since $\{U_\varepsilon g : gH \in X-N\}$ is a covering of G. Hence there exists $x_1, \cdots x_k \in X-N$ such that $U_\varepsilon x_1, \cdots U_\varepsilon x_k$ cover X. Moreover

$$|f(gx_i)-f(hx_i)| < \varepsilon \text{ if } g, h \in U_\varepsilon - N_{x_i}.$$

In other words

$$|f(x)-f(y)| < \varepsilon \text{ if } x, y \in (U_\varepsilon - N_{x_i})x_i.$$

If we consider these inequalities for $\varepsilon = 1, 1/2, \cdots$ we see that there is a null set $M \subset X$ and a finite open covering $U_n^1, \cdots U_n^{m_n}$ such that $|f(x)-f(y)| < 1/n$ if $x, y \in U_n^i - M$. Hence f is uniformly continuous on $X-M$ and $f|X-M$ extends uniquely to a continuous function f'. Obviously $f = f'$ a.e.

5. Factors

In order to show that the class of unipotent affine transformations on nilmanifolds is fully satisfactory as a class of models it is desirable that the class of transformations based on them should be closed under the operation of factoring. Unfortunately, as yet we have not been able to achieve this result in its full generality. Nevertheless our concluding result shows that factors of unipotent affine transformations are unipotent affines. It is a partial generalisation of the corresponding fact for transformations with quasidiscrete spectra [16].

THEOREM 5.1. *Let T be an ergodic unipotent affine transformation of a nilmanifold X and let T'be a measure preserving transformation of a Lebesgue space X'. Let F be a measure preserving transformation of X onto a.a. X' such that $FT = T'F$ a.e. Then T' is metrically isomorphic to a unipotent affine transformation of a nilmanifold.*

PROOF. Let $\mathscr{B}$, $\mathscr{B}'$ be the σ-algebras of measurable subsets of X, X'. We need only show that $\mathscr{A} = F^{-1}\mathscr{B}'$ is a σ-algebra defined by cosets of some closed subgroup of N $(N/D = X)$. To do this, by virtue of Proposition 4.4, we need to prove that $\mathscr{A}$ is N invariant (i.e. $n\mathscr{A} = \mathscr{A}$ for all $n \in N$) given that $T\mathscr{A} = \mathscr{A}$.

By (4.2) T has generalised discrete spectrum of finite type. Therefore T has generalised discrete spectrum mod $\mathscr{A}$ of finite type. Hence $\mathscr{A} \subset D_1(\mathscr{A}) \subset \cdots \subset D_n(\mathscr{A}) = \mathscr{B}$. Of course $n\mathscr{B} = \mathscr{B}$ for all $n \in N$. We prove that $\mathscr{A}$ is N invariant by induction.

If $D_k(\mathscr{A})$ is N invariant then $T|D_k(\mathscr{A})$ acts as a unipotent affine. We then need to prove $D_{k-1}(\mathscr{A})$ is N invariant. In other words, there is no loss in generality if we assume the above series has length one i.e. $D_1(\mathscr{A}) = \mathscr{B}$.

In this case by (4.1) there is a compact group G of measure preserving transformations acting measurably on X such that $TT_g x = T_g Tx$ and $\mathscr{A} = \{B : gB = B \text{ for all } g \in G\}$. By Theorem 4.3 each T_g, $g \in G$ is a translation by a central element g' of N. If $B \in \mathscr{A}$ and $n \in N$ then $T_g nB = g'nB = ng'B = nB$. Hence $nB \in \mathscr{A}$. We have therefore proved that $\mathscr{A}$ is N invariant and the proof of the theorem is complete.

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