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A CLASS OF SPACES LACKING NORMAL STRUCTURE

by

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In [5, Lemma 5], Goebel showed that if B is a Banach space with coefficient of convexity less than 1, then B has normal structure. This paper gives an example (Theorem 1) of a class of spaces which lack normal structure and have coefficient of convexity 1. Thus, Goebel's result is the best result possible. In Theorem 2 of this paper, it is shown that the duals of this class of spaces *have* normal structure and that their coefficients of convexity are between 1 and 2, so that normal structure is not self-dual.

We shall use the following definitions and notation. Let B be a Banach space with norm $\|\ \|$ and let K denote the unit sphere of B. The *modulus of convexity* of B is the function δ defined for t in $[0, 2]$ as follows:

$$2\delta(t) = \inf \{2-\|x+y\| : x, y \in K, \|x-y\| \geqq t\}$$

(see [4]). A space B is *uniformly convex* provided that its modulus of convexity is positive on $(0, 2]$ ([3], [4]). The *coefficient of convexity* of B, $\varepsilon_0 = \varepsilon_0(B)$, is $\sup \{t \in [0, 2] : \delta(t) = 0\}$ ([5]).

Let C be a bounded subset of B. The *diameter of* C, diam C, is $\sup \{\|x-y\| : x, y \in C\}$. A member x of C is a *non-diametral point* provided that diam $C > \sup \{\|x-u\| : u \in C\}$ and a *diametral point of* C is a point x for which the previous inequality is replaced by equality. For y in B, the *distance from* y *to* C, $d(y, C)$, is

$$\inf \{\|y-u\| : u \in C\}.$$

A space B *has normal structure* if each bounded convex subset of B with positive diameter has a non-diametral point ([2]). The convex hull of a subset A of B will be denoted by co A.

A space B is *uniformly non-square* if there is an $r > 0$ such that for x and y in X having norm 1, $\|x+y\|+\|x-y\| \leqq 4-r$ ([7]).

The examples in this paper are based on the l_p spaces. For $1 < p < \infty$ and x in l_p, define sequences x^+ and x^- as follows:

$$(x^+)_n = \sup (x_n, 0) = (|x_n|+x_n)/2$$
$$(x^-)_n = \sup (-x_n, 0) = (|x_n|-x_n)/2.$$

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For x in l_p, x^+ and x^- are in l_p and $x = x^+ - x^-$. Denote the l_p norm by $\| \ \|$. For $1 \leqq q < \infty$, let $l_{p,q}$ denote the set of elements of l_p with the norm:

$$|x| = (\|x^+\|^q + \|x^-\|^q)^{1/q}.$$

Let $l_{p,\infty}$ denote the set of elements of l_p with the norm:

$$|x| = \sup \{\|x^+\|, \|x^-\|\}.$$

It is easy to show that for $1 \leqq q \leqq \infty$, the function $| \ |$ is indeed a norm for l_p which is equivalent to the l_p norm. We shall show in Theorem 1 that for $1 < p < \infty$, $l_{p,\infty}$ *lacks* normal structure and $\varepsilon_0(l_{p,\infty}) = 1$ and in Theorem 2 that $l_{p,1}$ *has* normal structure and $2^{1/p} \leqq \varepsilon_0(l_{p,1}) < 2$. Using an argument which is considerably more complicated than the proof of Theorem 2, one can show that $\varepsilon_0(l_{p,1})$ is actually equal to $2^{1/p}$.

If $1 < p < \infty$ and $1 \leqq q \leqq \infty$ and p^* and q^* are the conjugate indices of p and q, then a straightforward argument shows that $(l_{p,q})^*$ is isometrically isomorphic to l_{p^*,q^*}. Thus, by combining Theorems 1 and 2, we obtain a class of reflexive spaces such that each space lacks (has) normal structure while its dual has (lacks) normal structure, so that normal structure is not self-dual.

A lack of connection between normal structure and reflexivity has been shown previously. In [1, p. 439] Belluce, Kirk and Steiner gave an example of a reflexive space lacking normal structure (the coefficient of convexity of this example is 2); consequently, reflexivity does not imply normal structure. On the other hand, Zizler has shown [8, proposition 2] that each separable Banach space B has an equivalent norm with respect to which B has normal structure; thus, normal structure does not imply reflexivity.

Using the methods of this paper, one can show that for $1 < p, q < \infty$, $l_{p,q}$ is uniformly convex and therefore has normal structure. We shall not analyze these spaces here.

Incidentally, it is not difficult to prove that in an arbitrary Banach space B with modulus of convexity δ, the limit from the left of δ at 2, $\delta(2^-)$, is equal to $1-(\varepsilon_0/2)$. In [5], Goebel has noted that B is uniformly convex if and only if $\varepsilon_0 = 0$. Consequently, we have that B is uniformly convex if and only if $\delta(2^-) = 1$. It is interesting to compare this with Goebel's observation that B is strictly convex if and only if $\delta(2) = 1$.

THEOREM (1). *For $1 < p < \infty$, $l_{p,\infty}$ lacks normal structure and its coefficient of convexity is* 1.

PROOF. The following theorem of Brodskii and Mil'man [2] is used to show that $B = l_{p,\infty}$ lacks normal structure:

A space X does not have normal structure if and only if there is a bounded sequence $\{x_n\}$ of elements of X such that the distance from x_{n+1} to co $\{x_1, \cdots, x_n\}$ tends to the diameter of the set of all x_i as $n \to \infty$ (such a sequence is called a *diametral sequence*).

We shall show that the sequence, $\{e_n\}$, of unit vectors of B (i.e., e_n is that member of B whose n-th coordinate is 1 with all other coordinates 0) is a diametral sequence. If m is a positive integer and $\alpha_1 + \cdots + \alpha_m = 1$ and $0 \leqq \alpha_i \leqq 1$ for $1 \leqq i \leqq m$, then

$$|e_{m+1} - \sum \alpha_i e_i|^p = \sup\{1, \sum \alpha_i^p\} = 1.$$

Thus, the distance from e_{m+1} to co $\{e_1, \cdots, e_m\}$ is 1. The above equation also shows that the diameter of the sequence $\{e_n\}$ is 1, so B lacks normal structure.

By Goebel's lemma 5 and the previous paragraph, $\varepsilon_0(B) \geqq 1$. We shall show that $\varepsilon_0 \leqq 1$. Suppose that $\delta(t) = 0$ for some t in $[0, 2]$. Then, there are sequences $\{x_n\}$ and $\{y_n\}$ in K, the unit sphere of B, such that for each n, $|x_n - y_n| \geqq t$ and $|x_n + y_n| \to 2$ as $n \to \infty$. For each u in B, $(-u)^+ = u^-$ and $(-u)^- = u^+$, so we may assume that for each n, $|x_n + y_n| = ||(x_n + y_n)^+||$ and $|x_n - y_n| = ||(x_n - y_n)^+||$. But, $2 \geqq ||x_n^+ + y_n^+|| \geqq |x_n + y_n| \to 2$; consequently, the uniform convexity of l_p implies that $||x_n^+ - y_n^+|| \to 0$. Therefore,

$$\begin{aligned} t \leqq |x_n - y_n| &\leqq ||(x_n^+ - y_n^+)^+|| + ||(y_n^- - x_n^-)^+|| \\ &\leqq ||x_n^+ - y_n^+|| + 1 \to 1. \end{aligned}$$

Thus, $\varepsilon_0 = 1$, and the proof is complete.

THEOREM (2). *For $1 < p < \infty$, $l_{p,1}$ has normal structure and $2^{1/p} \leqq \varepsilon_0(l_{p,1}) < 2$.*

PROOF. First, we establish the inequality for ε_0. Let $x = e_1$ and $y = -e_2$. Then $|x+y| = 2$ and $|x-y| = 2^{1/p}$, so that $\varepsilon_0 \geqq 2^{1/p}$. In [5], Goebel notes that a space B is uniformly non-square if and only if $\varepsilon_0(B) < 2$. It is easy to see that if B is uniformly non-square then B^* is also. By Theorem 1, $\varepsilon_0(l_{p^*, \infty}) = 1$, and thus, $\varepsilon_0(l_{p,1}) < 2$.

To show that $l_{p,1}$ has normal structure, we shall use the following theorem of Gossez and Lami Dozo [6]:

Let B be a Banach space with Schauder basis $\{e_n\}$. For each positive integer k and each x in B, let $U_k(x) = \Sigma_1^k x_n e_n$ and let $V_k(x) = x - U_k(x)$. Suppose that $\{k_n\}$ is a strictly increasing sequence of positive integers with the following property:

If $c > 0$, there is an $r > 0$ with the property that if x is in B and n is an integer such that $||U_{k_n}(x)|| = 1$ and $||V_{k_n}(x)|| \geqq c$, then $||x|| \geqq 1 + r$.

Then, each convex weakly relatively compact subset of B of at least two points has a non-diametral point.

The sequence $\{e_n\}$ of unit coordinate vectors is a Schauder basis for $l_{p,1}$. It follows from the Minkowski inequality that for each positive integer k and each x in $l_{p,1}$,

$$|x|^p \geqq |U_k(x)|^p + |V_k(x)|^p.$$

Therefore, the above theorem is applicable. Since this space is reflexive, each bounded convex subset is weakly relatively compact. Thus, $l_{p,1}$ has normal structure. I want to thank the referee for calling my attention to the paper of Gossez and Lami Dozo.

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