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SYLOW THEORY IN LOCALLY FINITE GROUPS

by

B. Hartley

Contents

1. Introduction

We make the convention that all groups occurring in this paper are assumed to be locally finite. Let π be a set of primes and G a group. We shall say that G is *Sylow π-sparse* if $|\mathrm{Syl}_\pi H| < 2^{\aleph_0}$ for every countable subgroup H of G, where $\mathrm{Syl}_\pi H$ denotes the set of Sylow, that is maximal, π-subgroups of H. In order for a group G to be Sylow π-sparse it obviously suffices that every countable subgroup H of G be contained in some subgroup K of G with $|\mathrm{Syl}_\pi K| < 2^{\aleph_0}$ (cf. [4] Lemma 2.2) and so a countable group G is Sylow π-sparse if and only if $|\mathrm{Syl}_\pi G| < 2^{\aleph_0}$.

We shall say that G is *Sylow π-connected* if the Sylow π-subgroups of G are conjugate in G, and *Sylow π-integrated* if every subgroup of G is Sylow π-connected. Thus the class of Sylow π-integrated groups is just the class $\mathfrak{D}_\pi^{\mathrm{S}}$ in the notation of [4].

In this paper one of our concerns is to investigate the relationship between $\mathfrak{D}_\pi^{\mathrm{S}}$ and the class of Sylow π-sparse groups. This relationship seems to be a very intimate one. Obviously every Sylow π-integrated group is Sylow π-sparse, and we know of no example to show that these two properties are distinct.

We prove

THEOREM (A). *If a group G contains a Sylow p-sparse subgroup of finite index then G is Sylow p-integrated.*

Here p, as always, denotes a prime.

THEOREM (B). *Let G be a π-separable group containing a Sylow π-sparse subgroup of finite index. Then G is Sylow π-integrated.*

Thus for π-separable groups (groups having a series of finite length in which each factor is either a π-group or a π'-group) the concepts 'Sylow π-sparse' and 'Sylow π-integrated' coincide, and 'Sylow p-sparse' and 'Sylow p-integrated' are always equivalent. Similar theorems were proved for countable groups in [4]. The above results may be established by modifying the arguments of [4] (I am indebted to Dr. M. J. Tomkinson for pointing this out) or may be deduced from [4] by considering the closure properties of the class of Sylow π-integrated groups. In the usual closure operation notation, as used for example in [4], we have

LEMMA (2.1). *The class $\mathfrak{D}_\pi^{\mathrm{S}}$ of Sylow π-integrated groups is $\{\mathrm{Q}, \mathrm{S}, \mathrm{D}_0, \mathrm{L}_\omega\}$-closed,*

where, if $\mathfrak{X}$ is some class of groups,
$G \in \mathrm{D}_0\mathfrak{X} \Leftrightarrow G$ is a direct product of finitely many $\mathfrak{X}$-groups,
$G \in \mathrm{L}_\omega\mathfrak{X} \Leftrightarrow$ every countable set of elements of G lies in some $\mathfrak{X}$-subgroup of G.

We also show that the set of Sylow π-subgroups of a Sylow π-integrated group is homomorphism invariant (Lemma 2.2). It does not seem to be known whether Sylow π-connectedness is sufficient to ensure this, and in fact not a great deal is known about even Sylow p-connected groups in general.

We are also interested in obtaining information about the structure of Sylow π-sparse groups. For any group G let $\sigma(G)$ denote the set of prime divisors of the orders of the elements of G. Then we have

THEOREM (D). *Let G be locally soluble and let $\sigma(G) = \pi_1 \cup \cdots \cup \pi_n$ be a partition of $\sigma(G)$ into finitely many pairwise disjoint subsets. Suppose that G is Sylow π_i-sparse for $1 \leqq i \leqq n$. Then*

(i) $G/\prod_{i=1}^n O_{\pi_i}(G) \in \mathfrak{R}_1$.
(ii) *If π is any union of sets π_i, then G is Sylow π-integrated.*

Subgroups such as $O_\pi(G)$, $O_{\pi,\pi'}(G), \cdots$ are defined by: $O_\pi(G)$ is the largest normal π-subgroup of G, $O_{\pi,\pi'}(G)/O_\pi(G) = O_{\pi'}(G/O_\pi/G), \cdots$. The class $\mathfrak{R}_1$ was introduced in [4]; a group G belongs to $\mathfrak{R}_1$ if and only if every locally nilpotent subgroup of G is almost abelian and of finite (Mal'cev special) rank. By ([4] Lemma 4.8), locally soluble $\mathfrak{R}_1$-groups are countable, metabelian-by-finite, and of finite rank. Consequently the conclusions of Theorem D imply that G is π_i-separable for $1 \leqq i \leqq n$.

As corollaries of Theorem D we have

COROLLARY (D1). *Let G be locally soluble. Suppose that $\sigma(G)$ is finite and G is Sylow p-sparse for all $p \in \sigma(G)$. Then $G/\rho(G)$ satisfies Min and G is Sylow π-integrated for any subset π of $\sigma(G)$.*

We write $\rho(G)$ for the Hirsch-Plotkin radical of G, Min for the minimal condition on subgroups and Min-p for the minimal condition on p-subgroups.

COROLLARY (D2). *Suppose that G is locally soluble and is Sylow p- and p'-sparse. Then $G/O_{p,p',p}(G)$ is finite and $G/O_p(G)$ satisfies Min-p.*

This result is only of interest when $p = 2$; for odd p a much stronger result, in which G is only assumed to be locally p-soluble and Sylow p-sparse, is proved in [5].

Finally we consider groups which are completely Sylow sparse, that is, are Sylow π-sparse for every set π of primes. In [4] and other places, a certain class $\mathfrak{U}$ is considered, whose definition may be given as follows: $G \in \mathfrak{U}$ if and only if

*U*1. *$G \in \mathrm{PL}\mathfrak{N}$, that is, G has a finite series with locally nilpotent factors.*

*U*2. *G is completely Sylow integrated (in the obvious sense).*

We prove that the second of these conditions implies the first by proving

THEOREM (E). *Suppose that G is locally soluble. Then $G \in \mathfrak{U}$ if and only if G is completely Sylow sparse.*

For by a well known theorem of P. Hall, a finite completely Sylow integrated group is soluble and so an arbitrary completely Sylow integrated group is locally soluble. Theorem E can easily be deduced from Corollary D2 and ([4] Theorem D); however we give an independent approach which avoids the use of Thompson's results on groups of automorphisms of soluble groups.

Theorems A and B may be found in Section 2, Theorem D and other structural results about Sylow π-sparse groups are established in Section 3, and Section 4 contains Theorem E and some other observations about the class $\mathfrak{U}$. It will be useful to write $\mathfrak{O}_\pi$ for the class of π-groups, and X^G denotes the normal closure of a subset X of a group G. The rest of our notation is either standard, to be found in [4], or will be explained when introduced.

2. Closure properties

LEMMA (2.1). *The class $\mathfrak{D}_\pi^{\mathrm{S}}$ of Sylow π-integrated groups is $\{\mathrm{Q}, \mathrm{S}, \mathrm{D}_0, \mathrm{L}_\omega\}$-closed.*

PROOF. The fact that $\mathfrak{D}_\pi^{\mathrm{S}}$ is s-closed is immediate from the definition, and the D_0-closure follows by the argument of ([6] Lemma 6.7).

Now since $\mathfrak{D}_\pi^{\mathrm{S}}$ is S-closed, a group belongs to $\mathrm{L}_\omega\mathfrak{D}_\pi^{\mathrm{S}}$ if and only if every countable subgroup of that group belongs to $\mathfrak{D}_\pi^{\mathrm{S}}$. Therefore, if $\mathfrak{D}_\pi^{\mathrm{S}}$ is not L_ω-closed, there exists a group G such that every countable subgroup of G is Sylow π-integrated, while G contains two non-conjugate Sylow π-subgroups S and T. Suppose then that this situation holds. We construct a sequence $x_0, x_1, \ldots$ of elements of G and integers $0 \leqq n_0 \leqq n_1 \leqq \ldots$ such that

(i) *the set* $\{x_0, \ldots, x_{n_i}\}$ *is a subgroup* F_i *of* G,
(ii) $\langle S^{x_i} \cap F_{i+1}, T \cap F_{i+1}\rangle \notin \mathfrak{O}_\pi$,
(iii) $i \leqq n_i$.
for each $i \geqq 0$.

Suppose that for some $j \geqq 0$ we have obtained $x_0, \ldots, x_{n_j}$ and $n_0, \ldots, n_j$ such that (i) and (iii) hold for $i \leqq j$ and (ii) for $i<j$. (We may begin the construction by putting $x_0 = 1$, $n_0 = 0$). Then by (iii) the element x_j has already been defined, and as S and T are not conjugate in G, we have $S^{x_j} \neq T$ and so $\langle S^{x_j}, T\rangle \notin \mathfrak{O}_\pi$. Therefore there exist finite subsets $S_0 \leqq S$, $T_0 \leqq T$ such that $\langle S_0^{x_j}, T_0\rangle \notin \mathfrak{O}_\pi$. Let $F_{j+1} = \langle F_j, S_0, T_0, y\rangle$ where y is an arbitrary element of G not lying in F_j; clearly such an element exists since G is infinite and even uncountable. Then the elements of F_{j+1} not lying in F_j may be indexed as $x_{n_j+1}, \ldots, x_{n_{j+1}}$, where $n_{j+1} > n_j$, and (i) and (iii) hold with $i = j+1$. Furthermore $\langle S^{x_j} \cap F_{j+1}, T \cap F_{j+1}\rangle$ contains $\langle S_0^{x_j}, T_0\rangle$, which is not a π-group. Thus (ii) holds and the construction can be carried out.

Let $H = \{x_0, x_1, \ldots\} = \bigcup_{i=0}^{\infty} F_i$. Then H is countable and so H is Sylow π-connected. Therefore there is an element $x \in H$ such that

$$\langle (S \cap H)^x, T \cap H\rangle = \langle S^x \cap H, T \cap H\rangle \in \mathfrak{O}_\pi.$$

But $x = x_i$ for some i and so we obtain $\langle S^{x_i} \cap F_{i+1}, T \cap F_{i+1}\rangle \in \mathfrak{O}_\pi$, contradicting (ii). This establishes that $\mathfrak{D}_\pi^{\mathrm{S}}$ is L_ω-closed.

To establish the Q-closure it now suffices to show that every homomorphic image of a countable $\mathfrak{D}_\pi^{\mathrm{S}}$-group lies in $\mathfrak{D}_\pi^{\mathrm{S}}$. This follows from ([4] Lemma 2.2). Alternatively we may use the following lemma, which yields more information. It has already been established under further assumptions elsewhere (for example [1] Lemma 2.1, [10] Lemma 9.13).

LEMMA (2.2). *Let G be Sylow π-integrated and $K \lhd G$. Then the Sylow π-subgroups of G/K are precisely the groups SK/K with $S \in Syl_\pi G$.*

This result will itself be deduced from the following special case:

LEMMA (2.3). *Let G be Sylow π-integrated, $K \lhd G$ and suppose that $K \in \mathfrak{O}_{\pi'}$, $G/K \in \mathfrak{O}_\pi$. Then G splits over K.*

PROOF OF 2.3. Suppose that the result is false. Then since any set of cardinal numbers is well ordered by the usual ordering of cardinals according to magnitude there is a counterexample $K \lhd G$ such that the index $|G : K| = \aleph$ is as small as possible. We notice first that $\aleph$ must be infinite. For otherwise $G = KF$ for some finite subgroup F and a well known theorem of Schur shows that F splits over $F \cap K$, giving a contradiction.

Let λ be the least ordinal of cardinal $\aleph$. Since $\aleph$ is infinite, λ is a limit ordinal. The elements of G/K may be indexed as $\{Kx_\alpha; \alpha < \lambda\}$, and if we write $G_\beta = \langle K, x_\alpha; \alpha < \beta\rangle$ for $\beta \leqq \lambda$, we have

(i) $G_0 = K$,
(ii) $G_\beta \leqq G_{\beta+1}$ *if* $\beta < \lambda$,
(iii) $G_\mu = \bigcup_{\alpha<\mu} G_\alpha$ *if* $\mu \leqq \lambda$ *is a limit ordinal*,
(iv) $G_\lambda = G$,
(v) $|G_\beta : K| < \aleph$ *if* $\beta < \lambda$.

In fact, (i)–(iv) are immediate from the definition. To see (v), let $\beta < \lambda$. Then by the choice of λ, G_β/K is generated by a set whose cardinal $\bar{\aleph}$ is smaller than $\aleph$. Therefore $|G_\beta/K|$ is either finite or equal to $\bar{\aleph}$, according as $\bar{\aleph}$ is finite or infinite, and is in any case smaller than $\aleph$.

We now construct inductively subgroups P_β of G_β $(\beta \leqq \lambda)$ such that

(vi) $G_\beta = KP_\beta$, $K \cap P_\beta = 1 (\beta \leqq \lambda)$,
(vii) $P_\beta \leqq P_{\beta+1}$ $(\beta < \lambda)$,
(viii) $P_\mu = \bigcup_{\alpha<\mu} P_\alpha$ *if* $\mu \leqq \lambda$ *is a limit ordinal.*

Then P_λ will complement K in G, and this contradiction will establish Lemma 2.3. We begin by putting $P_0 = 1$. Suppose $0 < \gamma \leqq \lambda$ and that the subgroup P_β have been obtained for $\beta < \gamma$. If γ is a limit ordinal we put $P_\gamma = \bigcup_{\alpha<\gamma} P_\alpha$. Otherwise γ has the form $\beta + 1$. Then as λ is a limit, we have $\gamma < \lambda$. Therefore by (v) and the choice of $\aleph$, there exists a subgroup P_γ^* of G_γ such that $KP_\gamma^* = G_\gamma$, $K \cap P_\gamma^* = 1$. Then $G_\beta = K(G_\beta \cap P_\gamma^*)$, and P_β and $G_\beta \cap P_\gamma^*$ are Sylow π-subgroups of G_β. Therefore, as G_β is Sylow π-connected, we have $P_\beta = (G_\beta \cap P_\gamma^*)^x$ for some $x \in K$. Let $P_\gamma = P_\gamma^{*x}$. Then $P_\beta \leqq P_\gamma$, and so the construction proceeds and can be completed.

Proof of Lemma 2.2. We have $G \in \mathfrak{D}_\pi^S$ and $K \lhd G$. Let $U/K \in \mathrm{Syl}_\pi G/K$ and $T \in \mathrm{Syl}_\pi K$. Then by the Frattini argument

$$U = KN, \tag{1}$$

where $N = N_U(T)$. Now T is a normal Sylow π-subgroup of $N \cap K$ and so $N \cap K/T \in \mathfrak{O}_{\pi'}$. Also $N/N \cap K \cong NK/K = U/K \in \mathfrak{O}_\pi$. We show that $N/T \in \mathfrak{D}_\pi^S$. In fact if $X/T \leqq N/T$ and R_1/T, R_2/T are Sylow π-subgroups of X/T, then R_1 and R_2 are Sylow π-subgroups of X (since $T \in \mathfrak{O}_\pi$)

and so are conjugate in X as $G \in \mathfrak{D}_\pi^S$. Therefore R_1/T and R_2/T are conjugate in X/T, whence N/T is Sylow π-integrated as claimed. Lemma 2.3 now shows that $N/T = (N \cap K/T) \cdot (W/T)$ for some π-subgroup W/T of N/T. Then W is a π-subgroup of N and as $N = (N \cap K)W$, we have from (1) that $U = KW$. It is now easy to see that $W \in \mathrm{Syl}_\pi G$, and in any case we may always ensure that this is the case by replacing W by a Sylow π-subgroup of G containing it.

We have now shown that every Sylow π-subgroup of G/K is the natural image of some Sylow π-subgroup of G. In particular G possesses Sylow π-subgroups whose images modulo K belong to $\mathrm{Syl}_\pi G/K$; by conjugacy, it follows that every Sylow π-subgroup of G has that property.

The proof of Lemma 2.2 becomes much simpler if one assumes in addition that the Sylow π-subgroups of $G/O_\pi(G)$ are countable. For it suffices, as above, to show that every Sylow π-subgroup U/K of G/K is the image of some Sylow π-subgroup of G, and in doing this we may assume, by passing to $G/O_\pi(G)$, that the Sylow π-subgroups of G are countable. Let $S \in \mathrm{Syl}_\pi U$ and suppose if possible that $SK < U$. Then as S is countable there is a countable subgroup U^*/K of U/K with $SK < U^* \leqq U$. By well known results ([4] Lemma 2.1) there exists a π-subgroup T^* of U^* such that $U^* = KT^*$, and if T is a Sylow π-subgroup of U containing T^* then we have $SK < TK \leqq U$. Since G is Sylow π-integrated we have that $T = S^x$ for some $x \in U$ and so SK/K and TK/K are conjugate in U/K, contradicting the fact that a subgroup of a periodic group cannot be conjugate to a proper subgroup of itself.

We know of no situation in which, in a Sylow π-integrated group G, the Sylow π-subgroups of $G/O_\pi(G)$ are uncountable, and it is tempting to conjecture that this cannot occur. Some results in that direction are given in Corollary C1.

Proofs of Theorems A and B. In proving Theorem A, we may suppose that G contains a normal Sylow p-sparse subgroup N of finite index. Let H be a countable subgroup of G. Then $H \cap N$ is a normal Sylow p-sparse subgroup of finite index of H. Thus $|\mathrm{Syl}_p H \cap N| < 2^{\aleph_0}$. Now it can be seen from the argument of ([4] Theorem A1) that this implies that $H \cap N$ is Sylow p-connected – it is not necessary to invoke the continuum hypothesis for this purpose as the statement of ([4] Theorem A1) might lead us to believe. Therefore by ([4] Theorem A3), H is Sylow p-connected. It follows that $G \in \mathrm{L}_\omega \mathfrak{D}_p^S$ and so by Lemma 2.1 $G \in \mathfrak{D}_p^S$, that is, G is Sylow p-integrated.

Theorem B follows from ([4] Theorem B) in the same way; the same remarks about the continuum hypothesis apply.

As an immediate corollary of Theorem B we have:

LEMMA (2.4). *Suppose that G contains a normal $\mathfrak{U}$-subgroup N such that G/N is finite and soluble. Then $G \in \mathfrak{U}$.*

This follows since $\mathfrak{U}$-groups are π-separable for all sets π. Lemma 2.4 was established as ([4] Lemma 6.6), but the present approach is rather more direct.

3. Sylow π-sparse groups and π-separability

We begin with a slight extension of Theorem C of [4]. A group G will be called *upper π-separable* if its upper π-series $\{P_\alpha\}$, defined by

$$P_0 = 1, P_{\alpha+1}/P_\alpha = O_{\pi'\pi}(G/P_\alpha), P_\mu = \bigcup_{\alpha<\mu} P_\alpha$$

for ordinals α and limit ordinals μ, ultimately reaches G. The class of upper π-separable groups is QS-closed, and if G is any non-trivial upper π-separable group then either $O_\pi(G) \neq 1$ or $O_{\pi'}(G) \neq 1$. The following lemma can then be established by the argument of ([4] Lemma 5.4).

LEMMA (3.1). *If G is upper π-separable and $O_\pi(G) = 1$ then $O_{\pi'}(G) \geqq C_G(O_{\pi'}(G))$.*

Our extension of ([4] Theorem C) is the following:

THEOREM (C). *Let G be upper π-separable, locally π-soluble, and Sylow π-sparse. Then $|G : O_\pi^4(G)| < \infty$ and $O_\pi^3(G)/O_\pi^2(G) \in \mathfrak{R}_1$.*

If furthermore π is finite, then $|G : O_\pi^3(G)| < \infty$ and $O_\pi^3(G)/O_\pi^2(G)$ satisfies Min.

A finite group X is called π-soluble, if every composition factor of X is either a π'-group or a cyclic π-group.

Here $O_\pi^2(G) = O_{\pi,\pi'}(G)$, $O_\pi^3(G) = O_{\pi,\pi',\pi}(G)$, and so on. The increase in generality comes in the removal of the countability assumption and the replacement of 'π-separable' by 'upper π-separable'. The argument given in [4] requires only a few modifications to establish the more general result, and since we require the more general theorem in the sequel we now briefly describe these modifications.

The following elementary fact will be useful:

LEMMA (3.2). *Suppose that $G \in \mathrm{L}\mathfrak{D}_\pi^{\mathrm{S}}$, G is Sylow π-sparse, and $K \lhd G$. Then G/K is Sylow π-sparse.*

PROOF. Let H/K be a countable subgroup of G/K. Then $H = KL$ for some countable subgroup L of G and so $H/K \cong L/L \cap K$. By ([4] Lemma 2.1) every Sylow π-subgroup of $L/L \cap K$ is the natural image of some Sylow π-subgroup of L and so, since $|\mathrm{Syl}_\pi L| < 2^{\aleph_0}$, we have $|\mathrm{Syl}_\pi\, H/K| = |\mathrm{Syl}_\pi L/L \cap K| < 2^{\aleph_0}$. Hence G/K is Sylow π-sparse.

Next we obtain some information about locally π-soluble groups in which the abelian π-subgroups have finite rank.

LEMMA (3.3). *There exists a function f from the set of non-negative integers into itself with the following property: if π is a set of primes and G is a locally π-soluble group in which, for $p \in \pi$, every abelian p-subgroup of G has rank $\leqq n$, then $|G : O_{\pi',\pi,\pi'}(G)| \leqq f(n)$.*

PROOF. We begin by considering a finite π-soluble group G in which, for $p \in \pi$, every abelian p-subgroup of G has rank not exceeding n. Let $p \in \pi$, let P be any p-subgroup of G and let A be any maximal abelian normal subgroup of P. Then A can be generated by n elements. It is well known that $A = C_P(A)$; hence P/A is isomorphic to a subgroup of Aut A and so, by ([8] Lemma 5), P/A can be generated by $\frac{1}{2}n(5n-1)$ elements. Hence P can be generated by $f_1(n) = n+\frac{1}{2}n(5n-1)$ elements, and so, for $p \in \pi$, every p-subgroup of G can be generated by $f_1(n)$ elements.

Let U/V be any chief factor of G. Then U/V is either a π'-group or an elementary abelian p-group with $p \in \pi$. In the latter case U/V has rank at most $f_1(n)$ and so $G/C_G(U/V)$ can be viewed as an irreducible π-soluble group of linear transformations of a vector space of dimension at most $f_1(n)$ over $\boldsymbol{Z}_p$; also $p \in \pi$. It follows from the argument of ([4] Lemma 5.5) that $G/C_G(U/V)$ has an abelian normal subgroup of index bounded by a function of $f_1(n)$ alone and so there is a function $f_2(n)$ such that

$$(G^{f_2(n)})' \leqq C_G(U/V), \text{ where } G^m = \langle x^m : x \in G\rangle.$$

This holds for every π-chief factor U/V of G. Therefore, as G is π-soluble, it follows that $(G^{f_2(n)})' \leqq O_{\pi'\pi}(G)$ and hence that $G^{f_2(n)} \leqq O_{\pi',\pi,\pi'}(G)$.

Let $L = G/O_{\pi',\pi,\pi'}(G)$. Then L is π-soluble and of exponent dividing $f_2(n)$; also if $p \in \pi$, every p-subgroup of L can be generated by $f_1(n)$ elements. Let p be a prime divisor of $|L|$ belonging to π, let S_p be a Sylow p-subgroup of L and A_p a maximal abelian normal subgroup of S_p. Then $|A_p| \leqq f_2(n)^{f_1(n)}$ and since A_p is self-centralizing in S_p we have that S_p/A_p is isomorphic to a subgroup of Aut A_p and so $|S_p/A_p|$ is bounded in terms of $|A_p|$. Hence $|S_p| \leqq f_3(n)$ for some suitable function f_3, and since this holds for every prime divisor of $|L|$ belonging to π, and every such prime divisor divides $f_2(n)$, it follows that there is a function f_4 such that every π-subgroup of L has order at most $f_4(n)$. In particular $|O_\pi(L)| \leqq f_4(n)$. Since $O_{\pi'}(L) = 1$ we have that $O_\pi(L) \geqq C_L(O_\pi(L))$ (Lemma 3.1) whence we obtain a function f such that $|L| \leqq f(n)$. Then f has the property required by Lemma 3.3 for finite π-soluble groups G.

A standard inverse limit argument now gives the general result. Let

G be an arbitrary locally π-soluble group in which, for $p \in \pi$, every abelian p-subgroup of G has rank $\leqq n$. Let $\boldsymbol{F}$ be the set of all finite subgroups of G and for each $F \in \boldsymbol{F}$ let $\Sigma(F)$ denote the set of all normal subgroups N of F such that $|F : N| \leqq f(n)$ and $N = O_{\pi', \pi, \pi'}(N)$. By the first part of the discussion $\Sigma(F) \neq \emptyset$. If $F_1, F_2, \in \boldsymbol{F}$, $F_1 \geqq F_2$ and $N_1 \in \Sigma(F_1)$, then clearly $N_1 \cap F_2 \in \Sigma(F_2)$; thus the sets $\Sigma(F)$ form an inverse system of finite non-empty sets indexed by the set $\boldsymbol{F}$. The inverse limit of such a system is well known to be non-empty and so we may choose a family $\{N_F : F \in \boldsymbol{F}\}$ such that $N_F \in \Sigma(F)$ and $N_{F_2} = N_{F_1} \cap F_2$ whenever $F_1 \geqq F_2$. Then $N = \bigcup_{F \in \boldsymbol{F}} N_F$ is a normal subgroup of G of index at most $f(n)$ and $N = O_{\pi', \pi, \pi'}(N)$ (cf. [1] Corollary 3.10). Hence f has the properties required of it.

Corollary (3.4). *Let G be a locally π-soluble group in which every abelian π-subgroup has finite rank. Then $|G : O_{\pi', \pi, \pi'}(G)| < \infty$.*

Proof. Since every π-subgroup of G is locally soluble, a theorem of Gorčakov [2] shows that every π-subgroup of G has finite rank. By Lemma 3.3 it suffices to deduce from this that the ranks of the π-subgroups of G are bounded. If this is not the case then there exists, for each integer $n > 0$, a finite π-subgroup Q_n of G of rank at least n. Let $F_n = \langle Q_1, \ldots, Q_n \rangle$ $(n \geqq 1)$. Then F_n is finite and π-soluble, and $F_1 \leqq F_2 \leqq \ldots$.

Let $P_1 \leqq P_2 \leqq \cdots$ be a tower with $P_i \in \mathrm{Syl}_\pi F_i$. Then $P = \bigcup_{i=1}^{\infty} P_i$ is a π-subgroup of G. However, since finite π-soluble groups are Sylow π-connected ([3] Theorem 6.3.6), Q_n is conjugate to a subgroup of P_n and so P_n has rank at least n. Therefore P has infinite rank, a contradiction.

We now deal with a special case of Theorem C.

Lemma 3.5. *Suppose that $G = HK$, $H \lhd G$, $H \cap K = 1$, where $H \in \mathfrak{D}_{\pi'}$ and $K \in \mathfrak{D}_\pi$. Suppose further that $C_K(H) = 1$, K is locally soluble, and G is Sylow π-sparse. Then $K \in \mathfrak{R}_1$. If in addition π is finite, then K satisfies Min.*

Proof. Let K_1 be any countable subgroup of K. For each $1 \neq k \in K$ there exists an element $h_k \in H$ such that $[h_k, k] \neq 1$. Then $H_1 = \langle h_k : 1 \neq k \in K_1 \rangle^{K_1}$ is a countable subgroup of H normalized by K_1 and such that $C_{K_1}(H_1) = 1$. Since G is Sylow π-sparse, we have that $|\mathrm{Syl}_\pi H_1 K_1| < 2^{\aleph_0}$. The arguments used to prove ([4] Lemma 4.7) then yield without modification that $K_1 \in \mathfrak{R}_1$. Therefore by ([4] Lemma 4.8) every countable subgroup of K has finite rank. Hence K itself has finite rank, and so is countable by a theorem of Kargapolov [7]. Hence $K \in \mathfrak{R}_1$.

If π is finite, then ([4] Lemma 4.8) shows that K satisfies Min.

Now Theorem C can be established without difficulty.

Proof of Theorem C. Let $\bar{G} = G/O_\pi(G)$. We show that every π-subgroup of $\bar{G}$ lies in $\mathfrak{R}_1$. In fact let $\bar{K}$ be any π-subgroup of $\bar{G}$ and $\bar{H} = O_{\pi'}(\bar{G})$. Then by Lemma 3.1 $C_{\bar{K}}(\bar{H}) = 1$. Since finite π-soluble groups are Sylow π-integrated we have from Lemma 3.2 that $\bar{G}$, and hence $\bar{H}\bar{K}$, is Sylow π-sparse. Therefore by Lemma 3.5, $\bar{K} \in \mathfrak{R}_1$. It follows from ([4] Lemma 4.8) that every π-subgroup of $\bar{G}$ has finite rank and hence Corollary 3.4 gives that $|\bar{G} : O_{\pi',\pi,\pi'}(\bar{G})| < \infty$, that is $|G : O_\pi^4(G)| < \infty$. Furthermore we have by ([4] Lemma 2.1) that every countable subgroup of $O_{\pi',\pi}(\bar{G})/O_{\pi'}(\bar{G})$ is the natural image of some π-subgroup of $\bar{G}$ and so has finite rank. Therefore $O_{\pi',\pi}(\bar{G})/O_{\pi'}(\bar{G})$ has finite rank, and so is countable by Kargapolov's theorem [7]. Hence $O_{\pi',\pi}(\bar{G})$ splits over $O_{\pi'}(\bar{G})$ and it follows that $O_{\pi',\pi}(\bar{G})/O_{\pi'}(\bar{G}) \in \mathfrak{R}_1$. Thus $O_\pi^3(G)/O_\pi^2(G) \in \mathfrak{R}_1$, and the rest of the theorem follows from [4] Lemma 4.8.

Corollary (C1). *Let G be upper π-separable, locally π-soluble and Sylow π-sparse. Then the Sylow π-subgroups of $G/O_\pi(G)$ are countable. If π is finite then $G/O_{\pi,\pi'}(G)$ is countable.*

This observation is relevant to the remarks made after the proof of Lemma 2.2. We notice that the hypotheses of Corollary C1 do not imply that $G/O_{\pi,\pi'}(G)$ is countable if π is infinite. For let p be a prime and let q_1, $q_2, \dots$ be an infinite sequence of distinct primes satisfying $p|q_i - 1$ for all i. Such a sequence exists by Dirichlet's theorem on the primes in an arithmetic progression. Let $Q = Q_1 \times Q_2 \times \dots$ where $|Q_i| = q_i$ for each i. Then the automorphism group of Q contains a subgroup P which is isomorphic to the Cartesian or complete direct product of a countable infinity of cyclic groups of order p. Let H be the semidirect product QP. Now Q isomorphic to a subgroup L of the multiplicative group of the algebraic closure K of $\boldsymbol{Z}_p$. We can view the additive group of K as a $\boldsymbol{Z}_p L$-module by allowing L to act by multiplication. Then every non-trivial element of L acts fixed point freely. Therefore there is $\boldsymbol{Z}_p Q$-module W such that $C_W(x) = 0$ for every $1 \neq x \in Q$, and if V is the induced module W^H, then we also have $C_V(x) = 0$ for every $1 \neq x \in Q$, as $Q \lhd H$.

Now let G be the semidirect product $G = VH = VQP$, V being written multiplicatively, and let $\pi = \{q_1, q_2, \dots\}$. Any countable subgroup of VQ lies in one of the form $V_0 Q$, where V_0 is a countable subgroup of V normalized by Q. The fact that $C_{V_0}(x) = 1$ for all $1 \neq x \in Q$, combined with ([4] Lemma 4.3) shows that $V_0 Q$ is Sylow π-sparse. Hence VQ is Sylow π-sparse, and since G/VQ is a π'-group, G is Sylow π-sparse (and so Sylow π-integrated by Theorem B). Obviously G is π-separable and locally π-soluble; in fact G is soluble of derived length three.

Now $C_H(Q) = Q$ and so every non-trivial normal subgroup of H meets Q non-trivially. Hence $O_{\pi'}(H) = 1$ and also, since $C_Q(V) = 1$, we have $C_H(V) = 1$. Therefore $C_G(V) = V$. It follows that $O_\pi(G) = 1$, $O_\pi^2(G) = V$ and $O_\pi^3(G) = VQ$. Thus we even have that $G/O_\pi^3(G)$ is uncountable.

We now consider Theorem D, which will be deduced from the following two results; these may be of some independent interest.

LEMMA (3.6). *Let G be a countable and non-trivial locally π-soluble group. Suppose that G is Sylow π-sparse and $O_{\pi'\pi}(G) = 1$. Then G contains a non-trivial normal subgroup H with the following property: if τ is a set of primes such that $\tau \cap \pi = \emptyset$ and H is Sylow τ-sparse, then every abelian τ-subgroup of H has finite rank.*

Notice that by ([5] Theorem A) the hypotheses of Lemma 3.6 cannot be satisfied if $\pi = \{p\}$ and p is odd, and it may be that they cannot be satisfied at all.

COROLLARY (3.7). *With the hypotheses of Lemma 3.6, suppose further that τ is a set of primes such that $\tau \cap \pi = \emptyset$ and G is locally τ-soluble and Sylow τ-sparse. Then $O_{\tau'\tau}(G) \neq 1$.*

PROOF OF COROLLARY 3.7. Let H be a normal subgroup of G with the properties given by Lemma 3.6. Then since G is Sylow τ-sparse so is H. Therefore every abelian τ-subgroup of H has finite rank and so by Corollary 3.4 H is τ-separable. Since $H \neq 1$ this gives $O_{\tau'\tau}(H) \neq 1$, and since $H \lhd G$ this gives $O_{\tau'\tau}(G) \neq 1$. Thus Corollary 3.7 follows from Lemma 3.6.

PROOF OF THEOREM D. We now deduce Theorem D from the foregoing results. We have that G is locally soluble, $\sigma(G) = \pi_1 \cup \cdots \cup \pi_n$ is a partition of $\sigma(G)$ and G is Sylow π_i-sparse for $1 \leqq i \leqq n$. With these hypotheses, we first prove

(*) *G is π_i-separable for $1 \leqq i \leqq n$.*

If this result does not hold then there exists a group G which satisfies our hypotheses but is not π_1-separable, and we may suppose that G is chosen to make n as small as possible subject to these conditions. Then $n > 1$. Since G is not π_1-separable the π_1-lengths of the finite subgroups of G must be unbounded ([1] Corollary 3.10) and so G contains a countable subgroup which is not π_1-separable. In obtaining a contradiction we may therefore suppose without loss of generality that G is countable.

Let X be the terminus of the upper π_1-series of G, as defined at the beginning of this section. Then X is upper π_1-separable, locally soluble and Sylow π_1-sparse and so is π_1-separable by Theorem C. Hence $X < G$. By considering G/X instead of G and using Lemma 3.2, we may therefore suppose that $G \neq 1$ but $O_{\pi'_1\pi_1}(G) = 1$. It follows from Corollary 3.7 that $O_{\pi'_2\pi_2}(G) = 1$. Consequently, since $O_{\pi_2}(G) \leqq O_{\pi'_1}(G) = 1$, we must have $R = O_{\pi'_2}(G) \neq 1$. Now R is Sylow π_i-sparse for $1 \leqq i \leqq n$, and $\pi'_2 \cap \sigma(G) = \pi_1 \cup \pi_3 \cup \cdots \cup \pi_n$. Therefore by the choice of n, R is π_1-separable. In particular $O_{\pi'_1\pi_1}(R) \neq 1$. But $R \lhd G$ and so we have $O_{\pi'_1\pi_1}(G) \neq 1$, a contradiction. This establishes (*).

Now the class $\mathfrak{R}_1$ is evidently closed under finite extensions and so it now follows from Theorem C that, if $1 \leqq i \leqq n$, then the π_i-subgroups of $G/O_{\pi_i}(G)$ all belong to $\mathfrak{R}_1$. Let $L = \prod_{i=1}^n O_{\pi_i}(G)$ and let M/L be any locally nilpotent subgroup of G/L. Then $M/L = M_1/L \times \cdots \times M_n/L$, where M_i/L is the Sylow π_i-subgroup of M/L. Let $X_i = O_{\pi_i, \pi'_i}(G)$. Then $O_{\pi_i}(G) \leqq L \leqq X_i$ and so X_i/L is a π'_i-group. Hence $M_i \cap X_i = L$ and so $M_i/L \cong M_iX_i/X_i$, a locally nilpotent π_i-subgroup of G/X_i. By Theorem C, M_iX_i/X_i is abelian-by-finite and of finite rank. It follows that M/L has these properties and so $G/L \in \mathfrak{R}_1$.

In establishing (ii), we may suppose without loss of generality that $\pi = \pi_1 \cup \cdots \cup \pi_j$ $(1 \leqq j \leqq n)$. We have to prove that $G \in \mathfrak{D}_\pi^S$ and by Lemma 2.1 we may assume that G is countable; also since the hypotheses on G are inherited by subgroups it suffices to prove that $G \in \mathfrak{D}_\pi$. It follows from (*) that G is π-separable and so G has a finite series of normal subgroups in which each factor is either a π-group or a π'-group. By using Lemma 3.2 and induction on the length of such a series we may assume that G contains a normal subgroup N which is either a π-group or a π'-group and is such that $G/N \in \mathfrak{D}_\pi$. If N is a π-group it is then immediate that $G \in \mathfrak{D}_\pi$. Otherwise let $S, T \in \mathrm{Syl}_\pi G$. Then as G/N is Sylow π-connected there is an element $x \in G$ such that $\langle S^xN/N, TN/N\rangle = U/N$ is a π-group. Since U/N is countable there is a π-subgroup V of U such that $U = NV$, $N \cap V = 1$. Let Y be any subgroup of V. Then since Y is countable and locally soluble we have $Y = \langle Y_1, \cdots, Y_j\rangle$, where Y_i is a π_i-group. Now NY_i is Sylow π_i-sparse by hypothesis, and so by the argument of ([4] Lemma 4.2) there is a finite subgroup F_i of Y_i such that $C_N(Y_i) = C_N(F_i)$. Then $C_N(Y) = \bigcap_{i=1}^j C_N(Y_i) = \bigcap_{i=1}^j C_N(F_i) = C_N(F)$, where $F = \langle F_1, \cdots, F_n\rangle$. Since F is finite it now follows from ([4] Lemma 4.3) that $U = NV$ is Sylow π-connected. Therefore S^x and T, which are Sylow π-subgroups of U, are conjugate in U. This completes the deduction of Theorem D from Lemma 3.6.

Proofs of Corollaries D1 and D2. To obtain Corollary D1 we apply Theorem D to the partition of $\sigma(G)$ into one-element sets. If $\sigma(G) = \{p_1, \cdots, p_n\}$ then evidently $\rho(G) = \prod_{i=1}^n O_{p_i}(G)$; also by ([4] Lemma 4.8) an $\mathfrak{R}_1$-group X such that $\sigma(X)$ is finite satisfies Min.

If G is Sylow p- and p'-sparse then Theorem D gives that G is p-separable. The desired conclusion then follows from Theorem C.

It remains to establish Lemma 3.6. In order to do this we require one or two technical results, the first of which is similar to ([4] Lemma 3.2).

LEMMA (3.8). *Let G be Sylow π-sparse and let S be a π-subgroup of G. Then there exists a finite subgroup F of S such that if $x \in G$ and $F^x \leqq S$, then $\langle S, S^x \rangle \in \mathfrak{D}_\pi$.*

PROOF. Suppose that G is a group containing a π-subgroup S to which no such F corresponds. We shall exhibit a countable subgroup H of G such that $|\mathrm{Syl}_\pi H| = 2^{\aleph_0}$. To do this we recursively construct finite subgroups $F_0, F_1, \dots$ of G and sets Σ_n ($n \geqq 0$) of 2^n π-subgroups of F_n such that:

(i) Each member of Σ_n has the form $S^x \cap F_n$ with $x \in F_n$.

(ii) No two distinct members of Σ_n generate a π-group.

(iii) Each member of Σ_{n-1} is contained in two distinct members of Σ_n ($n > 0$).

To begin the construction put $F_0 = 1$, $\Sigma_0 = \{1\}$. Assume that we have obtained Σ_m, and let

$$\Sigma_m = \{S^{x_i} \cap F_m;\ 1 \leqq i \leqq 2^m\},$$

with $x_i \in F_m$. Now the subgroup $S \cap F_m$ does not have the property of F in the statement of the lemma, and so there is an element $y \in G$ such that $(S \cap F_m)^{y^{-1}} \leqq S$ but $\langle S, S^{y^{-1}} \rangle$ is not a π-group. Therefore $\langle S, S^y \rangle = \langle S, S^{y^{-1}} \rangle^y$ is not a π-group, and so there is a finite subgroup $F_{m+1} \geqq \langle F_m, y \rangle$ such that

$$(1) \qquad \langle S \cap F_{m+1}, S^y \cap F_{m+1} \rangle \notin \mathfrak{D}_\pi.$$

Let Σ_{m+1} consist of the subgroups $S^{x_i} \cap F_{m+1}$ and $S^{yx_i} \cap F_{m+1}$ ($1 \leqq i \leqq 2^m$). Then (i) holds. Now since $(S \cap F_m)^{y^{-1}} \leqq S$ and $y \in F_{m+1}$ we have $(S \cap F_m)^{y^{-1}} \leqq S \cap F_{m+1}$, whence $S \cap F_m \leqq (S \cap F_{m+1})^y = S^y \cap F_{m+1}$. Therefore, as $x_i \in F_m$, $S^{x_i} \cap F_m = (S \cap F_m)^{x_i}$ is contained in both $S^{x_i} \cap F_{m+1}$ and $S^{yx_i} \cap F_{m+1}$. Therefore by (ii) with $n = m$, the group generated by $S^{x_i} \cap F_{m+1}$ and either of $S^{x_j} \cap F_{m+1}$ and $S^{yx_j} \cap F_{m+1}$ ($j \neq i$) is not a π-group. Nor is $\langle S^{x_i} \cap F_{m+1}, S^{yx_i} \cap F_{m+1} \rangle = \langle S \cap F_{m+1}, S^y \cap F_{m+1} \rangle^{x_i}$, by (1). Therefore (ii) holds with $n = m+1$. In particular the subgroups $S^{x_i} \cap F_{m+1}$ and $S^{yx_i} \cap F_{m+1}$ are distinct, and so (iii) holds by the remarks above.

Now let $H = \bigcup_{i=0}^{\infty} F_i$, a countable subgroup of G. If $S_0 \leqq S_1 \leqq \cdots$ is a tower with $S_i \in \Sigma_i$ then $\bigcup_{i=0}^{\infty} S_i$ is a π-subgroup of G. By (iii) there are $2^{\aleph_0}$ such towers, and by (ii), two subgroups obtained as the unions of distinct towers cannot generate a π-group. Hence H has a set Σ of $2^{\aleph_0}$ π-subgroups, no two of which generate a π-group. Each member of Σ lies in some Sylow π-subgroup of H and the Sylow π-subgroups obtained thus are all distinct. Hence $|\mathrm{Syl}_\pi H| = 2^{\aleph_0}$, as required.

The next lemma is essentially the key step in establishing Lemma 3.6 and Theorem D.

LEMMA (3.9). *Let* $1 = F_0 \leqq F_1 \leqq \cdots$ *be a tower of finite subgroups of a group* $G = \bigcup_{i=0}^{\infty} F_i$. *Let* A *be an abelian* π-*subgroup of* G *and for* $i = 0, 1, \ldots$ *let* R_i *be a* π'-*subgroup of* G *normalized by* $A_i = A \cap F_i$. *Assume that the ranks of the groups* $A_i/C_{A_i}(R_i)$ *are unbounded. Then* $|\mathrm{Syl}_\pi G| = 2^{\aleph_0}$.

PROOF. We begin with some preliminary remarks. Suppose that $l \geqq 0$ and $D \leqq A_l$. For each $i \geqq 0$ let S_i be a subgroup of R_i normalized by A_i. Then since A_i centralizes D, A_i normalizes $C_{S_i}(D) = S_i \cap C_G(D)$.
Suppose further that A_l contains subgroups E such that the groups $A_i/C_{A_i}(C_{S_i}(E))$ $(i \geqq 0)$ have unbounded ranks and that D is chosen maximal among these subgroups. We than claim that

$$A_l/D \text{ is cyclic.} \tag{2}$$

In fact, suppose that $\bar{A}_l = A_l/D$ is not cyclic. Then $\bar{A}_l$ acts naturally on $C_{S_i}(D)$ if $i \geqq l$ and we have from ([4] Lemma 4.4) that $C_{S_i}(D)$ is generated by the centralizers in it of the non-trivial elements of $\bar{A}_l$. Therefore

$$C_{S_i}(D) = \langle C_{S_i}(D^*);\ D < D^* \leqq A_l \rangle \ (i \geqq l). \tag{3}$$

Now the subgroups $C_{S_i}(D^*)$ are A_i-invariant if $i \geqq l$ and it follows from (3) that $C_{A_i}(C_{S_i}(D)) = \cap\, C_{A_i}(C_{S_i}(D^*))$, the intersection running over subgroups D^* with $D < D^* \leqq A_l$. Therefore $A_i/C_{A_i}(C_{S_i}(D))$ is a subdirect product of the groups $A_i/C_{A_i}(C_{S_i}(D^*))$ with $D < D^* \leqq A_l$. Since there are only finitely many such subgroups D^*, and since the groups $A_i/C_{A_i}(C_{S_i}(D))$ $(i \geqq l)$ have unbounded ranks, there must be such a D^* for which the groups $A_i/C_{A_i}(C_{S_i}(D^*))$ $(i \geqq l)$ have unbounded ranks, contradicting the choice of D.

We now recursively construct elements $1 = x_0, x_1, \ldots$ of A and integers $n_0, n_1, \ldots$ such that, if $B_j = \langle x_0, \ldots, x_j \rangle$ and $R_{ij} = C_{R_i}(B_j)$, then

(i) $B_j \leqq A_{n_j}$ $(j \geqq 0)$.
(ii) $R_{n_j, j-1} > R_{n_j, j}$ $(j > 0)$.
(iii) For each $j \geqq 0$, the groups $A_i/C_{A_i}(R_{ij})$ $(i > 0)$. have unbounded ranks.

We begin by putting $x_0 = 1$, $n_0 = 0$. Then (i) holds. We have $R_{i,0} = R_i$, and so (iii) holds with $j = 0$ by hypothesis.

Assume now that we have $x_0, \ldots, x_k$ and $n_0, \ldots, n_k$ such that (i)–

(iii) hold for $j \leqq k$. Then in particular we have from (iii) that there exists an integer $n_{k+1} \geqq n_k$ such that

$$A_{n_{k+1}}/C_{A_{n_{k+1}}}(R_{n_{k+1},k}) \textit{ has rank } \geqq 2. \tag{4}$$

Now by (iii) with $j = k$, we can choose a subgroup D of $A_{n_{k+1}}$ maximal subject to the condition that the groups $A_i/C_{A_i}(C_{R_{i,k}}(D))$ have unbounded ranks. It follows from (2) and (4) that D does not centralize $R_{n_{k+1},k}$. Let x_{k+1} be an element of D which does not centralize $R_{n_{k+1},k}$ and let $B_{k+1} = \langle B_k, x_{k+1}\rangle$. Then (i) holds with $j = k+1$, and since $R_{n_{k+1},k+1}$ is just the centralizer in $R_{n_{k+1},k}$ of x_{k+1}, (ii) holds by the choice of x_{k+1}. Finally $R_{i,k+1} = C_{R_{i,k}}(x_{k+1}) \geqq C_{R_{i,k}}(D)$ and so (iii) holds with $j = k+1$, by the choice of D. Thus the construction can be carried out.

Let $S = \langle x_0, x_1, \ldots\rangle$. We apply Lemma 3.8 to S. Let F be any finite subgroup of S. Then $F \leqq B_{j-1}$ for suitable j, and by (ii) there exists an element $x \in R = R_{n_j}$ centralized by B_{j-1} but not by B_j. In particular $F^x \leqq S$. On the other hand $\langle S, S^x\rangle$ contains $\langle B_j, B_j^x\rangle$, which contains $[x, B_j]$. This is a non-trivial subgroup of the π'-group R, since B_j normalizes R by (i). Therefore $\langle S, S^x\rangle$ is not a π-group, and so no finite subgroup of S has the property of F in Lemma 3.7. Therefore $|\mathrm{Syl}_\pi H| = 2^{\aleph_0}$ for some subgroup H of G, and so $|\mathrm{Syl}_\pi G| = 2^{\aleph_0}$.

Proof of Lemma 3.6. We have a countable and nontrivial locally π-soluble group G with $|\mathrm{Syl}_\pi G| < 2^{\aleph_0}$ and $O_{\pi'\pi}(G) = 1$. We begin by constructing finite subgroups $1 = F_0 \leqq F_1 \leqq \cdots$ of G such that $G = \bigcup_{i=0}^{\infty} F_i$ and $F_i \cap O_{\pi'\pi}(F_{i+1}) = 1$. Let $1 = x_0, x_1, \ldots$ be the elements of G and suppose that F_n is a finite subgroup of G containing $\langle x_0, \ldots x_n\rangle$. Since $O_{\pi'\pi}(G) = 1$ we have that, for each $1 \neq x \in O_{\pi'\pi}(F_n)$, $x^G \notin \mathfrak{D}_{\pi'}\mathfrak{D}_\pi$. Therefore there is a finite subgroup G_x of G such that $x^{G_x} \notin \mathfrak{D}_{\pi'}\mathfrak{D}_\pi$. Put $F_{n+1} = \langle x_{n+1}, F_n, G_x; 1 \neq x \in O_{\pi'\pi}(F_n)\rangle$. Then $F_{n+1} \geqq \langle x_0, \ldots, x_{n+1}\rangle$ and $F_n \cap O_{\pi'\pi}(F_{n+1}) = 1$.

Since G is locally π-soluble but not π-separable it follows from Corollary 3.4 that G contains an abelian π-subgroup A of infinite rank. Let $R_i = O_{\pi'}(F_i)$ $(i \geqq 0)$. Then $A_i = A \cap F_i$ normalizes R_i for each i, and it follows from Lemma 3.9 that the ranks of the groups $A_i/C_{A_i}(R_i)$ are bounded by some integer n. Let X be a finite subgroup of rank $n+1$ of A. Then for each sufficiently large i there is a non-trival element of X which centralizes R_i. Since X is finite it follows that there is an element $1 \neq x \in X$ which centralizes infinitely many of the subgroups R_i. Since $O_{\pi'\pi}(F_j) \cap F_i = 1$ whenever $j > i$ we may assume, by omitting some of the F_i and renumbering the rest, that $x \in F_1$ and x centralizes every R_i.

Now let $H = x^G$. Then $1 \neq H \lhd G$ and $H = \bigcup_{i=0}^{\infty} H_i$, where $H_i = x^{F_i}$.

Now since $R_i \lhd F_i$ we have that $x^{F_i} = H_i$ centralizes R_i, and so $S_i = H_i \cap R_i$ is central in H_i. Since $H_i \lhd F_i$ we have $S_i = O_{\pi'}(H_i)$, and it follows that $O_{\pi'\pi}(H_i) = S_i \times T_i$ where$T_i = O_\pi(H_i)$. Since H_i is π-separable it now follows from Lemma 3.1 that $C_{H_i}(T_i) \leqq S_i \times T_i$. Now let τ be a set of primes with $\tau \cap \pi = \emptyset$ and suppose that H contains an abelian τ-subgroup B of infinite rank. Now $H_i \cap O_{\pi'\pi}(H_{i+1}) \leqq F_i \cap O_{\pi'\pi}(F_{i+1}) = 1$ and so if $B_i = B \cap H_i$ then $B_i \cap (S_{i+1} \times T_{i+1}) = 1$. Therefore B_i normalizes the τ'-subgroup T_{i+1} and acts faithfully on it. Since the groups B_i have unbounded ranks, Lemma 3.9 shows that $|\mathrm{Syl}_\tau H| = 2^{\aleph_0}$, so that H is not Sylow τ-sparse. This establishes Lemma 3.6, and completes the proof of Theorem D.

4. The class $\mathfrak{U}$

The following result is immediate from Corollary D1:

LEMMA (4.1). *Suppose that G is locally soluble, that $\sigma(G)$ is finite, and that G is Sylow p-sparse for all $p \in \sigma(G)$. Then $G \in \mathfrak{U}$.*

It is easy to see, by considering suitable direct products of finite groups, that in general a locally soluble group G may be Sylow p-sparse for all $p \in \sigma(G)$ without belonging to $\mathfrak{U}$. However, we now prove Theorem E, according to which if such a G is completely Sylow sparse, then $G \in \mathfrak{U}$.

PROOF OF THEOREM E. Let G be a locally soluble group. If $G \in \mathfrak{U}$ then G is completely Sylow integrated and so completely Sylow sparse, and so we assume G completely Sylow sparse in order to prove the converse.

We recall that a group H is called a radical group (in the sense of Plotkin [9]) if the upper L$\mathfrak{N}$-series of H, defined by $R_0 = 1$, $R_{\alpha+1}/R_\alpha = \rho(G/R_\alpha)$, $R_\mu = \bigcup_{\alpha<\mu} R_\alpha$ for ordinals α and limit ordinals μ, reaches H. It is well known and easy to prove that if H is a radical group then $R_1 \geqq C_H(R_1)$ ([9] Theorem 7); hence if R_1 is finite, so is H. Suppose now that H is a completely Sylow sparse radical group. Then R_4 is obviously π-separable for any set π, and so Theorem B gives that R_4 is completely Sylow integrated. Hence $R_4 \in \mathfrak{U}$ and so, by ([4] Theorem E), R_4/R_3 is finite. It follows from the above remarks that H/R_3 is finite, and so $R_n = H$ for some finite n. This gives $H \in \mathfrak{U}$. After that brief digression we now return to the proof of Theorem E and take H to be the terminus of the upper L$\mathfrak{N}$–series of our completely Sylow sparse group G. Then by our digression it suffices to prove $H = G$. We therefore assume $H < G$; then $L = G/H \neq 1$ but $\rho(L) = 1$. Also L is completely Sylow sparse and locally soluble. Therefore by Corollary D2 and the fact that $O_p(L) \leqq \rho(L)$, we have that H satisfies Min-p for all primes p. Let M be the subgroup of

L generated by the radicable abelian subgroups of L. Then M has no proper subgroup of finite index and by local solubility and Min-p, every chief factor of M is a finite elementary abelian group. Therefore every chief factor of M is central and M is locally nilpotent. Since $M \lhd G$ we must have $M = 1$, and since every p-subgroup of L contains a radicable abelian subgroup of finite index it follows that the Sylow p-subgroups of L are finite for each prime p.

Now L contains an abelian subgroup A of infinite rank, otherwise Gorčakov's theorem [2] shows that L has finite rank and a theorem of Kargapolov [7] gives that $L \in (\mathrm{L}\mathfrak{N})\mathfrak{A}\mathfrak{F}$, contradicting the fact that $L \neq 1$ but $\rho(L) = 1$. We may suppose that A is countably infinite, and by arguing as in the proof of Lemma 3.6 we can then construct a tower $1 = F_0 \leqq F_1 \leqq \cdots$ of finite subgroups of L such that $A \leqq \bigcup_{i=0}^{\infty} F_i = K$ and $F_i \cap \rho(F_{i+1}) = 1$. Here, of course, $\rho(F_i)$ is the Fitting subgroup of F_i.

Let $i > 0$ be a given integer and σ_i, π_i be given finite sets of primes with $\sigma_i \cap \pi_i = \emptyset$. Then since the Sylow p-subgroups of A are finite for each prime p but A has infinite rank there exists a prime q not belonging to $\sigma_i \cup \pi_i$ such that the Sylow q-subgroup A_q of A has rank at least i. Furthermore $A_q \leqq F_{n_i}$ for some n_i, which we may choose sufficiently large to ensure that F_{n_i} contains a Sylow $\sigma_i \cup \pi_i \cup \{q\}$-subgroup of K. Then A_q normalizes $R_i = \rho(F_{n_i+1})$, and since $F_i \cap R_i = 1$ we have that $C_{A_o}(R_i) = 1$ and that no prime in $\sigma_i \cup \pi_i \cup \{q\}$ divides $|R_i|$. We put $\pi_{i+1} = \pi_i \cup \{q\}$, $\sigma_{i+1} = \sigma_i \cup \sigma(R_i)$ and continue in this way. Finally let $\pi = \bigcup_{i=0}^{\infty} \pi_i$, $\sigma = \bigcup_{i=0}^{\infty} \sigma_i$, and let B be the Sylow π-subgroup of A. Then we have that $B_i = B \cap F_{n_i}$ normalizes R_i and that the rank of $B_i/C_{B_i}(R_i)$ is at least i. Also $\sigma(R_i) \leqq \sigma$ for all i, and $\sigma \cap \pi = \emptyset$. Therefore by Lemma 3.9, $|\mathrm{Syl}_\pi K| = 2^{\aleph_0}$. However, K is a countable subgroup of L, which is completely Sylow sparse, and so we have a contradiction. Thus Theorem E is established.

We conclude with the following remark, which is independent of what goes before.

LEMMA (4.2). $\mathfrak{U} \leqq (\mathrm{L}\mathfrak{N})\mathfrak{R}_1$.

PROOF. Let $R = \rho(G)$ and let A/R be a locally nilpotent subgroup of G/R. By ([4] Theorem E), A/R has finite rank, and so we only need show that A/R is abelian-by-finite. Since $G/R \in \mathfrak{A}^2\mathfrak{F}$ (by ([4] Theorem E)), we may evidently suppose that $G/R \in \mathfrak{A}^2$. Then $A' \leqq R_2$, the second term of the upper $\mathrm{L}\mathfrak{N}$-series of G. For each prime p let S_p be a Sylow p-subgroup of A. Now $A \in (\mathrm{L}\mathfrak{N})^2 \cap \mathfrak{U}$ and so by ([4] Lemma 6.4), $A/\rho(A) \in \mathfrak{R}_1$. Hence the Sylow p-subgroup of $A/\rho(A)$ is abelian for all but finitely many primes p, and so there is a finite set σ of primes such that $S'_p \leqq \rho(A)$ if $p \notin \sigma$. Since $\rho(A) \geqq R$, we have $[S'_p, R_{p'}] = 1$ if $p \notin \sigma$, where $R_{p'}$ is the Sylow p'-subgroup of R. However $S'_p \leqq R_2$, and by ([4] Lemma 6.3) we have that $C_{S_p \cap R_2}(R_{p'}) \leqq \rho(R_2) = R$. Therefore $S'_p \leqq R$ if $p \notin \sigma$, and hence the Sylow p-subgroup S_pR/R of A/R is abelian if $p \notin \sigma$. Therefore A/R is abelian-by-finite, and the lemma is proved.

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