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**The Clifford algebra and the Spinor group
of a Hilbert space**

THE CLIFFORD ALGEBRA
AND THE SPINOR GROUP OF A HILBERT SPACE

by

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1. Introduction

Clifford algebras constitute an essential tool in the study of quadratic forms. However, for infinite dimensional problems, it seems often natural to ask for some extra topological structure on these algebras. The present note studies such structures on an example which is the analogue for Hilbert spaces of arbitrary dimension of the standard algebras Cl^k for finite dimensional real vector spaces (the C_k's of Atiyah-Bott-Shapiro [2]). In particular, we give an explicit construction of the covering group of the nuclear orthogonal group; the main result appears in section 10. The present work relies heavily on a theorem due to Shale and Stinespring [23] and on general results on Banach-Lie groups which can be found in Lazard [19].

2. The Clifford algebra of a quadratic form

This section is only to fix notations. For a more detailed exposition, see Atiyah-Bott-Shapiro [2], Bourbaki [5] or Chevalley [6].

Let H be a real vector space of arbitrary dimension and let $Q : H \to R$ be a quadratic form on H. The *Clifford algebra of Q*, denoted by $Cl(Q)$, is a real algebra with unit e_0, defined as a quotient of the tensor algebra of H. There is a *canonical injection* $i_Q : H \to Cl(Q)$ by which H can be identified to a subspace of $Cl(Q)$. The Z-graduation on the tensor algebra of H induces a Z_2-graduation on $Cl(Q)$, and the corresponding decomposition will be denoted by $Cl(Q) = Cl^+(Q) \oplus Cl^-(Q)$. On $Cl(Q)$, there is defined the *principal automorphism* α and the *principal antiautomorphism*

β. Both α and β are involutions on $Cl(Q)$; the restriction of α to $Cl^+(Q)$ is the identity, the restriction of α to $Cl^-(Q)$ is minus the identity; the restriction of β to H is the identity.

Let E be a subspace of H and let Q_E be the restriction of Q to E. The injection $E \to H$ induces an injection $Cl(Q_E) \to Cl(Q)$ by which $Cl(Q_E)$ can be identified to a subalgebra of $Cl(Q)$. Let $(e_i)_{i \in I}$ be a basis of H, and suppose I is given a total order. Let $S(I)$ be the set of strictly increasing finite sequences of elements of I; for $s = (i_1, \cdots, i_k) \in S(I)$, let e_s be the product $e_{i_1} \cdots e_{i_k}$ in $Cl(Q)$; then $(e_s)_{s \in S(I)}$ is a basis of the vector space $Cl(Q)$; the identity e_0 of $Cl(Q)$ corresponds to the empty sequence. We will write $S(k)$, or even S when there is no risk of confusion, instead of $S(\{1, \cdots, k\})$ for any strictly positive integer k.

The rest of this note is concerned with the case in which H is a real prehilbert space where the scalar product is denoted by $\langle,\rangle$, and in which Q is the form $x \mapsto \varepsilon|x|^2$ ($\varepsilon = \pm 1$). We will denote by $X \mapsto X^*$ the composition of β and α if $\varepsilon = -1$ (the involutions α and β commute) and the antiautomorphism β if $\varepsilon = +1$.

PROPOSITION 1. *Let H be a real prehilbert space of infinite dimension and let Q be the form $x \mapsto \varepsilon|x|^2$ as above. Then $Cl(Q)$ and $Cl^+(Q)$ are simple and their centers both coincide with the scalar multiples of the identity.*

PROOF. Let E be a finite even dimensional subspace of H; it is a classical result that $Cl(Q_E)$ is simple. Hence $Cl(Q)$ is locally simple, i.e. any two of its elements are contained in some simple subalgebra of it. It follows that $Cl(Q)$ is simple. The other affirmations of the proposition are proved the same way. •

Proposition 1 is well-known: Doplicher-Powers [10], Størmer [26].

3. Hilbert algebra structure on the Cl^k's and Cl'^k's

In this section, $Cl^{\varepsilon, k}$ denotes the Clifford algebra of the quadratic form $Q : x \mapsto \varepsilon|x|^2$ defined on the euclidean space R^k ($\varepsilon = \pm 1$).

LEMMA 2. *There exists a unique linear form λ^ε on $Cl^{\varepsilon, k}$ such that*

(i) $\lambda^\varepsilon(e_0) = 1$

(ii) $\lambda^\varepsilon(XY) = \lambda^\varepsilon(YX)$ for all $X, Y \in Cl^{\varepsilon, k}$

(iii) $\lambda^\varepsilon(\alpha(X)) = \lambda^\varepsilon(X)$ for all $X \in Cl^{\varepsilon, k}$.

PROOF. *Existence*: Let $(e_i)_{1 \leq i \leq k}$ be the canonical basis of R^k and let $(e_s)_{s \in S}$ be the associated basis of $Cl^{\varepsilon, k}$. Let $X = \sum_{s \in S} X^s e_s$ be in $Cl^{\varepsilon, k}$; set $\lambda^\varepsilon(X) = X^0$. Then λ^ε enjoys properties (i) to (iii).

Unicity: Let $e_s = e_{i_1} \cdots e_{i_p}$ be a basis element in $Cl^{\varepsilon,k}$. If p is odd, $\lambda^\varepsilon(e_s)$ must be zero by (iii). If p is even and not zero,

$$\lambda^\varepsilon(e_{i_1} \cdots e_{i_p}) = \lambda^\varepsilon(e_{i_2} \cdots e_{i_p} e_{i_1}) = \lambda^\varepsilon(-e_{i_1} e_{i_2} \cdots e_{i_p})$$

by (ii), so that $\lambda^\varepsilon(e_s) = 0$. The unicity follows. •

By lemma 2 and a canonical procedure (see e.g. Dieudonné [8], chap. XV, § 6), one can define a scalar product on $Cl^{\varepsilon,k}$:

$$\langle X, Y \rangle = \lambda^\varepsilon(XY^*) \text{ for all } X, Y \in Cl^{\varepsilon,k}.$$

Equivalently, for any basis $(e_s)_{s \in S}$ of $Cl^{\varepsilon,k}$ associated to an orthonormal basis $(e_i)_{1 \le i \le k}$ of R^k, and if $X = \sum_{s \in S} X^s e_s$ and $Y = \sum_{s \in S} Y^s e_s$ are two elements of $Cl^{\varepsilon,k}$, one can define $\langle X, Y \rangle = \sum_{s \in S} X^s Y^s$.

PROPOSITION 3. *The scalar product $\langle , \rangle$ and the involution $X \mapsto X^*$ turn $Cl^{\varepsilon,k}$ into a real Hilbert algebra.*

PROOF. It suffices to check the identities $\langle X^*, Y^* \rangle = \langle Y, X \rangle$ and $\langle XZ, Y \rangle = \langle X, YZ^* \rangle$ for all $X, Y, Z \in Cl^{\varepsilon,k}$, and they follow trivially from lemma 2. •

NOTATIONS. Instead of $Cl^{\varepsilon,k}$ and λ^ε, one writes as well Cl^k and λ when $\varepsilon = -1$, and Cl'^k and λ' when $\varepsilon = +1$.

REMARKS.

1. The definition of a real Hilbert algebra is analogue to that of a complex Hilbert algebra as given for example in Diximer [9], chap. I, § 5.

2. Complete Hilbert algebras have been introduced by Ambrose [1] under the name of H^*-algebras. The finite dimensional $Cl^{\varepsilon,k}$ are clearly H^*-algebras (see however Ambrose's remark following his condition 3 on page 366).

3. The canonical injection mentioned in section 2 is an isometry of the euclidean space R^k into the Hilbert algebra $Cl^{\varepsilon,k}$, so that it is still safe to identify R^k to a subspace of $Cl^{\varepsilon,k}$.

4. If $j \le k$, the inclusion $Cl^{\varepsilon,j} \to Cl^{\varepsilon,k}$ induced by the inclusion $R^j \to R^k$ is an isometry and the involution in $Cl^{\varepsilon,j}$ is the restriction of the involution in $Cl^{\varepsilon,k}$, so that one can still identify $Cl^{\varepsilon,j}$ to a subalgebra of $Cl^{\varepsilon,k}$.

5. If $(e_i)_{1 \le i \le k}$ is an orthonormal basis of R^k, then $(e_s)_{s \in S(k)}$ is an orthonormal basis of $Cl^{\varepsilon,k}$.

4. Hilbert algebra structure on $Cl(H)$ and $Cl'(H)$

In this section, H is an infinite dimensional real prehilbert space and $Cl^\varepsilon(H)$ is the Clifford algebra of the form $Q : x \mapsto \varepsilon|x|^2$ ($\varepsilon = \pm 1$).

Define the linear form λ^ε on $Cl^\varepsilon(H)$ as follows: Let $X \in Cl^\varepsilon(H)$; let E be a finite dimensional subspace of H such that $X \in Cl(Q_E)$; define $\lambda^\varepsilon(X)$ to be the value of λ^ε on X as calculated in $Cl(Q_E)$, which clearly does not depend on E. Then lemma 2 holds when $Cl^{\varepsilon,k}$ is replaced by $Cl^\varepsilon(H)$. For $X, Y \in Cl^\varepsilon(H)$, define as in section 3 $\langle X, Y \rangle = \lambda^\varepsilon(XY^*)$.

PROPOSITION 4. *The scalar product $\langle,\rangle$ and the involution $X \mapsto X^*$ turn $Cl^\varepsilon(H)$ into a real Hilbert algebra.*

PROOF. The only non-trivial point to check is the separate continuity of the multiplication in $Cl^\varepsilon(H)$. We will check that $\sup\langle XY, XY \rangle < \infty$ for all $X \in Cl^\varepsilon(H)$, where the supremum is taken over all Y's inside the unit ball of $Cl^\varepsilon(H)$.

Let X be fixed, and let Y be an arbitrary element of $Cl^\varepsilon(H)$. Let E [resp. $E(Y)$] be a finite dimensional subspace of H such that $X \in Cl(Q_E)$ [resp. $Y \in Cl(Q_{E(Y)})$]. Let $(e_1, \cdots, e_n)$ be an orthonormal basis of $E + E(Y)$ such that $(e_1, \cdots, e_k)$ is an orthonormal basis of E. Write $X = \sum_{s \in S(k)} X^s e_s$ and $Y = \sum_{s \in S(n)} Y^s e_s$. The inequality

$$\langle XY, XY \rangle \leqq \left(\sum_{s \in S(k)} |X^s| \right)^2 \langle Y, Y \rangle$$

can now be easily proved by induction on the number of non zero coordinates X^s of X; hence the map

$$\begin{cases} Cl^\varepsilon(H) \to Cl^\varepsilon(H) \\ \quad Y \quad \mapsto \quad XY \end{cases} \text{ is continuous. } \bullet$$

PROPOSITION 5.
(i)   The injection $H \to Cl^\varepsilon(H)$ is an isometry.
(ii) For any subspace E of H, the injection $Cl(Q_E) \to Cl^\varepsilon(H)$ is an isometry (Q_E as in section 2).
(iii) E is dense in H if and only if $Cl(Q_E)$ is dense in $Cl^\varepsilon(H)$.

The proofs are trivial. $\bullet$

NOTATIONS.
Let $\mathfrak{H}$ be the completion of the prehilbert space $Cl^\varepsilon(H)$; the norm on $\mathfrak{H}$ will be denoted by $\| \ \|_2$. When we shall want to emphasize the Hilbert algebra structure of $Cl^\varepsilon(H)$, we shall use the more complete notation $(\mathfrak{H}, Cl^\varepsilon(H), *)$. The involution α [resp. β, $X \mapsto X^*$] extends uniquely to an isometric involution on $\mathfrak{H}$ which will again be denoted by the same symbol. Note that as vector spaces $\mathfrak{H} \neq Cl^\varepsilon(H)$.

The maximal *Hilbert algebra of bounded elements* in $(\mathfrak{H}, Cl^\varepsilon(H), *)$ will be denoted by $(\mathfrak{H}, Cl_2^\varepsilon(H), *)$ (see Dixmier [9], chap I, § 5, no 3 for the definition). For any $A \in Cl_2^\varepsilon(H)$, L_A will denote the bounded operator

on $\mathfrak{H}$ which extends the left multiplication by A on $Cl_2(H)$; one has $(L_A)^* = L_{(A^*)}$. Similarly for the right multiplication R_A. For any $X \in \mathfrak{H}$, L_X' will denote the (not necessarily continuous) operator on $\mathfrak{H}$ with domain $Cl_2(H)$ defined by $L_X'(A) = R_A(X)$ for all $A \in Cl_2^\varepsilon(H)$. Similarly for the right multiplication R_X'.

Instead of $(\mathfrak{H}, Cl_2^\varepsilon(H), *)$, one writes as well $(\mathfrak{H}, Cl_2(H), *)$ when $\varepsilon = -1$, and $(\mathfrak{H}, Cl_2'(H), \beta)$ when $\varepsilon = +1$.

5. The involutive Lie algebras spin$(H; C_0)$ and spin$H; C_2)$

In this section, H is an infinite dimensional real *Hilbert* space and $Cl^\varepsilon(H)$ is the Clifford algebra of the form $Q : x \mapsto \varepsilon|x|^2$ $(\varepsilon = \pm 1)$.

Let $o(H; C_0)$ be the Lie algebra of all skew-adjoints finite rank operators on H. This Lie algebra is simple, and is the 'classical compact Lie algebra of finite rank operators on H, of type BD' as defined in [14]. Its completion with respect to the Hilbert-Schmidt norm $\| \ \|_2$ is the compact L^*-algebra denoted by $o(H; C_2)$. The purpose of this section is to identify these two Lie algebras as sub Lie algebras of the ad hoc Clifford algebras.

PROPOSITION 6. *Let* **spin**$(H; C_0)$ *be the subspace of* $Cl^\varepsilon(H)$ *span by the elements of the form* $[x, y]$ *where x, y are in H. Then*:

(i)   **spin**$(H; C_0)$ *is a sub Lie algebra of* $Cl^\varepsilon(H)$.

(ii) $X + X^* = 0$ *for all* $X \in$ **spin**$(H; C_0)$.

(iii) *If* $X \in$ **spin**$(H; C_0)$, *then* $Xy - yX$ *is in H for all y in H; the operator defined in this way on H, denoted by* $D\rho_0(X)$, *is in* $o(H; C_0)$.

(iv) *The map*

$$D\rho_0 : \begin{cases} \mathbf{spin}\,(H; C_0) \to & o(H; C_0) \\ \quad X & \mapsto (y \mapsto Xy - yX) \end{cases}$$

is an isomorphism of involutive Lie algebras.

PROOF.

Step one. Suppose for convenience that $\varepsilon = -1$ and let $x, y, z, t \in H$. Then the two relations

$$\tfrac{1}{4}[y, [x, z]] = \langle x, y \rangle z - \langle y, z \rangle x$$

$$[[y, t], [x, z]] = 4\langle x, y \rangle [z, t] - 4\langle y, z \rangle [x, t]$$
$$+ 4\langle x, t \rangle [y, z] - 4\langle t, z \rangle [y, x]$$

can be proved as in finite dimensions (see Jacobson [16], chap. VII, § 6, formulas (29) and (30)). The first one clearly implies the first half of (iii) and the second one implies (i); statement (ii) can be readily checked.

Step two. Let $X \in \mathbf{spin}\,(H; C_0)$; proposition 4 says in particular that the adjoint of the operator $D\rho_0(X)$ is $D\rho_0(X^*)$. It follows from (ii) that $D\rho_0(X)$ is skew-adjoint, hence that it is an element of $o(H; C_0)$, which ends the proof of (iii).

Step three. The map $D\rho_0$ of (iv) is clearly a homomorphism of involutive Lie algebras. Let $X \in \mathbf{spin}\,(H; C_0)$ and let $(e_i)_{i \in N^*}$ be an orthonormal basis of H such that X can be expressed as a finite sum:

$$X = \sum_{1 \leq i < j \leq k} X^{i,j}[e_i, e_j] = \sum_{1 \leq i < j \leq k} 2 X^{i,j} e_i e_j.$$

Considered as an element of $\mathfrak{H}$, the norm of X is given by

$$(\|X\|_2)^2 = 4 \sum_{1 \leq i < j \leq k} (X^{i,j})^2.$$

Now the matrix representation of $D\rho_0(X)$ with respect to the basis $(e_i)_{i \in N^*}$ of H is

$$\begin{pmatrix} 0 & -4X^{1,2} & -4X^{1,3} & \cdots \\ 4X^{1,2} & 0 & -4X^{2,3} & \cdots \\ 4X^{1,3} & 4X^{2,3} & 0 & \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}.$$

Hence the Hilbert-Schmidt norm of $D\rho_0(X)$ is given by

$$(\|D\rho_0(X)\|_2)^2 = 32 \sum_{1 \leq i < j \leq k} (X^{i,j})^2 = (2\sqrt{2}\|X\|_2)^2.$$

It follows that $D\rho_0$ is injective.

Step four. Let $Y \in o(H; C_0)$. Again by writing Y as a matrix with finitely many non zero entries, with respect to an ad hoc orthonormal basis of H, one can easily find $X \in \mathbf{spin}\,(H; C_0)$ such that $D\rho_0(X) = Y$. This ends the proof of (iv). ●

COROLLARY 7. *Let* $\mathbf{spin}\,(H; C_2)$ *be the closure of* $\mathbf{spin}\,(H; C_0)$ *in* $\mathfrak{H}$. *Then* $\mathbf{spin}\,(H; C_2)$ *is a real* L^*-*algebra, the map* $D\rho_0$ *has a unique continuous extension*

$$D\rho_2 : \mathbf{spin}\,(H; C_2) \to o(H; C_2)$$

which is an isomorphism of involutive Lie algebras, and the equality

$$\|D\rho_2(X)\|_2 = 2\sqrt{2}\|X\|_2$$

holds for all $X \in \mathbf{spin}\,(H; C_2)$.

The proof is immediate; for the notion of a real L^*-algebra, see [14] and the references given there. ●

REMARK.
Sections 2 to 5 (and sections 7 and 8 below) hold in the same way when

$H = H^p \oplus H^q$ is the orthogonal sum of two prehilbert spaces (Hilbert spaces in section 5) and when Q is the form

$$\begin{cases} H^p \oplus H^q \to \quad \mathbf{R} \\ x \oplus y \quad \mapsto -|x|^2 + |y|^2. \end{cases}$$

Then the Clifford algebra of Q, say $Cl^{p,q}(H)$, is a Hilbert algebra for the scalar product defined as above and for the involution defined by $X^* = \gamma(\beta(X)) = \beta(\gamma(X))$, where γ is the automorphism of the Clifford algebra which extends

$$\begin{cases} H^p \oplus H^q \to H^p \oplus H^q \\ x \oplus y \quad \mapsto (-x) \oplus y \end{cases}$$

(the dimension of H^p is p, that of H^q is q, p and q are either positive integers or ∞). In proposition 6, the subspace generated by the $[x, y]$'s is now a Lie algebra $\mathbf{spin}(H, p, q; C_0)$, and $D\rho_0$ is an isomorphism of involutive Lie algebras

$$\mathbf{spin}(H, p, q; C_0) \to o(H, p, q; C_0)$$

where the last Lie algebra is defined as in [14]. Corollary 7 holds with the corresponding modification.

Clearly, the complexification of $Cl^{p,q}(H)$ does not depend of the pair (p, q) but only of the sum $p + q = \dim(H) \in N \cup \{\infty\}$. This suggests that the consideration of the $Cl^{p,q}(H)$'s in general is not essential from a physical point of view (canonical anticommutation relations; see for example Guichardet [11] and Slawny [24]). Moreover, the complexity in sections 6 and 9-10 below would be very much increased if arbitrary pairs (p, q) were to be dealt with.

Consequently, we restrict ourselves to the cases $(p, q) = (\dim H, O)$ and $(p, q) = (O, \dim H)$. About the $Cl^{p,q}(H)$'s when $p + q$ is finite, see however Karoubi [17].

6. Bogoliubov automorphisms

In this section, H is a Hilbert space.

DEFINITION. Let $(\mathfrak{H}, Cl_2^\varepsilon(H), *)$ be as in section 4. An *automorphism* of this Hilbert algebra is an orthogonal map $\varphi : \mathfrak{H} \to \mathfrak{H}$ whose restriction to $Cl_2^\varepsilon(H)$ is an automorphism of involutive algebra which preserves the identity; the group of all these φ's is denoted by $\mathrm{Aut}(Cl_2^\varepsilon(H))$. A *Bogoliubov automorphism* of $(\mathfrak{H}, Cl_2^\varepsilon(H), *)$ is an automorphism φ such that $\varphi(H) = H$; the corresponding group is denoted by $\mathrm{Bog}(Cl_2^\varepsilon(H))$.

Let $O(H)$ be the group of all orthogonal operators on H. Let $U \in O(H)$;

by the universal property of Clifford algebras, U extends to an automorphism $Cl(U)$ of $Cl^\varepsilon(H)$, which is clearly a *-automorphism. If λ^ε is defined as in the beginning of section 4, then the linear form $\lambda^\varepsilon \cdot Cl(U)$ still enjoys properties (i) to (iii) of lemma 2; hence $\lambda^\varepsilon \cdot Cl(U) = \lambda^\varepsilon$ and $Cl(U)$ defines an orthogonal operator on $\mathfrak{H}$, which will again be denoted by the same symbol. It is moreover elementary to check that $Cl(U)$ is a *-automorphism of the algebra of bounded elements $Cl_2^\varepsilon(H)$. In other words, there is a canonical group-isomorphism

$$Cl : O(H) \rightarrow \text{Bog}(Cl_2^\varepsilon(H)).$$

LEMMA 8. *Let U be in $O(H)$. Then the following are equivalent:*

(i) *There exists an invertible element $u \in Cl_2^\varepsilon(H)$ such that $Cl(U) = L_u \cdot R_{u^{-1}}$.*

(ii) *There exists a unique (up to multiplication by a scalar) invertible element $u \in Cl_2^\varepsilon(H)$ such that $Cl(U) = L_u \cdot R_{u^{-1}}$, and one has either $u \in Cl_2^{\varepsilon+}(H)$, or $u \in Cl_2^{\varepsilon-}(H)$.*

PROOF. It is easy to check that the center of $Cl_2^\varepsilon(H)$ is the same as that of $Cl^\varepsilon(H)$, namely that it consists of the multiples of the identity only (proposition 1). Hence the only thing to prove is the second part of (ii). But as $U = (-I)U(-I)$, one has $Cl(U) = \alpha \cdot Cl(U) \cdot \alpha$; if $Cl(U) = L_u R_{u^{-1}}$, then $\alpha \cdot Cl(U) \cdot \alpha = L_{\alpha(u)} R_{\alpha(u)^{-1}}$ and $\alpha(u)$ must be a multiple of u; as α is an involution, it follows that $\alpha(u) = \pm u$. •

DEFINITION. Let $Cl(U)$ be a Bogoliubov automorphism of $(\mathfrak{H}, Cl_2^\varepsilon(H), *)$. Then $Cl(U)$ is said to be *inner* if it satisfies conditions (i) and (ii) of lemma 8, and *outer* if not. Notations being as in (ii), $Cl(U)$ is said to be *even* if $u \in Cl_2^{\varepsilon+}(H)$ and *odd* if $u \in Cl_2^{\varepsilon-}(H)$.

COROLLARY 9. *Let $Spin(H; vN)$ be the set of all unitary even elements $u \in Cl_2^{\varepsilon+}(H)$ such that $uHu^{-1} = H$. Then $Spin(H; vN)$ is a subgroup in the (abstract) group of all invertible elements of $Cl_2^\varepsilon(H)$.*

For $u \in Spin(H; vN)$, let $\rho_{vN}(u)$ be the orthogonal operator

$$\begin{cases} H \rightarrow H \\ x \mapsto uxu^{-1}. \end{cases}$$

Then the image of ρ_{vN} is a normal subgroup of $O(H)$, and its kernel, consisting of e_0 and $-e_0$, is isomorphic to Z_2. •

The proof is trivial.

Corollary 9 indicates that $Spin(H; vN)$ must support the covering group of some *topological* subgroup of $O(H)$. As one would like to use the general theory of Banach-Lie groups, the Hilbert space structure on $Cl_2^\varepsilon(H)$ is not convenient, because it is never complete in the infinite

dimensional case. In order to remove this impediment, we furnish $Cl_2^\varepsilon(H)$ with a C^*-algebra structure in section 7; we consider an ad hoc sub-C^*-algebra and the ad hoc subgroup of $\mathrm{Spin}\,(H; vN)$; after having proved a local result in section 8 and recalled a theorem in section 9, we can establish the main result of this work in section 10.

7. C^*-algebra structure on the Clifford algebra

In this section, H is a prehilbert space as in section 4.

Consider the Hilbert algebra $(\mathfrak{H}, Cl_2^\varepsilon(H), *)$, and for each $X \in Cl_2^\varepsilon(H)$ let L_X be the operator on $\mathfrak{H}$ defined as in section 4. Define a new norm on $Cl_2^\varepsilon(H)$ by

$$\|X\|_\infty = \text{norm of } L_X = \{\sup_{Y \in \mathfrak{H}, \|Y\|_2 \le 1} \langle XY, XY\rangle\}^{\frac{1}{2}}$$
$$= \{\sup_{Y \in Cl^\varepsilon(H), \|Y\|_2 \le 1} \langle XY, XY\rangle\}^{\frac{1}{2}}.$$

Let $Cl_\infty^\varepsilon(H)$ be the involutive algebra $Cl_2^\varepsilon(H)$ furnished with the norm $\|\ \|_\infty$.

PROPOSITION 10. $Cl_\infty^\varepsilon(H)$ is a real C^*-algebra.

PROOF. The only point to check is the completeness of $Cl_\infty^\varepsilon(H)$. Let $(X_n)_{n \in N}$ be a Cauchy sequence in $Cl_\infty^\varepsilon(H)$; it is a fortiori a Cauchy sequence with respect to the norm $\|\ \|_2$, so that it converges with respect to $\|\ \|_2$ towards an element X in $\mathfrak{H}$. Let M be a bound for $(\|X_n\|_\infty)_{n\in N}$; for any $Y \in Cl_2^\varepsilon(H)$:

$$\|R_Y(X_n)\|_2 = \|L_{X_n}(Y)\|_2 \le M\|Y\|_2$$

for every $n \in N$. Hence

$$\|R_Y(X)\|_2 = \|L'_X(Y)\|_2 \le M\|Y\|_2$$

and $X \in Cl_\infty^\varepsilon(H)$.

It must yet be shown that $(X_n)_{n\in N}$ converges towards X with respect to $\|\ \|_\infty$. Chose $\delta > 0$ and let $Y \in Cl_\infty^\varepsilon(H)$ with $\|Y\| \le 1$. As $(X_n)_{n\in N}$ is a Cauchy sequence with respect to $\|\ \|_\infty$, there is an integer n_0 independent of Y such that $\|(X_n - X_m)Y\|_2 \le \delta$ for $n, m \ge n_0$. As $(X_n)_{n\in N}$ converges towards X with respect to $\|\ \|_2$ and as the multiplication is separately continuous in $Cl_2^\varepsilon(H)$, $\|(X - X_m)Y\|_2 \le \delta$ for $m \ge n_0$. As n_0 is independent of Y:

$$\sup_{\|Y\|_2 \le 1} \|(X - X_m)Y\|_2 = \|X - X_m\|_\infty \le \delta$$

for $m \ge n_0$. $\bullet$

Let $Cl_1^\varepsilon(H)$ be the C^*-algebra which is the closure of $Cl^\varepsilon(H)$ in $Cl_\infty^\varepsilon(H)$. It will be called *the C^*-algebra of the form $x \mapsto \varepsilon|x|^1$ on H.*

Sections 2 to 4 imply straightforwardly a certain number of properties of $Cl_1^\varepsilon(H)$ collected now for convenience in the case $\varepsilon = -1$.

SCHOLIE 11. Let H be an infinite dimensional real prehilbert space and let $Cl_1(H)$ be the C^*-Clifford algebra of the form $x \mapsto -|x|^2$ on H. Then:

(i) $Cl_1(H)$ is a Z_2-graded real C^*-algebra with unit e_0.
(ii) $Cl_1(H)$ and $Cl_1^+(H)$ are topologically simple and their common center is equal to $\boldsymbol{R}e_0$.
(iii) The injection $H \to Cl_1(H)$ is an isometry.
(iv) If $(e_n)_{n \in N^*}$ is an orthonormal basis of H and if Cl^k is the standard Clifford algebra of the space $R^k = \mathrm{span}\ \{e_1, \cdots, e_k\}$ in H, then the union Cl^∞ of the Cl^k's is dense in $Cl_1(H)$.
(v)   If E is a dense subspace of H, then the inclusion $E \to H$ induces an isomorphism onto the image $Cl_1(E) \to Cl_1(H)$.

REMARK. Similar statements hold for the C^*-Clifford algebras $Cl_1^{p,q}(H)$, as indicated at the end of section 5.

8. The involutive Banach-Lie algebra spin $(H; C_1)$
and the Banach-Lie group Spin $(H; C_1)$

From now on, H is always a *separable Hilbert* space. The completion of the Lie algebra $o(H; C_0)$ defined in section 5 with respect to the nuclear norm $\|\ \|_1$ is an involutive (compact) real Banach-Lie algebra denoted by $o(H; C_1)$.

PROPOSITION 12. *Let $\mathbf{spin}(H; C_1)$ be the closure of $\mathbf{spin}(H; C_0)$ in $Cl_1^\varepsilon(H)$. Then $\mathbf{spin}(H; C_1)$ is an involutive real Banach-Lie algebra, the map $D\rho_0$ (see proposition 6) has a unique continuous extention*

$$D\rho_1 : \mathbf{spin}(H; C_1) \to o(H; C_1)$$

which is an isomorphism of involutive Lie algebras, and the equality

$$\|D\rho_1(X)\|_1 = 4\|X\|_\infty$$

holds for all $X \in \mathbf{spin}(H; C_1)$.

PROOF. It is clearly sufficient to show that $\|D\rho_0(X)\|_1 = 4\|X\|_\infty$ for all $X \in \mathbf{spin}(H; C_0)$. We consider the case $\varepsilon = -1$.

Step one. Let $(e_n)_{n \in N}$ be an orthonormal basis of H and let X be an
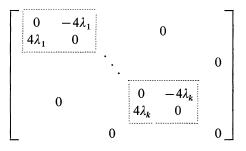
element in **spin**$(H; C_0)$ of the form $\sum_{n=1}^{k} \lambda_n [e_{2n-1}, e_{2n}]$. From the proof of proposition 4: $\|X\|_\infty \leq 2\sum_{n=1}^{k}|\lambda_n|$; from step three in the proof of proposition 6: $\|D\rho_0(X)\|_1 = 8\sum_{n=1}^{k}|\lambda_n|$. Hence $\|D\rho_0(X)\|_1 \geq 4\|X\|_\infty$.

Consider now X as an element of the complexified C^*-Clifford algebra $Cl_1(H)^C$. For each $n \in \{1, \cdots, k\}$, define

$$e_{\sigma(n)} = \begin{cases} e_{2n-1} + ie_{2n} & \text{if } \text{sign}(\lambda_n) = -1 \\ e_{2n-1} - ie_{2n} & \text{if } \text{sign}(\lambda_n) = +1 \end{cases}$$

(it is supposed that none of the λ_n's is zero), and let e be the product $\prod_{n=1}^{k} e_{\sigma(n)}$. An elementary computation then shows that $Xe = 2i(\sum_{n=1}^{k}|\lambda_n|)e$. Hence $2i(\sum_{n=1}^{k}|\lambda_n|)$ is an eigenvalue for the operator of left multiplication by X in the C^*-algebra $Cl_1(H)^C$. It follows that the norm of X in $Cl_1(H)^C$ is at least as large as the modulus of $2i(\Sigma|\lambda_n|)$, hence that $\|X\|_\infty \geq 2\sum_{n=1}^{k}|\lambda_n|$, hence that $\|D\rho_0(X)\|_1 = 4\|X\|_\infty$.

Step two. Let X be any element of **spin**$(H; C_0)$. Then there exists an orthonormal basis of H with respect to which the operator $D\rho_0(X)$ has the matrix representation

$$\begin{bmatrix} \begin{matrix} 0 & -4\lambda_1 \\ 4\lambda_1 & 0 \end{matrix} & & 0 & \\ & \ddots & & 0 \\ 0 & & \begin{matrix} 0 & -4\lambda_k \\ 4\lambda_k & 0 \end{matrix} & \\ & 0 & & 0 \end{bmatrix}$$

where $\lambda_1, \cdots, \lambda_k$ are non zero real numbers. By proposition 6(iv), it follows that the element X is equal to $\sum_{n=1}^{k} \lambda_n [e_{2n-1}, e_{2n}]$, so that the argument of step one applies. The case $\varepsilon = +1$ is similar. ●

Let $Cl_1^\varepsilon(H)$ be the involutive real Banach-Lie algebra defined by the C^*-algebra $Cl_1^\varepsilon(H)$, and let $Cl_1^\varepsilon(H)^{\text{inv}}$ be the group of invertible elements in $Cl_1^\varepsilon(H)$, which is a Banach-Lie group with Lie algebra $Cl_1^\varepsilon(H)$. Let Spin$(H; C_1)$ be the group of unitary even elements $u \in Cl_1^{\varepsilon +}(H)$ such that $uHu^{-1} = H$.

PROPOSITION 13. *The group Spin$(H; C_1)$ defined above has a unique structure of sub Banach-Lie group of $Cl_1^\varepsilon(H)^{\text{inv}}$, with Lie algebra* **spin**$(H; C_1)$.

PROOF. It is evident that $\exp X \in$ Spin$(H; C_1)$ for all $X \in$ **spin**$(H; C_1)$. On the other hand, the Lie algebra **spin**$(H; C_1)$ is stable by $Ad(u)$ for any $u \in$ Spin$(H; C_1)$; indeed, for any $X \in$ **spin**$(H; C_1)$: $Ad(u)(X) = uXu^{-1}$; if X is of the particular form $[y, z]$ with $y, z \in H$, then $uXu^{-1} = $

$[uyu^{-1}, uzu^{-1}] \in$ **spin** $(H; C_1)$; by linearity and by continuity of the multiplication in $Cl_1^\varepsilon(H)$, it follows that uXu^{-1} is in **spin** $(H; C_1)$ in general. Proposition 13 follows now from general principles in the theory of Banach-Lie groups (see Lazard [19], corollaire 21.10). ●

9. A result of Shale and Stinespring

The group of those *-automorphisms φ of $Cl_1^\varepsilon(H)$ such that $\varphi(H) = H$ is denoted by $\text{Bog}(Cl_1^\varepsilon(H))$ and is clearly isomorphic to the group $\text{Bog}(Cl_2^\varepsilon(H))$ defined in section 6. The canonical group isomorphism

$$O(H) \to \text{Bog}(Cl_1^\varepsilon(H))$$

is again denoted by Cl.

The group of those orthogonal operators on H of the form $\text{id}_H + T$ where T is trace-class is a Banach-Lie group denoted by $O(H; C_1)$, whose connected component is denoted by $O^+(H; C_1)$, and whose Lie algebra is that $o(H; C_1)$ defined in section 8 (see [14]).

PROPOSITION 14. *Let* $U \in O(H)$. *Then* $Cl(U)$ *is inner and even if and only if* $U \in O^+(H; C_1)$.

INDICATIONS FOR THE PROOF.

Step one. Let $U \in O^+(H; C_1)$. Then there exists $Y \in o(H; C_1)$ such that $U = \exp Y$ (it is an easy corollary of proposition II.15.B in [14], of which the proof follows closely Putnam and Wintner [22]). Let $D\rho_1$ be as in proposition 12, let $X = (D\rho_1)^{-1}(Y) \in$ **spin** $(H; C_1)$ and let $u = \exp X \in \text{Spin}(H; C_1)$. Then, for any $z \in H$:

$$uzu^{-1} = (\exp X)z(\exp X)^{-1} = (Ad(\exp X))(z)$$
$$= (\exp(adX))(z) = (\exp D\rho_1(X))(z) = U(z)$$

where adX is as usually the map

$$\begin{cases} Cl_1^\varepsilon(H) \to Cl_1^\varepsilon(H) \\ \quad Z \quad \mapsto [X, Z]. \end{cases}$$

It follows that $Cl(U) = L_u R_{u^{-1}}$, so that $Cl(U)$ is inner and even.

Remark. Let $U \in O(H)$ be such that $Cl(U)$ is inner and even, and suppose moreover that one can chose u in the connected component of the Banach-Lie group $\text{Spin}(H; C_1)$ such that $Cl(U) = L_u R_{u^{-1}}$. Then there exists a finite number of elements $X_1, \cdots, X_n$ in **spin** $(H; C_1)$ such that $u = \exp(X_1) \cdots \exp(X_n)$; hence, by a straightforward computation:

$$U(z) = (\exp D\rho_1(X_1)) \cdots (\exp D\rho_1(X_n))(z)$$

for all $z \in H$, so that $U \in O^+(H; C_1)$. We will see later that the Banach-Lie group $\mathrm{Spin}(H; C_1)$ is indeed connected, but we do not know that at this point of our argument. Hence we must call on:

Step two. Let $U \in O(H)$ be such that $Cl(U)$ is inner and even; then $U \in O^+(H; C_1)$. We refer to Shale and Stinespring [23] or Slawny [24] for the proof of step two, which is a consequence of results about those canonical transformations which are implementable in all Fock representations of the complex C^*-algebra $Cl_\infty^\varepsilon(H)^C$ (terminology as in [24]). Alternatively, it seems that a direct proof of step two could follow from lemmas 1 and 2 in Blattner [4]. •

10. The universal covering $\rho_1 : \mathrm{Spin}(H; C_1) \to O^+(H; C_1)$ and the homotopy type of the Spin group

Corollary 9 and proposition 14 imply that the diagram

$$
\begin{array}{ccc}
\mathbf{spin}(H; C_1) & \xrightarrow{\ D\rho_1\ } & o(H; C_1) \\
{\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\
\mathrm{Spin}(H; C_1) & \xrightarrow[\ \rho_1\ ]{} & O^+(H; C_1)
\end{array}
$$

is commutative and that ρ_1 is onto. As $D\rho_1$ is a continuous (hence smooth) isomorphism by proposition 12, and as each of the exponential maps is a local diffeomorphism whose derivative at the origin of the Lie algebra is the identity, it follows that ρ_1 is smooth, that it is a local diffeomorphism, and that the derivative of ρ_1 at the identity of $\mathrm{Spin}(H; C_1)$ is precisely $D\rho_1$.

LEMMA 15. *The Banach-Lie group $\mathrm{Spin}(H; C_1)$ is connected and simply connected.*

PROOF. Let κ be the homomorphism of Z_2 in $\mathrm{Spin}(H; C_1)$ whose image consists of e_0 and $-e_0$. As the sequence

$$\{1\} \to Z_2 \xrightarrow{\kappa} \mathrm{Spin}(H; C_1) \xrightarrow{\rho_1} O^+(H; C_1) \to \{1\}$$

is exact, as ρ_1 is smooth, and as the fundamental group of $O^+(H; C_1)$ is precisely Z_2, it is sufficient to check that the two points e_0 and $-e_0$ in the kernel of ρ_1 can be connected by an arc. But let x and y be two orthogonal unit vectors in H; then $xy = [\frac{1}{2}(x-y), \frac{1}{2}(x+y)] \in \mathbf{spin}(H; C_1)$ and the continuous arc in $\mathrm{Spin}(H; C_1)$ given by $\exp(txy) = (\cos t)e_0 + (\sin t)xy$ connects e_0 $(t = 0)$ to $-e_0$ $(t = \pi)$. •

We can now sum up the results obtained so far as follows, in the case $\varepsilon = -1$:

THEOREM. *Let $Cl_\infty(H)$ be the C*-Clifford algebra of the quadratic form $x \mapsto -|x|^2$ on the real Hilbert space H, which is separable and infinite dimensional. Let $Spin(H; C_1)$ be the subgroup of the group of invertible elements $Cl_1(H)^{inv}$ consisting of those even unitary elements $u \in Cl_1^+(H)$ such that $uHu^{-1} = H$. Let* **spin** $(H; C_1)$ *be the closed subalgebra of the Banach-Lie algebra defined by $Cl_1(H)$ which is generated as a vector space by the products $[x, y]$ with $x, y \in H$. Then there exists on $Spin(H; C_1)$ a unique structure of sub Banach-Lie group of $Cl_1(H)^{inv}$ with Lie algebra* **spin** $(H; C_1)$; *this Banach-Lie group is connected in its own topology and the map*

$$\rho_1 \begin{cases} Spin(H; C_1) \to O^+(H; C_1) \\ \quad u \qquad \mapsto (x \mapsto uxu^{-1}) \end{cases}$$

is the universal covering of $O^+(H; C_1)$; moreover, if $\| \ \|_1$ is the nuclear norm on operators on H and if $\| \ \|_\infty$ is the norm defined on the C*-algebra $Cl_1(H)$, then $\|D\rho_1(X)\|_1 = 4\|X\|_\infty$ for all $X \in$ **spin** $(H; C_1)$.

Similar statements hold in the case $\varepsilon = +1$.
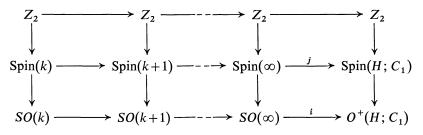
Let now $(e_k)_{k \in N*}$ be an orthonormal basis of H. Let Cl^k be the Clifford algebra of the form

$$\begin{cases} span (e_1, \cdots, e_k) \to R \\ \quad x \qquad \mapsto -|x|^2. \end{cases}$$

For all $k \in N^*$, the inclusion $Cl^k \to Cl_1(H)$ induces an inclusion $Spin(k) \to Spin(H; C_1)$. These define an inclusion j of the inductive limit $Spin(\infty)$ of the $Spin(k)$'s in $Spin(H; C_1)$.

PROPOSITION 16. *The inclusion $Spin(\infty) \xrightarrow{j} Spin(H; C_1)$ is a homotopy equivalence.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccc}
Z_2 & \longrightarrow & Z_2 & \dashrightarrow & Z_2 & \longrightarrow & Z_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Spin(k) & \longrightarrow & Spin(k+1) & \dashrightarrow & Spin(\infty) & \xrightarrow{j} & Spin(H; C_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SO(k) & \longrightarrow & SO(k+1) & \dashrightarrow & SO(\infty) & \xrightarrow{i} & O^+(H; C_1)
\end{array}$$

All vertical lines are Serre fibrations; as i is a homotopy equivalence (see references in [14], section II.6), the five lemma implies that j is a weak homotopy equivalence. But $Spin(\infty)$ and $SO(\infty)$ have both the homotopy type of an ANR (see Hansen [13], corollary 6.4), and $Spin(H; C_1)$ and $O^+(H; C_1)$ are ANR (see Palais [21], th. 4); hence Whitehead lemma applies (see [21], lemma 6.6) and j is a homotopy equivalence. ●

11. Hilbert spin manifolds

Let M be a smooth real manifold, locally diffeomorphic to a Banach space B. (We suppose M paracompact, connected and without boundary; and B separable and having smooth partitions of unity; it follows that M is separable and has smooth partitions of unity.) Let $\tilde{\xi} : E \to M$ be a vector bundle over M whose fiber is a separable real Hilbert space H. A *Riemannian nuclear structure on* $\tilde{\xi}$ [resp. an oriented Riemannian nuclear structure, a spin nuclear structure] is a reduction of the structural group of $\tilde{\xi}$ to $O(H; C_1)$ [resp. $O^+(H; C_1)$, $\mathrm{Spin}(H; C_1)$], and the bundle with such a structure will be denoted by ξ [resp. ξ_+, ξ_{Spin}]. Any bundle such as $\tilde{\xi}$ being trivialisable (Kuiper's theorem), such a reduction of structure is always possible. However, if $\tilde{\xi}$ has already been furnished with one of these structures, it may not be possible to reduce the structural group further. For example:

PROPOSITION 17. *Let ξ be a Riemannian nuclear vector bundle over M. Then ξ is orientable if and only if the first Stiefel-Whitney class $w_1(\xi) \in H^1(M; Z_2)$ vanishes.*

Proposition 17 is standard; in fact, much more complete results in this direction have been proved by Koschorke [18] (his propositions 6.2. and 6.3). A possible method of proof is that used below for proposition 18.

PROPOSITION 18. *Let ξ_+ be an oriented Riemannian nuclear vector bundle over M. Then ξ_+ has (at least) one spin structure if and only if the second Stiefel-Whitney class $w_2(\xi_+) \in H^2(M; Z_2)$ vanishes.*

PROOF. The exact sequence of topological groups

$$\{1\} \to Z_2 \to \mathrm{Spin}(H; C_1) \to O^+(H; C_1) \to \{1\}$$

induces an exact sequence of the cohomology sets (see Hirzebruch [15], 3.1 and 2.10.1)

$$H^1(M; \mathrm{Spin}(H; C_1)) \to H^1(M; O^+(H; C_1)) \overset{v}{\to} H^2(M; Z_2)$$

and ξ_+ can be considered as an element of $H^1(M; O^+(H; C_1))$. Hence it is sufficient to prove that $v = w_2$. By naturality, it is sufficient to do so when M is the classifying space B_{O^+} of the group $O^+(H; C_1)$. As the usual inclusions $SO(\infty) \to O^+(H; C_1)$ are homotopy equivalences, $H^1(B_{O^+}; Z_2) = Z_2$. Consequently, depending of its value on the classifying bundle over B_{O^+}, the map v is either identically zero, or is the class w_2. In order to exclude the first alternative, it is sufficient to make sure that there exists at least one manifold M and one bundle ξ_+ over M which has no spin structure. Such examples being well known, it follows that $v = w_2$. This result, in finite dimensions, is due to Haefliger [12]. ●

Proposition 18 does not depend on the explicit construction of Spin(H; C_1), but only on its universal property as covering group of $O^+(H; C_1)$. In fact, this proposition is a preliminary step for the following *project*, the carrying out of which will indeed depend on our explicit construction: investigate the properties of the differential operator defined over Spin(H; C_1)-manifolds as the Dirac operator is defined over finite dimensional spin manifolds (see for example Milnor [20] and Atiyah-Singer [3], section 5). Motivations for studying infinite dimensional elliptic operators can be found in Dalec'kiĭ [7].

REFERENCES

W. Ambrose
  [1] Structure theorems for a special class of Banach algebras. Trans. AMS 57 (1945) 364–386.

M. F. Atiyah, R. Bott and A. Shapiro
  [2] Clifford modules. Topology vol. 3 suppl. 1 (1964) 3–38.

M. F. Atiyah and I. M. Singer
  [3] Index of elliptic operators III. Ann. of Math. 87 (1968) 546–604.

R. J. Blattner
  [4] Automorphic group representations. Pacific J. Math. 8 (1958) 665–677.

N. Bourbaki
  [5] Algèbre, chapitre 9. Hermann 1959.

C. Chevalley
  [6] The algebraic theory of spinors. Columbia University Press, New York 1954.

Ja. L. Dalec'kiĭ
  [7] Infinite dimensional elliptic operators and parabolic equations connected with them. Russian Math. Surveys 29 (1970) no 4, 1–53.

J. Dieudonné
  [8] Eléments d'analyse, 2. Gauthiers-Villars 1968.

J. Dixmier
  [9] Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann), deuxième édition. Gauthier-Villars 1969.

S. Doplicher and R. T. Powers
  [10] On the simplicity of the even CAR algebra and free field models. Commun. Math. Phys. 7 (1968) 77–79.

A. Guichardet
  [11] Produits tensoriels infinis et représentations des relations d'anticommutation. Ann. Scient. Ec. Norm. Sup. (3) 83 (1966) 1–52.

A. Haefliger
  [12] Sur l'extension du groupe structural d'un espace fibré. C. R. Acad. Sci. Paris 243 (1956) 558–560.

V. L. Hansen
  [13] Some theorems on direct limits of expanding sequences of manifolds. Math. Scand. 29 (1971) 5–36.

P. de la Harpe
  [14] Classical Banach-Lie algebras and Banach-Lie groups in Hilbert space. Springer Lecture Notes, 285 (1972).

F. HIRZEBRUCH
[15] Topological methods in algebraic geometry. Third edition, Springer 1966.

N. JACOBSON
[16] Lie algebras. Interscience 1962.

M. KAROUBI
[17] Algèbres de Clifford et K-théorie. Ann. Scient. Ec. Norm. Sup. (4) 1 (1968) 161-270.

U. KOSCHORKE
[18] Infinite dimensional K-theory and characteristic classes of Fredholm bundle maps. Global Analysis, Berkeley, July 1968. Proc Sympos. Pure Math., XV (1970) 95–133.

M. LAZARD
[19] Groupes différentiables. 'Neuvième leçon' of a course to be published in book-form.

J. MILNOR
[20] Spin structures on manifolds. Enseignement Math. (2) 9 (1963) 198–203.

R. S. PALAIS
[21] Homotopy theory of infinite dimensional manifolds. Topology 5 (1966) 1–16.

C. R. PUTNAM and A. WINTNER
[22] The orthogonal group in Hilbert space. Amer. J. Math. 74 (1952) 52–78

D. SHALE and W. F. STINESPRING
[23] Spinor representations of infinite orthogonal groups. J. Math. Mech. 14 (1965) 315–322.

J. SLAWNY
[24] Representations of canonical anticommutation relations and implementability of canonical transformations. Thesis, The Weizmann Institute of Science, Rehovot 1969.

J. SLAWNY
[25] Same title. Commun. Math. Phys. 22 (1971) 104–114.

E. STØRMER
[26] The even CAR-algebra. Commun. Math. Phys. 16 (1970) 136–137.

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