COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 25, nº 1 (1972), p. 79-86
<http://www.numdam.org/item?id=CM_1972__25_1_79_0>

COMPOSITIO MATHEMATICA, Vol. 25, Fasc. 1, 1972, pag. 79-86
Wolters-Noordhoff Publishing
Printed in the Netherlands

THE LIE ALGEBRA OF ENDOMORPHISMS OF AN INFINITE-DIMENSIONAL VECTOR SPACE

by

Ian Stewart

1. Introduction

The structure of the Lie algebra of all endomorphisms of a finite-dimensional vector space is well known. The purpose of this paper is to investigate the infinite-dimensional case, and in particular to find the lattice of Lie ideals. Rosenberg [6] has carried out the analogous programme for the infinite general linear group.

Notation for Lie algebras will follow that of [9, 10]. Let $\mathfrak{k}$ be any field. Let c be any infinite cardinal, with successor c^+. Let V be a vector space over $\mathfrak{k}$ of dimension c, and for any infinite cardinal $d \leqq c^+$ define $E(c, d)$ to be the set of all linear transformations $\alpha : V \to V$ such that the image of α has dimension $< d$. Then $E(c, d)$ is an associative $\mathfrak{k}$-algebra. Under commutation $[\alpha, \beta] = \alpha\beta - \beta\alpha$ $(\alpha, \beta \in E(c, d))$ it becomes a Lie algebra which we shall denote $L(c, d)$.

Inside $L(c, c^+)$ we let $F = L(c, \aleph_0)$, $T =$ the set of endomorphisms of trace zero (in the sense of [9] p. 306), $S =$ the set of scalar multiplications $v \to vk$ $(v \in V,\ k \in \mathfrak{k})$. We shall prove:

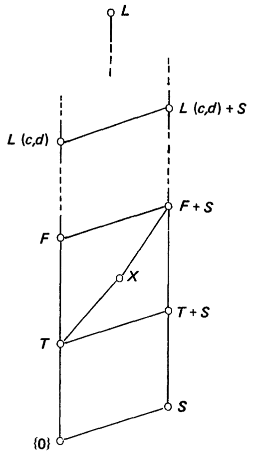
THEOREM (A). *Let $L = L(c, c^+)$. Then the ideals of L are precisely the following*:

a) $L(c, d)$ *for* $\aleph_0 \leqq d \leqq c^+$
b) $L(c, d) + S$ *for* $\aleph_0 \leqq d \leqq c$
c) *Any subspace* X *of* L *such that* $T \leqq X \leqq F + S$
d) S
e) $\{0\}$.

The lattice of ideals has the form as shown on the next page.
Further, every subideal of L is an ideal, so that L lies in the class $\mathfrak{T}$ *of* [9].

An immediate corollary of theorem A is that $L(c, c^+)$ satisfies the minimal condition for subideals, Min-si. We shall use this to show that theorem 3.3 of [9] p. 305 is in a sense best possible.

Finally we apply our results to prove that any Lie algebra can be embedded in a simple Lie algebra.



I am grateful to the referee for many helpful remarks which have simplified and improved the exposition.

2. The endomorphism algebra

We attack the problem through the associative ideal structure of $E(c, d)$, which is easily determined. By Jacobson [5] p. 108 an associative algebra A is simple if and only if it is simple considered as a ring. This remark combines with a theorem of Herstein [3] (see also Baxter [1]) to yield:

LEMMA (1). *If A is a simple associative $\mathfrak{k}$-algebra and $[A, A] = A$ then any proper Lie ideal of the Lie algebra associated with A is contained in the centre of A, unless A is of dimension 4 over its centre which is a field of characteristic 2.*

In the sequel all algebras considered will be infinite-dimensional over their centres, so the exceptional case never arises. By a slight extension of Jacobson [5] p. 93 theorem 1 we have:

LEMMA (2). *Let c, d be infinite cardinals with $d \leqq c^+$. Then any non-zero associative ideal of $E(c, d)$ is of the form $E(c, e)$ where $\aleph_0 \leqq e \leqq d$.*

COROLLARY. *If $c \geqq d$ are infinite cardinals then*

$$E(c, d^+)/E(c, d)$$

is a simple non-commutative associative algebra.

LEMMA (3). *Let $E = E(c, d)$ where $\aleph_0 < d \leqq c^+$. Then*

$$[E, E] = E.$$

PROOF. Let $a \in E$. Decompose V into a direct sum

$$V = X \oplus \bigoplus_{i \in \mathbf{Z}} V_i$$

in such a way that $\dim V_i = \dim im(a)$ for all i and that $im(a) \leqq W = \bigoplus_{i \in \mathbf{Z}} V_i$. For each i let $t_i : V_i \to V_{i+1}$ be an isomorphism. Let the automorphism $u : W \to W$ be defined by $u|_{V_i} = t_i$ and let $t : V \to V$ be defined by $t|_W = u$ and $t(X) = \{0\}$. We shall show that there exists $b \in E$ such that

$$[b, t] = a. \tag{1}$$

More precisely we show that there is a unique endomorphism b of V satisfying (1) such that

$$b(V_0) = \{0\} \tag{2}$$

and

$$b(V) \leqq W \tag{3}$$

(hence $b \in E$).

We set $a_i = a|_{V_i}$ and $b_i = b|_{V_i}$. In view of (3) the restrictions of (1) to X, to V_{i-1} ($i > 0$) and to V_i ($i < 0$) are respectively equivalent to the following equations:

$$b|_X = -u^{-1}a|_X \tag{4}$$

$$b_i = (a_{i-1} + t_i b_{i-1})t_{i-1}^{-1} \qquad (i > 0) \tag{5}$$

$$b_i = t_{i+1}^{-1}(a_i + b_{i+1} t_i) \qquad (i < 0) \tag{6}$$

and now the assertion is obvious since (5) and (6) constitute inductive definitions for the b_i.

Note that if $d = \aleph_0$ the lemma is false, for then $[E, E]$ is the set of trace zero maps which is smaller than E.

For any associative algebra A we let $Z(A)$ denote the centre of A. We then have:

LEMMA (4). *If $c \geqq d$ are infinite cardinals, then*

$$Z(E(c, d^+)/E(c, d))$$

is trivial except when $c = d$. It then has dimension 1 and consists of scalar multiplications (modulo $E(c, d)$).

This follows from:

LEMMA (5). *If $c \geqq d$ are infinite cardinals and $z \in L(c, c^+)$ satisfies*

$$[z, L(c, d^+)] \leqq L(c, d) + S$$

then $z \in L(c, d) + S$.

The proof of this lemma is more intricate than one might wish, and will be postponed until later.

Putting together the results so far obtained we have:

LEMMA (6). *If $c \geqq d$ are infinite cardinals then the Lie algebra*

$$L(c, d^+)/L(c, d)$$

is simple unless $c = d$; when its only nontrivial proper ideal is the centre, which has dimension 1 and consists of scalar multiplications (modulo $L(c, d)$).

The next result is implicit in [9] (p. 310):

LEMMA (7). *Let L be a Lie algebra, σ an ordinal, and $(G_\alpha)_{\alpha \leqq \sigma}$ an ascending series of ideals such that for all $\alpha < \sigma$*

1) *$G_{\alpha+1}/G_\alpha$ is simple non-abelian,*
2) *$C_{L/G_\alpha}(G_{\alpha+1}/G_\alpha) = G_\alpha/G_\alpha$.*

Then the only subideals of L are the G_α. Consequently $L \in$ Min-si $\cap\ \mathfrak{T}$.

PROOF. Let M be a proper subideal of L and let α be the least ordinal such that $G_\alpha \nleqq M$. It is easy to see that α cannot be a limit ordinal, so $\alpha = \beta+1$ for some β. Thus $(M+G_\beta)/G_\beta$ is a subideal of L/G_β not containing $G_{\beta+1}/G_\beta$. As the latter is a simple non-abelian ideal of L/G_β we have

$$(M+G_\beta)/G_\beta \cap G_{\beta+1}/G_\beta = G_\beta/G_\beta$$

so by [9] lemma 4.6 p. 309 M centralises $G_{\beta+1}/G_\beta$. By part (2) of the hypothesis $M \leqq G_\beta$, whence $M = G_\beta$.

Obviously $L \in \mathfrak{T}$, and $L \in$ Min-si since the ordinals are well-ordered.

Now we shall show that $L(c, d) \in$ Min-si $\cap\ \mathfrak{T}$. The presence of trace zero and scalar maps causes complications, so we study a suitable quotient algebra. Let $L = L(c, d)$, let F, S, T be as in theorem A, and put $I = F+S$. Then $L^* = L/I$ has an ascending series of ideals

$$O = L_0^* \leqq L_1^* \leqq \cdots \leqq L_\alpha^* \leqq \cdots \leqq L_\delta^* = L^*$$

for a suitable ordinal δ; the L_α^* being the ideals $(L(c, e)+S)/I$ arranged in ascending order.

Now I has a series $O \leqq T \leqq F \leqq I$ of ideals. But T is simple ([9] lemma 4.1 p. 306) and F/T and I/F are 1-dimensional. Therefore $I \in$ (Min-si)($\mathfrak{F}$)($\mathfrak{F}$) $\leqq$ Min-si, by [9] lemma 2.2 p. 303. By the same lemma, in order to prove that $L \in$ Min-si, it suffices to show that $L^* \in$ Min-si. This will follow from lemma 7 provided we can prove that

$$C_{L^*/L_\alpha^*}(L_{\alpha+1}^*/L_\alpha^*) = L_\alpha^*/L_\alpha^*$$

which is equivalent to the statement of lemma 5.

We now come to the proof of lemma 5. To simplify the notation we let $L = L(c, c^+)$, $E = L(c, d)$, $G = L(c, d^+)$. To prove lemma 5 we must show that if $z \in L$ and $[z, G] \leqq E+S$, then $z \in E+S$.

If V is a vector space with basis $(v_\lambda)_{\lambda \in \Lambda}$ and a is an endomorphism of V, we define $a_{\alpha\beta}(\alpha, \beta \in \Lambda)$ by:

$$v_\alpha a = \sum a_{\alpha\beta} v_\beta .$$

LEMMA (8). *If V is a vector space with basis $(v_\lambda)_{\lambda \in \Lambda}$ where Λ is infinite, and if a is an endomorphism of V such that* dim $im(a) = e$ *is infinite, then the set*

$$B = \{\beta : a_{\alpha\beta} \neq 0 \textit{ for some } \alpha \in \Lambda\}$$

has cardinality $|B| = e$.

PROOF. Let $W = \sum_{\lambda \in B} \mathfrak{k} v_\lambda$. By definition $\dim(W) = |B|$, and since $im(a) \leqq W$ we have $e \leqq |B|$. If $(i_\mu)_{\mu \in M}$ is a basis for $im(a)$, then each i_μ is a linear combination of finitely many v_λ $(\lambda \in B)$. Therefore $|B| \leqq |\boldsymbol{Z} \times M| = \aleph_0 \cdot e = e$ since e is infinite.

We now suppose that z is as above, and that V is a vector space with basis $(v_\lambda)_{\lambda \in \Lambda}$ where $|\Lambda| = c$.

LEMMA (9). *There exists z' such that $z'_{\alpha\alpha} = 0$ $(\alpha \in \Lambda)$, $[z', G] \leqq E+S$, and $z - z' \in E+S$.*

PROOF. Let $\mathscr{M}$ be the set of all pairs $(M, <)$ where M is a subset of Λ and $<$ is a well-ordering on M, such that if $\alpha \in M$ then $z_{\alpha\alpha} \neq z_{\alpha+1,\alpha+1}$ (where $\alpha+1$ is the successor to α in the ordering $<$). Then $\mathscr{M}$ is partially ordered by $\ll$, where $(M_1, <_1) \ll (M_2, <_2)$ if and only if M_1 is an initial segment of M_2. Clearly $\mathscr{M}$ is not empty and satisfies the hypotheses of Zorn's lemma. Let $(M, <)$ be a maximal element of $\mathscr{M}$. Suppose for a contradiction that $|M| \geqq d$. Take an initial segment I of M with $|I| = d$, and consider

$$t = [z, \sum_{\alpha \in I} e_{\alpha,\alpha+1}]$$

where $e_{\alpha\beta}$ $(\alpha, \beta \in \Lambda)$ is the elementary transformation sending v_α to v_β and all other basis elements to zero. By hypothesis $t \in E+S$, yet

$$\begin{aligned} t &= \sum z_{\alpha\beta} e_{\alpha\beta} e_{\beta,\beta+1} - \sum z_{\alpha\beta} e_{\alpha-1,\alpha} e_{\alpha\beta} \\ &= \sum (z_{\alpha,\beta-1} - z_{\alpha+1,\beta}) e_{\alpha\beta} \end{aligned}$$

(where terms involving $\alpha-1$ for limit ordinals α are deemed to be zero). Now the coefficient of $e_{\alpha,\alpha+1}$ is $z_{\alpha\alpha} - z_{\alpha+1,\alpha+1}$ which is non-zero for d values of α. By lemma 8 $t \notin E+S$ which is a contradiction.

Thus after choosing fewer than d values of α all the remaining $z_{\alpha\alpha}$ are equal. Thus $\sum z_{\alpha\alpha} e_{\alpha\alpha} \in E+S$. Define $z' = z - \sum z_{\alpha\alpha} e_{\alpha\alpha}$.

LEMMA (10). *Suspose that $z' \notin E+S$. Then there exist subsets A, A' of Λ and a bijection $\phi : A \to A'$ such that*

1) $A \cap A' = \emptyset$
2) *If* $\phi(\alpha) = \alpha'$ $(\alpha \in A)$ *then* $z_{\alpha\alpha'} \neq 0$
3) $|A| = |A'| = d.$

PROOF. Let $\mathscr{S}$ be the collection of all triples (A, A', ϕ) satisfying (1) and (2). Partially order $\mathscr{S}$ by $\ll$ where $(A, A', \phi) \ll (B, B', \Psi)$ if and only if $A \leqq B$, $A' \leqq B'$, and $\Psi|_A = \phi$. By Zorn's lemma there is a maximal element (A, A', ϕ) of $\mathscr{S}$. For brevity let $\phi(\alpha) = \alpha'$ $(\alpha \in A)$. We claim that $|A| = d$.

Suppose not. Then $|A| = d' < d$. Let

$$D = \{\delta : z'_{\gamma\delta} \neq 0, \gamma \in A \cup A'\}.$$

Since d is infinite we have $|D| < d$. By lemma 8 there must exist $\gamma' \notin (A \cup A' \cup D)$ such that $z'_{\gamma\gamma'} \neq 0$ for some $\gamma \neq \gamma'$ (since $z' \notin E+S$). Then $\gamma \notin (A \cup A')$ since $\gamma' \notin D$. Therefore $\gamma \neq \gamma'$, $\gamma \notin (A \cup A')$, $\gamma' \notin (A \cup A')$. Define

$$B = A \cup \{\gamma\}$$
$$B' = A' \cup \{\gamma'\}$$
$$\Psi(\beta) = \beta' \ (\beta \in A)$$
$$\Psi(\gamma) = \gamma'.$$

Then $(B, B', \Psi) \in \mathscr{S}$ and is greater than (A, A', ϕ), a contradiction. Hence $|A| \geqq d$ as claimed.

We may now derive the final contradiction required to prove lemma 5.

Suppose for a contradiction that $z' \notin E+S$. Then there exists (A, A', ϕ) as in lemma 10. Define $\pi : V \to V$ by

$$v_\alpha \pi = v_{\alpha'} \qquad (\alpha \in A)$$
$$v_{\alpha'} \pi = v_{\alpha'} \qquad (\alpha' \in A')$$
$$v_\beta \pi = 0 \qquad (\beta \in \Lambda \setminus (A \cup A')).$$

By definition $\pi \in G$. By hypothesis $u = [z', \pi] \in E+S$. But for $\alpha \in A$ we have

$$v_\alpha(z'\pi - \pi z') = \sum z'_{\alpha\beta} v_\beta \pi - \sum z'_{\alpha'\beta} v_\beta .$$

The coefficient of $v_{\alpha'}$ is

$$z'_{\alpha\alpha} + z'_{\alpha\alpha'} - z'_{\alpha'\alpha'} = z'_{\alpha\alpha'} \neq 0$$

so that $u_{\alpha\alpha'} \neq 0$ if $\alpha \in A$. Since $|A| = d$ and $\alpha \neq \alpha'$ we have $u \notin E+S$, a contradiction.

Hence $z' \in E+S$, whence $z \in E+S$, and lemma 5 is proved. By lemma 7 we have:

LEMMA (11).

1) $L(c, c^+) \in$ Min-si,

2) *Every subideal of* $L(c, c^+)$ *which contains* $F+S$ *is of the form* $L(c, d)+S$.

LEMMA (12). $L(c, c^+) \in \mathfrak{T}$.

PROOF. Suppose $L = L(c, c^+)$ has a proper ideal J of finite codimension. Now L has an ascending series, the finite-dimensional factors of which are abelian, the rest simple. Hence L/J is soluble, so that $[L, L] < L$, contrary to lemma 3. Therefore by theorem 3.1 of [9] p. 305 we have $L \in \mathfrak{T}$.

We now proceed to the:

PROOF OF THEOREM (A).

All the subalgebras listed are ideals; the only case requiring comment being (c). Since $L = L(c, c^+)$ has no ideals of finite codimension (proof of lemma 12) the factor $(F+S)/T$ is central (see [9] p. 305, proof of theorem 3.1). Therefore any subspace X between T and $F+S$ is an ideal.

Suppose now that I is an ideal of L. If $I \geqq F+S$ then by lemma 11 I is in the given list. Therefore we may assume $I \ngeqq F+S$. If $I \cap T = \{0\}$ then $[I, T] = \{0\}$. But it is easy to see that the only elements of L centralising every elementary transformation $e_{\alpha\beta}$ ($\alpha \neq \beta$) are the elements of S. Hence $I \leqq S$. Since $\dim S = 1$ we have $I = \{0\}$ or S. But T is simple ([9] lemma 4.1 p. 306) so if $I \cap T \neq \{0\}$ then $T \leqq I$. Now $I+F+S \lhd L$, and by lemma 11 $I+F+S = L(c, d)+S$ for some d. If $d = \aleph_0$ then $T \leqq I \leqq F+S$, which is case (c) of the list. There remains the case $d > \aleph_0$. Then we have $(I+F+S)/(T+S) = (L(c, d)+S)/(T+S)$ so that $(I+T+S)/(T+S)$ is of codimension $\leqq 1$ in $(L(c, d)+S)/(T+S) \cong (L(c, d))/T$ which has no proper ideals of finite codimension by the argument of lemma 12. Therefore $I+T+S = L(c, d)+S$. Now $T \leqq I$ so we have $I+S = L(c, d)+S$. If $I \neq L(c, d)+S$ and $I \neq L(c, d)$ then $I \cap L(c, d)$ is of codimension 1 in $L(c, d)$, contradicting lemma 3. Hence $I = L(c, d)$ or $I = L(c, d)+S$.

We have already remarked (in lemma 12) that $L \in \mathfrak{T}$; which completes the proof of the theorem.

3. Applications

In [9] it is proved that any Lie algebra satisfying Min-si and having no ideals of finite codimension has an ascending series of ideals whose factors are either infinite-dimensional simple or 1-dimensional central. The results of theorem A show that the 1-dimensional central factors cannot in general be dispensed with. In [9] this question was left open. The algebras $L(c, d)$ also provide new examples of Lie algebras in Min-si $\cap\ \mathfrak{T}$.

Following the general lines of Scott [7] p. 316 section 11.5.4 (for groups) we can prove:

THEOREM (B). *Any Lie algebra can be embedded in a simple Lie algebra.*

PROOF. Let K be a Lie algebra over a field $\mathfrak{k}$. By Jacobson [4] p. 162 cor. 4 K has a faithful representation by endomorphisms of a vector space V over $\mathfrak{k}$. By enlarging V if necessary we may embed K in $L(c^+, c^+)$ for some infinite cardinal c. If we split V into c subspaces of dimension c^+ and copy the K-action on each of these we may assume that K is represented by endomorphisms whose image has dimension $\geqq c$. Then the composite embedding

$$K \to L(c^+, c^+) \to L(c^+, c^+)/L(c^+, c)$$

maps K into a simple Lie algebra.

One might ask about Lie analogies of other embedding theorems for groups. For example, Dark [2] has proved that every group can be embedded as a subnormal subgroup of a perfect group. Strangely, the analogue of this is false for Lie algebras – an example may be found in [8], p. 98

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(Oblatum 7–I–1971
29–VII–1971)

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