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SOME LIMIT THEOREMS FOR LOCAL TIME

by

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1. Introduction

Let $X = \{X(t), t \geqq 0\}$ be a standard Markov process with state space E. Assume that for each $x \in E$, x is regular for itself: i.e., if $T_x = \inf\{t > 0 : X(t) = x\}$, then $P^x\{T_x = 0\} = 1$. Then according to the theory of Blumenthal and Getoor, there is for each x, a continuous, unique (up to constant multiples), increasing additive functional $\{L_t^x, t \geqq 0\}$, called the local time at x, which increases 'only' when the process X is in the state x (see [1], [2] for precise descriptions.) As such, L_t^x is supposed to give some indication of how much time before t the process X spends in the vicinity of x. In special cases, L_t^x admits representation as a limit of quantities that measure in some more direct way the amount of time spent near x. For example, if B_n is a sequence of neighborhoods of x with $\cap B_n = x$, then it often turns out that

$$L_t^x = \lim_{n\to\infty} [\mu(B_n)]^{-1} \int_0^t I_{B_n}[X(s)]ds$$

(see Griego, [6]) so that L_t^x is a limit of 'occupation times' averaged over smaller and smaller neighborhoods of x. (Here I_B is the indicator of B, μ is Lebesgue measure.) For certain diffusion processes, it is known that $L_t^0 = \lim_{\varepsilon \downarrow 0} \varepsilon\, d_\varepsilon(t)$, where $d_\varepsilon(t)$ is the number of times the real valued process crosses down from $\varepsilon > 0$ to 0 before time t (this result, conjectured by Lévy [8], was proved by Ito and McKean [7]). Finally, Blumenthal and Getoor ([1], see also [5]) showed that under fairly general circumstances, if X has a reference measure ξ, then an appropriate choice of the local time L_t^x could serve as a density for occupation time, in the sense that for every Borel set B, $\int_0^t I_B[X(s)]ds = \int_B L_t^x \xi(dx)$ a.s. In this paper, a new description of L_t^0, valid for a wide class of real valued Markov processes, is found which describes L_t^0 more or less in terms of 'the number of times' the process 'jumps across zero.'

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While the basic theorem of this paper can be formulated for quite general Markov processes (see section 2), the character of this result is most easily illustrated by a few special cases. Let $X = \{X(t), t \geqq 0\}$ be a real valued process with stationary independent increments and Lévy measure ν. Let $J_n(t)$ be the number of jumps $j(X, s) = X(s) - X(s-)$ before time t for which $X(s-) < 0 < X(s)$ and $2^{-n-1} < j(X, s) < 2^{-n}$. So $J_n(t)$ is the 'number of (upward) jumps across 0 having size in $(2^{-n-1}, 2^{-n})$'. Assume that $\nu(R) = \infty$, that 0 is regular for $\{0\}$, and that L_t^x is jointly continuous in (x, t). Then (theorem 3.2) there is a version l_t^0 of the local time at 0 (see section 2 for the precise description) such that for each T,

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq T} |J_n(t)/F_n - l_t^0| = 0 \qquad \text{a.s.}$$

provided $\Sigma[1/F_n] < \infty$, where $F_n = \int_{2^{-n-1}}^{2^{-n}} x\nu(dx)$. This result is very close in spirit to the result of Ito and McKean mentioned above. An illustration is the case where X is a stable process with index α, $1 < \alpha < 2$, in which case $J_n(t)/2^{n(\alpha-1)}$ converges a.s. to cl_t^0 uniformly, where c is a known constant. The basic result of section 2 also yields theorems of the following type. Let X be again a stable process with index α, $1 < \alpha < 2$, and set $Q_\varepsilon(t) = \sum_{s\leqq t}' |X(s) - X(s-)|$, where the prime indicates that the sum is over all $s \leqq t$ for which $X(s-) < 0 < X(s)$ and $|X(s) - X(s-)| < \varepsilon$. Thus $Q_\varepsilon(t)$ is the sum up to time t of all the upward jumps across 0 which have magnitude at most ε. Then for stable X with $1 < \alpha < 2$ (see theorem 3.1),

$$\lim_{\varepsilon\to 0} P\{\sup_{0\leqq t\leqq T} |Q_\varepsilon(t)/\varepsilon^{2-\alpha} - cl_t^0| > \delta\} = 0$$

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq T} |Q_{2^{-n}}(t)/2^{n(\alpha-2)} - cl_t^0| = 0 \qquad \text{a.s.}$$

for all $\delta > 0$, and $T > 0$.

Section 2 contains the statement and proof of the basic result, while section 3 presents a number of applications to processes with independent increments. The terminology referring to the theory of Markov processes will be that of [2].

2. Main result

Let $X = \{X(t), t \geqq 0\}$ be a standard, real valued Markov process. Let $T_x = \inf\{t > 0 : X(t) = x\}$, and assume from now on that each x is regular for itself. Define for $\alpha > 0$

$$\psi^\alpha(x, y) = E^x[\exp\{-\alpha T_y\}] \tag{2.1}$$

According to the theory of Blumentahl and Getoor [2], there is for each

x a continuous additive functional $\{L_t^x, t \geqq 0\}$, the local time at x, satisfying

$$E^x \int_0^\infty e^{-t} dL_t^y = \psi^1(x, y), \tag{2.2}$$

so that in particular $E^x \int_0^\infty e^{-t} dL_t^x = 1$. Assume from now on that $\psi^1(x, y)$ is jointly Borel measurable, and that there is a reference measure ξ for the process X. If U^α is the usual operator $U^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f[X(t)]dt$, then under the preceding assumptions Getoor and Kesten [5] have proved the following result which will be stated as a lemma for the convenience of the reader.

LEMMA 2.1 *There is a strictly positive finite Borel function g on R such that $l_t^x(\omega)$ defined by $l_t^x(\omega) = g(x)L_t^x(\omega)$ satisfies a.s.*

$$\int_0^t I_B[X(s)]ds = \int_B l_t^x(\omega)\xi(dx) \tag{2.3}$$

for all $t \geqq 0$ and Borel sets B simultaneously. Define for each $\alpha > 0$, $u^\alpha(x, y) = g(y)\, E^x \int_0^\infty e^{-\alpha t} dL_t^y$. Then

$$U^\alpha f(x) = \int u^\alpha(x, y) f(y) \xi(dy) \text{ for all Borel } f \geqq 0 \tag{2.4}$$

$$u^1(y, y) = g(y) \tag{2.5}$$

$$u^\alpha(x, y) = E^x \int_0^\infty e^{-\alpha t} dl_t^y \tag{2.6}$$

$$\psi^1(x, y) = u^1(x, y)/u^1(y, y). \tag{2.7}$$

Since a reference measure is assumed throughout this section, there exists, according to the theory of S. Watanabe [13], a Lévy system (N, A) for the process X. Here $N(x, dy)$ is a non-negative kernel such that for each $x \in R$, $N(x, \cdot)$ is a measure on the Borel sets of R and for each Borel set B, $N(\cdot, B)$ is a Borel function. $A = \{A(t), t \geqq 0\}$ is a finite, continuous additive functional having the following property: for every non-negative Borel function f on $R \times R$ vanishing on the diagonal

$$E^x \sum_{s \leqq t} f[X(s-), X(s)] = E^x \int_0^t Nf[X(s)]dA(s) \tag{2.8}$$

where $Nf(x) = \int N(x, dy) f(x, y)$. For simplicity, assume from now on that $A(t) = t$. This may always be achieved by a time change if necessary (see, for example [11]); in the case of processes with independent increments one may always take $A(t) = t$. as can be verified directly.

For the remainder of this section assume

(2.9) $$\gamma(x) = [1-\psi^1(x,0)\psi^1(0,x)]^{+(\frac{1}{2})} \to 0 \quad \text{as } x \to 0$$

(2.10) $$u^1(x,x) \to u^1(0,0) \textit{ as } x \to 0.$$

The following theorem is the main result of this section.

THEOREM 2.1. *Let $X = \{X(t), t \geqq 0\}$ be a standard, real valued Markov process. Assume that there is a reference measure ξ, that each point of the state space is regular for itself, that $\psi^1(x, y)$ is jointly Borel measurable, and that the additive functional A of the Lévy system (N, A) satisfies $A(t) = t$. For all sufficiently small ε or only for $\varepsilon \downarrow 0$ through a sequence, let $f_\varepsilon(x, y)$ be a non-negative Borel function vanishing on the diagonal such that there exists a compact interval $I(\varepsilon) = [a(\varepsilon), b(\varepsilon)]$ containing 0 and such that $Nf_\varepsilon(x) = 0$ outside $I(\varepsilon)$. Assume $a(\varepsilon) \to 0$ and $b(\varepsilon) \to 0$ as $\varepsilon \to 0$, and that*

$$F(\varepsilon) = \int Nf_\varepsilon(x)\xi(dx) = \int_{I(\varepsilon)} Nf_\varepsilon(x)\xi(dx)$$

satisfies $0 < F(\varepsilon) < \infty$. Assume (2.9) and (2.10). Finally, define

$$G(\varepsilon) = \int Nf_\varepsilon^2(x)\xi(dx) \qquad \textit{and}$$

$$Q_\varepsilon(t) = \sum_{s \leqq t} f_\varepsilon[X(s-), X(s)].$$

Then the following conclusions hold.

1. *If $\lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0$, then*

(2.11) $$\lim_{\varepsilon \to 0} P^0\{\sup_{0 \leqq t \leqq T} |Q_\varepsilon(t)/F(\varepsilon) - l_t^0| > \delta\} = 0$$

for each T and each $\delta > 0$.

2. *If $\{l_t^x\}$ is jointly continuous in (t, x) near 0 and if $\{\varepsilon_n, n \geqq 1\}$ is a sequence decreasing to 0 such that $\sum_{n \geqq 1} G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty$, then*

(2.12) $$\lim_{n \to \infty} \sup_{0 \leqq t \leqq T} |Q_{\varepsilon_n}(t)/F(\varepsilon_n) - l_t^0| = 0 \qquad a.s.$$

for each T.

REMARK. If $u^1(x, x)$ is continuous as a function of x, then it follows from lemma 2.1 that l_t^x will be jointly continuous in (t, x) if and only if L_t^x is jointly continuous. Conditions guaranteeing joint continuity of L_t^x have been given recently by Getoor and Kesten [5].

Before proceeding with the proof, note that, under the assumptions of theorem 2.1, Lemma 2.1 permits the following conclusion.

LEMMA 2.2. *Let $h(x)$ be a non-negative, finite Borel function. Then $A(t) = \int_0^t h[X(s)]ds$ and $B(t) = \int l_t^x h(x)\xi(dx)$ are equivalent stochastic processes.*

PROOF. It suffices to assume h bounded. Let u_A^α and u_B^α be the α potentials of A and B respectively. Then from (2.4),

$$\begin{aligned} u_A^1(x) &\overset{\text{def}}{=} E^x \int_0^\infty e^{-t} dA(t) \\ &= E^x \int_0^\infty e^{-t} h[X(s)]ds \\ &= U^1 h(x) = \int u^1(x, y)h(y)\xi(dy), \end{aligned}$$

and from (2.6)

$$\begin{aligned} u_B^1(x) &= E^x \int_0^\infty e^{-t} dB(t) \\ &= \int u^1(x, y)h(y)\xi(dy). \end{aligned}$$

Since A and B are continuous additive functionals having the same bounded 1-potentials, the conclusion follows (see [2], p. 157).

The proof of theorem 2.1 has the following structure. In the terminology of Meyer [10], the process $Q_\varepsilon(t)$ is increasing but not natural. By Meyer's decomposition theorem for supermartingales, there is a unique natural increasing process $V_\varepsilon(t)$ such that $Q_\varepsilon(t) - V_\varepsilon(t)$ is a martingale. The limit results on $Q_\varepsilon(t)$ are then deduced by analysing $V_\varepsilon(t)$ and the martingale separately.

PROOF OF THEOREM 2.1. First, observe that from 2.2

$$1 \geqq E^y \int_0^T e^{-t} dL_t^x \geqq e^{-T} E^y L_T^x,$$

so that

$$E^y L_T^x \leqq e^T \qquad \text{for every } T, \tag{2.13}$$

and from lemma 2.1

$$E^y l_T^x \leqq u^1(x, x) E^y L_T^x \leqq u^1(x, x) e^T. \tag{2.14}$$

Let $V_\varepsilon(t) = \int_0^t N f_\varepsilon[X(s)]ds$. From lemma 2.2, $V_\varepsilon(t) = \int l_t^x N f_\varepsilon(x)\xi(dx)$, and from (2.14), $E^y V_\varepsilon(t) = \int E^y(l_t^x) N f_\varepsilon(x)\xi(dx) \leqq e^t \int u^1(x,x) N f_\varepsilon(x)\xi(dx)$. By assumption (2.10), $u^1(x, x) \to u^1(0, 0)$ as $x \to 0$, so $u^1(x, x)$ is bounded for x near 0. Thus for ε small, and all y,

$$E^y V_\varepsilon(t) \leqq M e^t \int N f_\varepsilon(x)\xi(dx) < \infty$$

for some constant M. Observe that $Q_\varepsilon(t)$ is an additive functional and V_ε is a continuous additive functional. It then follows from (2.8) that

(2.15) $$M_\varepsilon(t) = Q_\varepsilon(t) - V_\varepsilon(t)$$

is an additive functional with mean zero, and so must be a martingale relative to each P^y.

Define $\mu^\varepsilon(dx) = Nf_\varepsilon(x)\xi(dx)/F(\varepsilon)$. By the assumptions of theorem 2.1, μ^ε is a probability measure carried by $I(\varepsilon)$, and μ^ε converges weakly to unit mass at 0. Moreover,

$$V_\varepsilon(t)/F(\varepsilon) = \int l_t^x \mu^\varepsilon(dx),$$

and

$$\sup_{0 \leqq t \leqq T} |V_\varepsilon(t)/F(\varepsilon) - l_t^0| \leqq \int \sup_{0 \leqq t \leqq T} |l_t^x - l_t^0| \mu^\varepsilon(dx).$$

If, as in case (2), l_t^x is jointly continuous in (t, x), then for almost all ω, $\sup_{0 \leqq t \leqq T} |l_t^x - l_t^0| \to 0$ as $x \to 0$ by uniform continuity, implying that $\sup_{0 \leqq t \leqq T} |V_\varepsilon(t)/F(\varepsilon) - l_t^0| \to 0$ a.s. in this case. To treat case 1, recall that according to a result of Meyer ([9], see also [2], V.3)

$$P^0\{\sup_{0 \leqq t \leqq T} |L_t^x - L_t^0| > 2\delta\} \leqq 2e^T e^{-\delta/\gamma(x)},$$

where $\gamma(x)$ was defined in (2.9). By virtue of the formula

$$E|Y| = \int_0^\infty P\{|Y| > s\} ds, \quad E^0 \sup_{0 \leqq t \leqq T} |L_t^x - L_t^0| \leqq 4e^T \gamma(x).$$

Since (see lemma 2.1)

$$\begin{aligned} l_t^x - l_t^0 &= u^1(x, x) L_t^x - u^1(0, 0) L_t^0 \\ &= u^1(x, x)[L_t^x - L_t^0] - L_t^0[u^1(x, x) - u^1(0, 0)], \end{aligned}$$

it follows from (2.13) and the calculation above that

(2.16) $$E^0 \sup_{0 \leqq t \leqq T} |l_t^x - l_t^0| \leqq u^1(x, x) 4e^T \gamma(x) + e^T[u^1(x, x) - u^1(0, 0)].$$

Hence

$$\begin{aligned} E^0\{\sup_{0 \leqq t \leqq T} |V_\varepsilon(t)/F(\varepsilon) - l_t^0|\} &\leqq \int E^0 \sup_{0 \leqq t \leqq T} |l_t^x - l_t^0| \mu^\varepsilon(dx) \\ &\leqq 4e^T \int u^1(x, x)\gamma(x)\mu^\varepsilon(dx) + e^T \int [u^1(x, x) - u^1(0, 0)] \mu_\varepsilon(dx). \end{aligned}$$

By assumptions (2.9), (2.10) the integrands above are continuous at 0 and are equal to 0 there. Since $\mu_\varepsilon(dx)$ converges weakly to unit mass at 0, this completes the treatment of $V_\varepsilon(t)$ in both case 1 and case 2.

Turning next to the martingale $M_\varepsilon(t)$, observe that a fundamental fact on Lévy systems (S. Watanabe [13], p. 63, eq. 3.11) implies that

$$E^x M_\varepsilon^2(T) = E^x \int_0^T N f_\varepsilon^2[X(s)]ds. \tag{2.17}$$

Since M_ε is a martingale, this together with the well-known martingale inequality of Doob ([4], p. 317) yields

$$\begin{aligned} E^0 \sup_{0 \leqq t \leqq T} [M_\varepsilon(t)]^2 &\leqq 4E^0[M_\varepsilon(T)]^2 \\ &= 4E^0 \int_0^T N f_\varepsilon^2[X(s)]ds \\ &= 4E^0 \int l_T^x\, N f_\varepsilon^2(x)\xi(dx) \qquad \text{(by lemma 2.2)} \\ &\leqq Me^T \int N f_\varepsilon^2(x)\xi(dx) \qquad \text{(if } \varepsilon \text{ is sufficiently small)} \\ &= Me^T G(\varepsilon). \end{aligned}$$

Since by hypothesis $\lim_{\varepsilon \to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0$ in case 1, it follows that $\sup_{0 \leqq t \leqq T} M_\varepsilon(t)/[F(\varepsilon)]^2 \to 0$ in probability. In case 2, since

$$E^0 \sup_{0 \leqq t \leqq T} [M_\varepsilon(t)/F(\varepsilon)]^2 < \text{const. } G(\varepsilon)/[F(\varepsilon)]^2$$

and $\Sigma G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty$, a Borel Cantelli argument shows

$$\sup_{0 \leqq t \leqq T} M_{\varepsilon_n}(t)/F(\varepsilon_n) \to 0 \quad \text{a.s.}$$

This completes the proof of theorem 2.1.

3. Processes with stationary independent increments

Theorem 2.1 yields a number of interesting results when $X = \{X(t), t \geqq 0\}$ is a process with independent increments. This section contains several of these applications.

Throughout this section, let $X = \{X(t), t \geqq 0\}$ be a real valued process with stationary independent increments having right continuous paths with left limits. Of course, $E^0 e^{iuX(t)} = \exp\{-t\phi(u)\}$ where

$$\phi(u) = iau + (\tfrac{1}{2})Su^2 + \int_R \{1 - e^{iux} + iux/[1+x^2]\}\nu(dx).$$

The measure ν is called the Lévy measure, and ϕ is called the exponent of X. Assume throughout that 0 is regular for itself; in the present circumstances this implies that each x is regular for $\{x\}$. Precise conditions under which 0 is regular for $\{0\}$ may be found in [3]. Assume also from now on that $\nu(R) = \infty$. (If $\nu(R) < \infty$, then it is obvious that it is impossible to represent local time (when it exists) as a limit of quantities depending only on the jumps about 0).

Under the last two assumptions, it is known ([3], [12]) that for each $\alpha > 0$ there exists a real, bounded, continuous function $u^\alpha(x)$ such that $U^\alpha f(x) = \int f(y)u^\alpha(y-x)dy$ for all bounded Borel f, and satisfying $u^\alpha(x) = u^\alpha(0)\psi^\alpha(0, x)$. It follows from this that Lebesgue measure is a reference measure and, in the notation of section 2, $u^\alpha(x, y) = u^\alpha(y-x)$; see [5] for more detail on this point. Since there is a reference measure, a Lévy system (N, A) exists which, in fact is given by $N(x, dy) = \nu(dy-x)$, $A(t) = t$. Actually, for the case of processes with independent increments one may show directly that this is a Lévy system for X even when there is no reference measure. It is clear that $\psi^1(x, y) = \psi^1(0, y-x)$ is jointly Borel measurable in (x, y) in the present case ($\psi^1(0, z)$ is continuous in z). Since $\psi^1(x, y) = u^1(y-x)/u^1(0)$, it follows again from the continuity of $u^1(\cdot)$ that $\lim_{x\to 0}\psi^1(x, 0) = \lim_{x\to 0}\psi^1(0, x) = u^1(0)/u^1(0) = 1$, so (2.9) holds. Finally, since $u^1(x, x) = u^1(0)$, (2.10) holds trivially and so all assumptions of section 2 are satisfied by a real valued process with stationary independent increments having 0 regular for $\{0\}$ and $\nu(R) = \infty$.

For theorem 3.1, let $F(\varepsilon) = F(g, \varepsilon) = \int_0^\varepsilon \int_x^\varepsilon g(y)\nu(dy)dx$, where g is a non-negative Borel function on $(0, \infty)$, bounded on finite intervals. If $0 < \delta < \varepsilon$, an integration by parts yields

$$\int_\delta^\varepsilon xg(x)\nu(dx) = \delta\int_\delta^\varepsilon g(x)\nu(dx)+\int_\delta^\varepsilon\int_x^\varepsilon g(y)\nu(dy)dx = \delta K(\delta)+\int_\delta^\varepsilon K(x)dx$$

where $K(x) = \int_x^\varepsilon g(y)\nu(dy)$ is a function that increases as x decreases. If $\delta \downarrow 0$ and $\int_0^\varepsilon xg(x)\nu(dx) < \infty$, then it follows, since all terms above are positive, that $\int_0^\varepsilon K(x)dx < \infty$ and $\lim_{\delta\to 0}\delta K(\delta)$ exists. Since

$$\infty > \int_0^\delta K(x)dx \geqq K(\delta)\delta,$$

it follows that $\lim_{\delta\to 0}\delta K(\delta) = 0$. Hence, if $\int_0^\varepsilon xg(x)\nu(dx) < \infty$, then $\infty > \int_0^\varepsilon\int_x^\varepsilon g(y)\nu(dy)dx = \int_0^\varepsilon xg(x)\nu(dx)$. Conversely it is not hard to see that if $F(\varepsilon) < \infty$, then $\infty > \int_0^\varepsilon xg(x)\nu(dx) = F(\varepsilon)$.

THEOREM 3.1 *Assume* $0 < F(\varepsilon) = \int_0^\varepsilon xg(x)\nu(dx) < \infty$, *and define* $G(\varepsilon) = \int_0^\varepsilon x[g(x)]^2\nu(dx)$. *Let* $Q_\varepsilon(t) = \sum'_{s\leqq t} g(|j(X, s)|)$ *where the prime means that the sum is over only those jumps* $j(X, s) = X(s)-X(s-)$ *for which* $X(s-) < 0 < X(s)$ *and* $|j(X, s)| < \varepsilon$. *Then:*

(a) *If* $\lim_{\varepsilon\to 0} G(\varepsilon)/[F(\varepsilon)]^2 = 0$, *then*

$$\lim_{\varepsilon\to 0} P\{\sup_{0\leqq t\leqq T} |Q_\varepsilon(t)/F(\varepsilon)-l_t^0| > \delta\} = 0$$

for every $T > 0$ *and* $\delta > 0$.

(b) *If* l_t^x *is jointly continuous in* (t, x) *and if* $\{\varepsilon_n, n \geqq 1\}$ *is a positive sequence converging to zero such that* $\Sigma G(\varepsilon_n)/[F(\varepsilon_n)]^2 < \infty$, *then*

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq T} |Q_{\varepsilon_n}(t)/F(\varepsilon_n)-l_t^0| = 0 \qquad \text{a.s.}$$

Conditions guaranteeing joint continuity of l_t^x can be found in [5]. A large number of functions g satisfy the hypothesis that $F(\varepsilon) < \infty$; in particular $g(x) = |x|$ always works, since $\int_{|x|<1} |x|^2 \nu(dx) < \infty$ for any Lévy measure ν. As a special case, suppose X is a stable process with index α, $1 < \alpha < 2$. Then the exponent is of the form

$$\begin{aligned}(3.1) \qquad \phi(u) = c_1 \int_0^\infty [e^{iux}-1+iux/(1+x^2)]x^{-\alpha-1}dx \\ +c_2 \int_{-\infty}^0 [e^{iux}-1+iux/(1+x^2)]|x|^{-\alpha-1}dx.\end{aligned}$$

where $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1+c_2 > 0$. Suppose $c_1 > 0$ for convenience. If $g(x) = |x|$, then $F(\varepsilon) = c\varepsilon^{2-\alpha}$, where $c = c_1(2-\alpha)^{-1}$, and

$$G(\varepsilon)/[F(\varepsilon)]^2 = \text{const. } \varepsilon^{\alpha-1}.$$

Then, as mentioned in the introduction, $\lim_{\varepsilon\to 0} Q_\varepsilon(t)/\varepsilon^{2-\alpha} = cl_t^0$ in probability, uniformly on compact intervals and

$$\lim_{n\to\infty} Q_{2^{-n}}(t)/2^{-n(2-\alpha)} = cl_t^0 \qquad \text{a.s.,}$$

uniformly on compact intervals. (That l_t^x is jointly continuous in the stable case is well-known - see [2] and the references there.) As another example, the asymmetric Cauchy processes are interesting to consider. Here the exponent ϕ is of the form (3.1) with $\alpha = 1$ and $c_1 \neq c_2$. Assume $c_1 > 0$ (if not, then one can establish the result below for $-X$ instead.) It was proved by Kesten and Getoor that no jointly continuous version of the local time exists for the asymmetric Cauchy processes ([5], example b, section 4). Choose g of theorem 3.1 to be

$$g(u) = [(-\log|u|) \vee 0]^a.$$

Then for sufficiently small ε,

$$F(\varepsilon) = c_1 \int_0^\varepsilon (-\log x)^a x^{-1} dx = [-c_1/(a+1)](-\log\varepsilon)^{a+1} < \infty \qquad \text{if } a < -1$$

and

$$G(\varepsilon) = c_1 \int_0^\varepsilon (-\log x)^{2a} x^{-1} dx = -[c_1/(2a+1)](-\log\varepsilon)^{2a+1}.$$

Thus $G(\varepsilon)/[F(\varepsilon)]^2 = [(a+1)^2/(2a+1)](1/-\log\varepsilon) \to 0$ as $\varepsilon \to 0$, so a limit theorem continues to hold even in this singular case.

PROOF OF THEOREM 3.1. Let $f_\varepsilon(x, y) = g(|x-y|)I\{x < 0 < y; 0 < y-x < \varepsilon\}$, where $I\{A\}$ is the indicator of the set A. Then $Q_\varepsilon(t) = \Sigma_{s \leqq t} f_\varepsilon[X(s-), X(s)]$. In the notation of theorem 2.1,

$$Nf_\varepsilon(x) = \int_{-x}^{\varepsilon} g(u)\nu(du) \text{ if } x \in [-\varepsilon, 0] \text{ and } Nf_\varepsilon(x) = 0,\ x \notin [-\varepsilon, 0].$$

Also,

$$\int Nf_\varepsilon(x)dx = \int_{-\varepsilon}^{0} dx \int_{-x}^{\varepsilon} g(u)\nu(du) = \int_{0}^{\varepsilon} dx \int_{x}^{\varepsilon} g(u)\nu(du)$$
$$= \int_{0}^{\varepsilon} xg(x)\nu(dx) = F(\varepsilon).$$

Similarly

$$\int Nf_\varepsilon^2(x)dx = \int_0^\varepsilon x[g(x)]^2\nu(dx) = G(\varepsilon).$$

The result now follows from theorem 2.1.

Next, consider the following choice of $f_\varepsilon : f_\varepsilon(x, y) = I\{x < 0 < y; \lambda(\varepsilon)\varepsilon < y-x < \varepsilon\}$, where $0 < \lambda(\varepsilon) < 1$. Then

$$J_\varepsilon(t) = \sum_{s \leqq t} f_\varepsilon[X(s-), X(s)]$$

is equal to the number of jumps $j(X, s) = X(s)-X(s-)$ across 0 up to time t for which $\varepsilon\lambda(\varepsilon) < j(X, s) < \varepsilon$. Here

$$Nf_\varepsilon(x) = \int \nu(dy) f_\varepsilon(x, y+x)$$
$$= \begin{cases} \nu[(-x, \varepsilon)], & -\varepsilon < x < -\varepsilon\lambda(\varepsilon) \\ \nu[(\varepsilon\lambda(\varepsilon), \varepsilon)], & -\varepsilon\lambda(\varepsilon) < x < 0 \\ 0, \text{ otherwise.} \end{cases}$$

Then

$$\int Nf_\varepsilon(x)dx = \int_{-\varepsilon}^{0} Nf_\varepsilon(x)dx$$
$$= \int_{\varepsilon\lambda(\varepsilon)}^{\varepsilon} \nu[(x, \varepsilon)]dx + \varepsilon\lambda(\varepsilon)\nu[(\varepsilon\lambda(\varepsilon), \varepsilon)]$$
$$= \int_{\varepsilon\lambda(\varepsilon)}^{\varepsilon} x\nu(dx) = F(\varepsilon),$$

since $f_\varepsilon = f_\varepsilon^2$, $\int Nf_\varepsilon^2 = \int Nf_\varepsilon = F(\varepsilon)$ in this case. An application of theorem 2.1 to the preceding calculations then yields the following result.

THEOREM 3.2. *Let λ be a function such that $0 < \lambda(\varepsilon) < 1$ for all ε and define $F(\varepsilon) = F(\lambda, \varepsilon) = \int_{\varepsilon\lambda(\varepsilon)}^{\varepsilon} x\nu(dx)$. Let $J_\varepsilon(t)$ be the number of jumps up to time t for which $X(s-) < 0 < X(s)$ and $\varepsilon\lambda(\varepsilon) < X(s)-X(s-) < \varepsilon$.*

(a) *If* $\lim_{\varepsilon\to 0} F(\varepsilon) = \infty$, *then*

$$\lim_{\varepsilon\to 0} P\{\sup_{0\leqq t\leqq T} |J_\varepsilon(t)/F(\varepsilon)-l_t^0| > \delta\} = 0.$$

(b) *If* l_t^x *is jointly continuous in* (x, t) *and if* ε_n *is a positive sequence converging to zero such that* $\sum_{n\geqq 1}[1/F(\varepsilon_n)] < \infty$ *then*

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq T} |J_{\varepsilon_n}(t)/F(\varepsilon_n)-l_t^0| = 0 \qquad a.s.$$

Let X be a stable process with index α, $1 < \alpha < 2$, and exponent (3.1) with $c_1 > 0$, and let $J_n(t)$ be the number of upward jumps across 0 up to time t having size $(2^{-n-1}, 2^{-n})$. According to theorem 3.2b,

$$\lim_{n\to\infty} \sup_{0\leqq t\leqq T} |J_n(t)/2^{n(\alpha-1)}-cl_t^0| = 0 \qquad \text{a.s.}$$

where $c = c_1(2^{\alpha-1}-1)/(\alpha-1)$, as mentioned in the introduction. (Take $\varepsilon_n = 2^{-n}$ and $\lambda(2^{-n}) = (\frac{1}{2})$ for all n.) If X is an asymmetric Cauchy process, $F(\varepsilon) = c_1\int_{\varepsilon\lambda(\varepsilon)}^{\varepsilon} x^{-1}dx = -c_1 \log\lambda(\varepsilon)$. If $\lambda(\varepsilon)\to 0$ as $\varepsilon\to 0$, then $\lim_{\varepsilon\to 0} F(\varepsilon) = +\infty$. Hence from theorem 3.2a,

$$\sup_{0\leqq t\leqq T} |J_\varepsilon(t)/[-\log\lambda(\varepsilon)]-c_1 l_t^0| \to 0$$

in probability. Again a limit theorem continues to hold in the singular Cauchy case.

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