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PSEUDOCONCAVE LIE GROUPS

by

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1. Introduction

The purpose of this note is to prove that every connected pseudoconcave complex Lie group is a complex torus. We outline the proof here, leaving the details for later sections.

Let G be a connected pseudoconcave complex Lie group of dimension n . The adjoint representation, Ad , maps G holomorphically into C^{n^2} . If f is a function holomorphic on C^{n^2} then $f \circ Ad$ is a holomorphic function on G . Every pseudoconcave complex manifold has only constant holomorphic functions. Thus, since the holomorphic functions on C^{n^2} separate points, Ad maps G onto the identity matrix. This implies that G is abelian. Hence there is a discrete subgroup of C^n , Γ such that G is biholomorphically isomorphic to C^n/Γ . Since G has only constant holomorphic functions, Γ has rank n over C . Thus we may assume that Γ is generated as a Z -module by the R -linearly independent vectors $v_1 \cdots, v_m, e_1, \cdots, e_n$, where e_i is the i -th unit vector in C^n and $m \leq n$.

If we take vectors v'_i near enough to v_i then the vectors $v'_1, \cdots, v'_m, e_1, \cdots, e_n$ will still be R -linearly independent. Let Γ' be the group generated by these vectors and $G' = C^n/\Gamma'$. Lemma 7, which uses some ideas of Morimoto [2], states that if $m < n$ then we can find vectors v'_i arbitrarily close to v_i so that there is a non-constant function which is holomorphic on G' . But Lemma 8 states that if the vectors v'_i are near enough to v_i then G' is pseudoconcave. The only alternative is that $m = n$ and therefore G is a complex torus.

2. Preliminary definitions and lemmata

For our purposes complex manifolds are assumed to be connected. Let X be an n -dimensional complex manifold and Y an open subset of X . Y is said to have smooth boundary if for every $p \in \partial Y$ there is an open neighborhood $U = U(p)$ and a real-valued C^∞ -function φ with nowhere vanishing gradient defined on U such that $\{u \in U : \varphi(u) < 0\} = Y \cap U$. The analytic tangent plane to ∂Y at p is the unique $(n-1)$ -dimensional

complex vector space contained in the real tangent space at p . The Levi form of φ at p , $L_p(\varphi)$, is the hermitian form determined by the matrix $(\partial^2\varphi(p))/(\partial z_\alpha\partial\bar{z}_\beta)$. The signature of $L_p(\varphi)$ restricted to the analytic tangent plane at p is a biholomorphic invariant and is independent of the defining function φ . We say ∂Y is pseudoconcave at p if $L_p(\varphi)$ has at least one negative eigenvalue when restricted to the analytic tangent plane at p .

Let $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. A closed disc in X with center x is the image of \bar{D} under a biholomorphic mapping into X with x corresponding to the origin. If ∂Y is pseudoconcave at p then there is a closed disc $\Delta \subset \bar{Y}$ with center p such that $\Delta \cap \partial Y = \{p\}$.

DEFINITION. Let X be a complex manifold of complex dimension ≥ 2 . We say X is pseudoconcave if there is an open, non-empty, relatively compact subset $Y \Subset X$ such that ∂Y is smooth and everywhere pseudoconcave.

LEMMA 1. *Every function f holomorphic on a pseudoconcave manifold X is constant.*

PROOF. Let Y display the pseudoconcavity of X . Then $|f|$ takes its maximum on \bar{Y} at $p \in \partial Y$. There is a closed disc $\Delta \subset \bar{Y}$ with center at p such that $\Delta \cap \partial Y = \{p\}$. $f|_\Delta$ is holomorphic on Δ and assumes its maximum modulus at the center of Δ . Thus $f|_\Delta$ is constant. Therefore $|f|$ takes its maximum on \bar{Y} at points of Y . Thus the restriction of f to the connected component of Y containing $\Delta \sim \{p\}$ is constant. Thus f is constant.

Let G be an n -dimensional complex Lie group. For every $x \in G$ define the holomorphic automorphism A_x on G by $A_x(g) = xgx^{-1}$. The Lie algebra of G , $L(G)$, is defined to be the tangent space of G at the identity. Thus, for every $x \in G$, $dA_x : L(G) \rightarrow L(G)$ is an automorphism of $L(G)$. We have, therefore, the holomorphic mapping $Ad : G \rightarrow \text{Aut}(L(G))$ defined by $Ad(x) = dA_x$. Since $\text{Aut}(L(G)) \subset \mathbb{C}^{n^2}$, $Ad : G \rightarrow \mathbb{C}^{n^2}$.

LEMMA 2. *Let G be an n -dimensional pseudoconcave complex Lie group. There is a discrete subgroup of \mathbb{C}^n of rank n over \mathbb{C} , Γ , such that G is biholomorphically isomorphic to \mathbb{C}^n/Γ .*

PROOF. Every function f holomorphic on \mathbb{C}^{n^2} gives us a function $f \circ Ad$ holomorphic on G . $f \circ Ad$ is constant by Lemma 1. The holomorphic functions on \mathbb{C}^{n^2} separate points. Thus $Ad[G] = \{I\}$, where I is the $n \times n$ identity matrix. Therefore G is abelian.

By standard results in Lie theory [1], G is biholomorphically isomorphic to \mathbb{C}^n/Γ , where Γ is a discrete subgroup of \mathbb{C}^n of rank l over \mathbb{C} . After making a \mathbb{C} -linear change of coordinates in \mathbb{C}^n we may assume that Γ

is generated as a \mathbf{Z} -module by the \mathbf{R} -linearly independent vectors $v_1, \dots, v_m, e_1, \dots, e_l$, where e_i is the i -th unit vector in \mathbf{C}^n . Thus $G \cong \mathbf{C}^l/\Gamma \times \mathbf{C}^{n-l}$. By Lemma 1, $l = n$ and thus Γ has rank n over \mathbf{C} .

If Γ has rank n over \mathbf{C} and is generated over \mathbf{Z} by the \mathbf{R} -linearly independent vectors $v_1 \cdots, v_m, e_1, \dots, e_n$ where e_i is the i -th unit vector in \mathbf{C}^n , then we follow the notation of [2] by writing $V = (v_1, \dots, v_m)$, $\Gamma = \Gamma(V)$, and $M^*(n, m; \mathbf{C})$ as the set of such matrices V .

A proof of the following classical lemma can be found in [2].

LEMMA 3. *Suppose $V, V' \in M^*(n, m; \mathbf{C})$. Then $\mathbf{C}^n/\Gamma(V)$ and $\mathbf{C}^n/\Gamma(V')$ are biholomorphically isomorphic if and only if there is a matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL(n+m, \mathbf{Z})$, with A an $n \times n$ matrix, such that $(A+V'B)V = (C+V'D)$. Further, if only V' (resp. V) is known to be in $M^*(n, m; \mathbf{C})$ and such a matrix exists then V (resp. V') $\in M^*(n, m; \mathbf{C})$.*

3. The main result

$V \in M^*(n, m; \mathbf{C})$ is said to satisfy the Morimoto condition if there exist non-zero vectors $a \in \mathbf{Z}^n$ and $c \in \mathbf{Z}^m$ such that $aV = c$. The following lemma is due to Morimoto [2], but the proof given here is somewhat simpler.

LEMMA 4. *Let $G = \mathbf{C}^n/\Gamma(V)$, where $V \in M^*(n, m; \mathbf{C})$ satisfies the Morimoto condition. Then there exists $V' \in M^*(n, m; \mathbf{C})$ such that $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$ and G is biholomorphically isomorphic to $\mathbf{C}^n/\Gamma(V')$.*

PROOF. Assume that we have constructed a matrix

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL(n+m, \mathbf{Z})$$

such that $BV-D$ is invertible and the last row of $C-AV$ is identically zero. Defining $V' = (C-AV)(BV-D)^{-1}$, we have $(A+V'B)V = C+V'D$. Thus, by Lemma 3, $V' \in M(n, m; \mathbf{C})$ and $\mathbf{C}^n/\Gamma(V') \cong \mathbf{C}^n/\Gamma(V)$. Since the last row of $C-AV$ is identically zero, $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$. It remains to construct such a matrix.

V satisfies the Morimoto condition. Thus there are nonzero vectors $a = (a_1, \dots, a_n) \in \mathbf{Z}^n$ and $c = (a_{n+1}, \dots, a_{n+m}) \in \mathbf{Z}^m$ such that $aV = c$. We may assume by renumbering the vectors v_i that $a_1 \neq 0$ and that $\gcd(a_1, \dots, a_{n+m}) = 1$. Set $\gcd(a_1, \dots, a_k) = p_k$ for $k = 2, \dots, n+m$ and $p_1 = a_1$. We can find relatively prime integers α_k and γ_k such that $p_{k-1}\alpha_k + a_k\gamma_k = p_k, k = 2, \dots, n+m$. For $j = 1, \dots, i-1$ and $i = 2, \dots, n$, the numbers $\beta_{ij} \equiv -a_j\gamma_i/p_{i-1}$ are integers, since p_{i-1} divides a_1, a_2, \dots, a_{i-1} . Define $D_k \equiv \det M_k$, where

$$M_k = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_k \\ \beta_{21} & \alpha_2 & & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ \beta_{k1} & \cdots & \beta_{k,k-1} & & & \alpha_k \end{pmatrix}.$$

Let $r_i = -\gamma_i/p_{i-1}$. Subtract the first row of M_k multiplied by r_i from its i -th row, $i = 2, \dots, k$. The resulting matrix is triangular with determinant

$$D_k = a_1 \prod_{i=2}^k (\alpha_i - r_i a_i) = p_1 \prod_{i=2}^k ((\alpha_i p_{i-1} + a_i \gamma_i)/p_{i-1}) = \prod_{i=1}^k p_i / \prod_{i=1}^{k-1} p_i = p_k,$$

$k = 2, \dots, m+n$. Let $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, with A an $n \times n$ -matrix, be M_{n+m} with first and n -th rows interchanged. Since $p_{n+m} = 1$, $M \in GL(n+m, \mathbf{Z})$.

Now

$$B = \begin{pmatrix} r_{n+1} a \\ \vdots \\ r_{n+m} a \end{pmatrix} \quad \text{and} \quad BV = \begin{pmatrix} r_{n+1} c \\ \vdots \\ r_{n+m} c \end{pmatrix}.$$

Let $c(k) = (a_{n+1}, \dots, a_{n+k}, 0, \dots, 0)$. Therefore

$$D = \begin{pmatrix} \alpha_{n+1} {}^t e_1 \\ \alpha_{n+2} {}^t e_2 + r_{n+2} c(1) \\ \vdots \\ \alpha_{n+m} {}^t e_m + r_{n+m} c(m-1) \end{pmatrix}.$$

where e_i is the i -th unit vector in \mathbf{C}^m .

Thus

$$BV - D = \begin{pmatrix} r_{n+1} a_{n+1} - \alpha_{n+1} & & & * \\ \cdot & \cdot & & \\ & & \cdot & \\ 0 & & r_{n+m} a_{n+m} - \alpha_{n+m} & \end{pmatrix}.$$

Using the same argument as in the case of M_k , we see that $BV - D$ is invertible. Since $aV = c$, the last row of $C - AV$ is identically zero.

COROLLARY 5 *If $V \in M^*(n, m; \mathbf{C})$ and satisfies the Morimoto condition then there is a non-constant function holomorphic on $G = \mathbf{C}^n/\Gamma(V)$.*

PROOF. By Lemma 4 there exists $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$ such that G is biholomorphically isomorphic to $\mathbf{C}^n/\Gamma(V')$. But $\mathbf{C}^n/\Gamma(V') \cong \mathbf{C}^{n-1}/\Gamma(V'') \times \mathbf{C}^*$.

Since there are non-constant functions holomorphic on C^* , there are non-constant functions holomorphic on G .

COROLLARY 6. *Let G be a pseudoconcave, complex Lie group. Then G is biholomorphically isomorphic to $C^n/\Gamma(V)$ where $V \in M^*(n, m; C)$ does not satisfy the Morimoto condition.*

PROOF. By Lemma 2 there exists $V \in M^*(n, m; C)$ such that G is biholomorphically isomorphic to $C^n/\Gamma(V)$. Suppose V satisfies the Morimoto condition. Then, by Corollary 5, there is a non-constant function holomorphic on G . But, by Lemma 1, every function holomorphic on a pseudoconcave manifold is constant. Thus V does not satisfy the Morimoto condition.

For $n \times m$ -matrices V and V' we define $d(V, V')$ as the euclidean distance between V and V' induced from C^{mn} . We topologize $M^*(n, m; C)$ via the metric d . V is said to have rational coordinates if $V = (v_{\alpha\beta}) = (x_{\alpha\beta} + iy_{\alpha\beta})$ with $x_{\alpha\beta}, y_{\alpha\beta} \in Q$. Define $M_Q^*(n, m; C)$ as the set of matrices in $M^*(n, m; C)$ with rational coordinates. $M_Q^*(n, m; C)$ is dense in $M^*(n, m; C)$.

LEMMA 7. *Suppose $m < n$. Then the matrices in $M_Q^*(n, m; C)$ satisfy the Morimoto condition. These are therefore dense in $M^*(n, m; C)$.*

PROOF. Suppose $V \in M_Q^*(n, m; C)$. Then $V = X + iY$, where $X = (x_{\alpha\beta})$, $Y = (y_{\alpha\beta})$ and $x_{\alpha\beta}, y_{\alpha\beta} \in Q$. Since $m < n$ there exists a non-zero row vector $a \in Q^m$ such that $aY = 0$. Define $c \equiv aX \in Q^m$. Thus $aV = c$. By clearing denominators we may take $a \in Z^m$ and $c \in Z^m$. Thus every matrix $V \in M_Q^*(n, m; C)$ satisfies the Morimoto condition.

The remaining step for the proof of the main theorem is to show that pseudoconcavity is invariant under small changes of the group.

LEMMA 8. *Suppose $V^0 \in M^*(n, m; C)$ and $G^0 = C^n/\Gamma(V^0)$ is a pseudoconcave Lie group. There exists $\varepsilon > 0$ such that if $V \in M^*(n, m; C)$ and $d(V, V^0) < \varepsilon$ then $G = C^n/\Gamma(V)$ is a pseudoconcave Lie group.*

PROOF. For every $V \in M^*(n, m; C)$ choose v_{m+1}, \dots, v_n such that $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n, e_1, \dots, e_n\}$ is a basis for R^{2n} . Define the real linear maps $S_V : R^{2n} \rightarrow R^{2n}$ by $v_i \rightarrow v_i^0$ and $e_i \rightarrow e_i$ for $i = 1, \dots, n$. Thus $S_V : C^n \rightarrow C^n$ is a diffeomorphism such that $S_V(V, E_n) = (V^0, E_n)$. Hence S_V induces a diffeomorphism $T_V : G \rightarrow G^0$ such that

$$\begin{array}{ccc}
 C^n & \xrightarrow{S_V} & C^n \\
 \pi_V \downarrow & & \downarrow \pi_{V^0} \\
 G & \xrightarrow{T_V} & G^0
 \end{array}$$

is commutative, where π_V (resp. π_{V^0}) is the quotient map which divides C^n by the group $\Gamma(V)$ (resp. $\Gamma(V^0)$).

If $d(V, V^0) < \varepsilon$ and ε is sufficiently small, then we may assume that $v_{m+1} = v_{m+1}^0, \dots, v_n = v_n^0$. If $\| \cdot \|$ is any norm on C^n , then there exists a constant $c > 0$ such that

$$\|Sz - z\| \leq c\varepsilon\|z\|$$

for any $z \in C^n$. This means that $\|S - I\| \leq c\varepsilon$ where I is the identity map of C^n onto itself.

Let Ω display the pseudoconcavity of G^0 . It is enough to show that there exists $\varepsilon > 0$ such that $T_V^{-1}[\Omega]$ is a pseudoconcave, open subset of G when $d(V^0, V) < \varepsilon$. Since $\partial T_V^{-1}[\Omega]$ is compact, it is enough to prove that $\partial T_V^{-1}[\Omega]$ is pseudoconcave at each $p \in \partial T_V^{-1}[\Omega]$ when $d(V^0, V) < \varepsilon(p)$. Now π_V and π_{V^0} are locally biholomorphic. Hence we only need to prove that $\partial(S_V^{-1} \circ \pi_V^{-1}[\Omega])$ is pseudoconcave at some point $q \in \pi_V^{-1}(p)$. We may assume that $q = 0$, because translation acts biholomorphically on G (resp. G^0). Thus we have reduced the proof of this lemma to the following:

LEMMA 9. *Let $S_V : C^n \rightarrow C^n$ be the real linear transformation defined above. Let U be a neighborhood of 0 and φ a real valued C^∞ -function defined on U with nowhere vanishing gradient and $\varphi(0) = 0$. Assume that 0 is a pseudoconcave boundary point of $U^- = \{u \in U : \varphi(u) < 0\}$. Then there exists $\varepsilon > 0$ such that $d(V^0, V) < \varepsilon$ implies that 0 is a pseudoconcave boundary point of $S_V^{-1}[U^-]$.*

PROOF. Set $\zeta = Sz$. By a suitable choice of the basis for coordinates in C^n , we may assume that

$$\varphi(\zeta_1, \dots, \zeta_n) = 2\text{Re}(\zeta_n + \sum \alpha_{ij} \zeta_i \zeta_j) + \sum \beta_{ij} \zeta_i \bar{\zeta}_j + 0(\|\zeta\|^3).$$

Since the origin is a pseudoconcave boundary point of U^- , we may as well assume that $\beta_{11} < 0$.

Let $\zeta = Sz$ be given by the equations

$$\zeta_i = \sum_{j=1}^n a_{ij} z_j + \sum_{j=1}^n b_{ij} \bar{z}_j, \quad 1 \leq i \leq n$$

Because of the assumption $d(V, V^0) < \varepsilon$, we must have $a_{ij} - \delta_{ij} = 0(\varepsilon)$ and $b_{ij} = 0(\varepsilon)$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$, as we have observed before.

Substituting the ζ_i by the above expressions, we obtain for the function $\varphi \circ S_V$ that the analytic tangent plane to $\varphi \circ S_V(z) = 0$ at the origin is given by the equations

$$\sum_{j=1}^n (a_{nj} + \bar{b}_{nj}) z_j = 0.$$

Let L be the complex line defined by the equations of the analytic tangent plane and $\{z_2 = \cdots = z_{n-1} = 0\}$. On L we have $(a_{n1} + \bar{b}_{n1})z_1 + (a_{nn} + b_{nn})z_n = 0$. Thus $z_n = 0(\varepsilon)z_1$ on L . Moreover the Levi form of $\varphi \circ S_V$ at the origin restricted to L reduces to

$$(\beta_{11} + 0(\varepsilon))|z_1|^2.$$

Recall that $\beta_{11} < 0$. Hence if ε is sufficiently small, the Levi form of $\varphi \circ S_V$ at the origin restricted to L has one negative eigenvalue.

THEOREM. *Every pseudoconcave, complex Lie group G^0 is a complex torus.*

PROOF. We have already shown that there exists $V^0 \in M^*(m, n; \mathbb{C})$ such that G^0 is biholomorphically isomorphic to $\mathbb{C}^n/\Gamma(V^0)$. By Lemma 8, there exists $\varepsilon > 0$ such that if $d(V^0, V) < \varepsilon$ then $G = \mathbb{C}^n/\Gamma(V)$ is pseudoconcave. If $m > n$ then, by Lemma 7, there exists $V \in M^*(n, m; \mathbb{C})$ satisfying $d(V^0, V) < \varepsilon$ and the Morimoto condition. But, by Corollary 6, this is absurd. Thus $m = n$ and G^0 is a complex torus.

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