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INTERMEDIATE FLEXIBILITY OF SURFACES

by

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1. Introduction

In Efimov's article [1] cohomology properties of a surface S immersed in Euclidean 3-space are related to existence and classification of infinitesimal isometric deformations of S. He defines the intermediate flexibility of S with respect to a subgroup F of the 1-dimensional homology group H of S; it becomes an isometric embedding invariant based on topological properties of S by means of the Rham cohomology.

So far, however, this relation between the classical rigidity problems and algebraic topology has remained of a hypothetical nature, since no surface having intermediate flexibility with respect to a non-trivial subgroup was known.

We exhibit here the first examples of truly intermediate flexibility. In addition we give a necessary and sufficient condition for surfaces in a certain class to admit intermediate flexibility.

Beside the obvious generalization, new phenomena of intermediate flexibility appear in higher dimensions; they will be discussed elsewhere.

2. Notations

We denote by τ the class of surfaces of revolution in E^3 homeomorphic to a torus with arbitrary smooth Jordan curves as meridians. More precisely:

DEFINITION 1. Let e_1, e_2, e_3 denote a fixed orthonormal system of vectors in E^3. A surface S belongs to the class τ if its position vector is representable as

$$x(u, v) = r(v)\cos u \cdot e_1 + r(v)\sin u \cdot e_2 + h(v)e_3,$$

$$0 \leqq u \leqq 2\pi,$$

where $r(v)$ and $h(v)$ are two functions of class C^1 defining by

$$x(0, v) = r(v)e_1 + h(v)e_3,$$

a simple closed meridian curve M which does not intersect the x_3-axis and for which the variable v plays the part of the arc length. Hence $r(v)$ and $h(v)$ are periodic of period L (perimeter of M) and satisfy $r(v) > 0$ and

$$r'^2(v)+h'^2(v) = 1, \qquad 0 \leqq v \leqq L, \tag{1}$$

accents denoting derivatives with respect to v.

In the following we consider vector fields y, z, s of E^3 defined on S only, and refer for motivation to [1].

Under an infinitesimal isometric deformation of first order every pencil of line-elements on S is moved as a rigid body. If the point x is thereby transformed into

$$x^* = x(u, v)+\varepsilon z(u, v) \tag{2}$$

(where ε is a parameter whose powers exceeding one are neglected), the deformation field z satisfies the local orthogonality condition

$$dx \cdot dz = 0,$$

and can be represented as

$$z = s+[y, x]$$

with

$$ds = [x, dy], dz = [y, dx]. \tag{3}$$

Here square brackets denote the vector product, s is the 'translation field', and y the 'rotation field' of the deformation.

Assuming the field z to be only locally defined (i.e., admitting 'multi-valued' fields z) we generalize (3) by

Definition 2. A rotation field on S is a C^2-vector field y of E^3 defined on S such that $[y, dx]$ is a closed (vector) differential.

In terms of the local coordinates (u, v) on S the components of y are twice continuously differentiable functions, periodic in u of period 2π and periodic in v of period L.

Every constant field $y = y_0$ is trivially a rotation field.

Definition 3. The surface S is non-rigid with respect to the subgroup F of the 1-dimensional homology group H over the reals if

a) there exists a non-trivial rotation field y such that the periods

$$P(y, C) = \oint_C [y, dx]$$

vanish for all closed curves C whose homology classes belong to F, and

b) there is no non-trivial rotation field having this property for a subgroup of H containing F.

If $F = H$ this concept corresponds to projective or strong flexibility; for $F = 0$ we have affine or weak flexibility. If there is a non-trivial subgroup F with this property we speak of intermediate flexibility.

3. The rotation fields

Let $y = \sum_{i=1}^{3} y_i e_i$ be an arbitrary rotation field on $S \in \tau$. The local exactness of $[y, dx]$ is expressed by the integrability condition

$$\left[\frac{\partial y}{\partial u}, \frac{\partial x}{\partial v}\right] = \left[\frac{\partial y}{\partial v}, \frac{\partial x}{\partial u}\right]. \tag{4}$$

Introducing the auxiliary functions

$$U(u, v) = y_1 \sin u - y_2 \cos u,$$

$$V(u, v) = y_1 \cos u + y_2 \sin u,$$

$$Z(u, v) = y_3,$$

we find that condition (4) is equivalent to the following system of partial differential equations:

$$h'\left(\frac{\partial U}{\partial u} - V\right) = r\frac{\partial Z}{\partial v}, \tag{5}$$

$$h'\left(U + \frac{\partial V}{\partial u}\right) = r'\frac{\partial Z}{\partial u}, \tag{6}$$

$$\frac{\partial}{\partial u}(r'U) = \frac{\partial}{\partial v}(rV). \tag{7}$$

Taking into account (1) we obtain as further consequences:

$$\frac{\partial U}{\partial u} = V + r\left(h'\frac{\partial Z}{\partial v} + r'\frac{\partial V}{\partial v}\right), \tag{8}$$

$$h'\frac{\partial V}{\partial v} = r'\frac{\partial Z}{\partial v}. \tag{9}$$

Under our assumptions the functions V and Z can be expanded in absolutely and uniformly convergent Fourier series with respect to u:

$$V(u, v) = \tfrac{1}{2}a_0(v) + \sum_{k=1}^{\infty} [a_k(v)\cos(ku) + b_k(v)\sin(ku)], \tag{10}$$

$$Z(u, v) = \tfrac{1}{2}c_0(v) + \sum_{k=1}^{\infty} [c_k(v)\cos(ku) + d_k(v)\sin(ku)], \tag{11}$$

where the coefficients a_k, b_k, c_k, d_k are C^2 functions of v with period L.

Substituting (10) and (11) into (8) and integrating we obtain:

$$a_0+rr'a_0'+rh'c_0' = 0. \tag{12}$$

(This expresses the condition that U is periodic in u.)

Then the Fourier expansion of U assumes the form

$$U(u, v) = U_0(v)+\sum_{k=1}^{\infty} \left[\frac{1}{k}(a_k+rr'a_k'+rh'c_k') \sin (ku) - \frac{1}{k}(b_k+rr'b_k'+rh'd_k') \cos (ku)\right], \tag{13}$$

where $U_0(v)$ is a function of period L.

Substituting these expansions in (5) to (9) leads to

PROPOSITION 1. *The most general rotation field on the surface $S \in \tau$ has the form*

$$y = (U \sin u+ V \cos u)e_1+(-U \cos u+ V \sin u)e_2+Ze_3,$$

where U, V, Z are determined by (10), (11), *and* (13), *and the coefficients are periodic solutions of the following system of first order linear homogeneous differential equations*:

$$\left.\begin{aligned}
&(14) & h'(k^2-1)a_k &= k^2r'c_k+rc_k',\\
&(15) & h'(k^2-1)b_k &= k^2r'd_k+rd_k',\\
&(16) & h'a_k' &= r'c_k',\\
&(17) & h'b_k' &= r'd_k'.
\end{aligned}\right\} \quad k = 1, 2, \cdots$$

For $k = 0$ the above relations must be replaced by

$$h'U_0(v) = 0, \tag{18}$$

$$h'a_0' = r'c_0', \tag{19}$$

$$r'a_0+ra_0' = 0, \tag{20}$$

$$h'a_0+rc_0' = 0. \tag{21}$$

Note that the two pairs of functions (a_k, c_k) and (b_k, d_k) satisfy for $k = 1, 2, \cdots$ the same conditions.

LEMMA. *For every rotation field y on every surface $S \in \tau$ we have*

$$a_0 = c_1 = d_1 \equiv 0, \qquad c_0 = \text{constant}.$$

PROOF. From equations (20) and (21) we deduce

$$a_0(v) = \frac{A}{r(v)}, \; c_0(v) = C - A\int_0^v \frac{h'(w)}{r^2(w)}\,dw,$$

with A and C constants.

Since $c_0(v)$ is periodic with period L we have $c_0(0) = c_0(L) = C$; hence

$$A\int_0^L \frac{h'(w)}{r^2(w)}\,dw = A\oint_M \frac{dh}{r^2} = 0. \tag{22}$$

By the Jordan-Brouwer Theorem the region m enclosed by the meridian M has positive measure (also denoted by m). Furthermore the function r^{-3} assumes a positive minimum ρ on the compact set $\bar{m}$. We conclude by Stokes' theorem:

$$\oint_M \frac{dh}{r^2} = -2\iint_m r^{-3}dm \leqq -2\,\rho\, m < 0. \tag{23}$$

Combining (22) and (23) we obtain $A = 0$, $a_0 = 0$, $c_0 = \text{constant}$.

Taking into account (14) we get

$$c_1(v) = \frac{D}{r(v)}, \; D = \text{constant}. \tag{24}$$

Since $h(v)$ assumes an extremum there exists some $\bar{v}$ with $h'(\bar{v}) = 0$. By (1) we have $r'(\bar{v}) = \pm 1$. Substituting (24) into (16) we find

$$c_1'(\bar{v}) = \pm\frac{D}{r^2(\bar{v})} = 0;$$

hence $D = 0$ and $c_1(v) \equiv 0$.

Similarly $d_1 \equiv 0$. Q.E.D.

As a consequence of this lemma and of the equations (16) and (17) we note the following relations which will be needed in § 5:

$$h'a_1' = 0, \tag{25}$$

$$h'b_1' = 0. \tag{26}$$

4. The periods of $[y, dx]$

Definition 4. As representatives of a homology basis for a surface $S \in \tau$ we choose the following cycles:

$$C_1 : x(u, 0) = r(0)\cos ue_1 + r(0)\sin ue_2 + h(0)e_3,$$

$$0 \leqq u \leqq 2\pi,$$

(a circle of latitude),

$$C_2 : x(0, v) = r(v)e_1 + h(v)e_3,$$
$$0 \leqq v \leqq L,$$

(a meridian).

Correspondingly we denote the periods of the vector differential $[y, dx]$ by

$$P_i = \oint_{C_i} [y, dx], \qquad i = 1, 2.$$

THEOREM 1. *For every rotation field y on every surface $S \in \tau$ the period P_1 vanishes.*

PROOF. Taking into account the orthogonality relations of the trigonometric functions and the lemma of § 3 we compute

$$\begin{aligned} P_1 &= [-r(0)\int_0^{2\pi} Z(u, 0) \cos u \, du]e_1 \\ &\quad -[r(0)\int_0^{2\pi} Z(u, 0) \sin u \, du]e_2 \\ &\quad +[r(0)\int_0^{2\pi} V(u, 0) du]e_3 = \\ &= -\pi r(0)c_1(0)e_1 - \pi r(0)d_1(0)e_2 + \pi r(0)a_0(0)e_3 \\ &= 0. \end{aligned}$$

Q.E.D.

The surface $S \diagdown C_1 \equiv S_1$ is homeomorphic to a cylinder, and theorem 1 entails that every closed differential $[y, dx]$ on S_1 is exact.

This means:

THEOREM 2. *If a surface of class τ is cut along a curve homologous to a circle of latitude we obtain a projectively flexible surface.*

Turning to P_2 we obtain

PROPOSITION 2. *For every rotation field on every surface of class τ we have*

$$P_2 = (\int_0^L h'b_1 \, dv)e_1 - (\int_0^L h'a_1 \, dv)e_2 + (\int_0^L U_0 \, r' \, dv) \, e_3.$$

Thus P_2 depends only on the lowest Fourier coefficients.

PROOF. From (14) and the periodicity of the functions $r(v)$ and $c_k(v)$ we deduce the following integral relations:

$$(k^2-1)\int_0^L (h'a_k - r'c_k)dv = \int_0^L (rc_k)'dv = 0, \qquad k = 1, 2, \cdots$$

Together with Lemma 1 they imply (after some computations) the desired result. Q.E.D.

The vector P_2 has the following geometric interpretation. Using (3) we find

$$P_2 = \oint_{C_2} dz = z_2 - z_1,$$

where z_1 and z_2 are the values of the deformation field on opposite sides of a cut in S homologous to C_1. For the relative shift between these opposite sides under the deformation (2) we obtain

$$x_2^* - x_1^* = \varepsilon(z_2 - z_1) = \varepsilon P_2.$$

Since this is independent of (u, v) we see that after the deformation of S_1 is carried out the two sides of our cut are separated by a gap 'of constant width', i.e., they can be made to coincide again by a translation through εP_2.

5. A criterion for intermediate flexibility

DEFINITION 5. Let F denote the subgroup of the 1-dimensional homology group of a surface $S \in \tau$ generated by the homology class containing the cycle C_1.

THEOREM 3. *A surface $S \in \tau$ has intermediate flexibility with respect to the group F if and only if its meridian contains a segment of non-zero length perpendicular to the axis of revolution.*

PROOF. Denote by $\mathscr{H}$ the subset of the closed interval $[0, L]$ consisting of those reals v satisfying $h'(v) = 0$. Denote by $C\mathscr{H}$ its complement with respect to $[0, L]$; furthermore let $\mathscr{B}$ be the subset of those $v \in [0, L]$ where $b_1'(v) = 0$.

The relation (26) then translates into the inclusion $C\mathscr{H} \subset \mathscr{B}$, valid for every rotation field on S.

a) If $\mathscr{H}$ does not contain an interval then $C\mathscr{H}$ and therefore $\mathscr{B}$ are dense in $[0, L]$. The continuity of b_1' implies $b_1' \equiv 0$ and $b_1 = $ constant everywhere. Similarly we obtain from (25) that $a_1 = $ constant, and from (18) that $U_0 \equiv 0$. Proposition 2 then implies $P_2 = 0$. By de Rham's theorem the vanishing of P_1 and P_2 means that $[y, dx]$ is exact; hence S has no intermediate flexibility.

b) Conversely, let $\mathscr{H}$ contain a non-zero interval $\mathscr{I}$. For $v \in \mathscr{I}$ we have either $r'(v) = 1$ or $r'(v) = -1$. We can satisfy (18) by constructing a function $U_0 \in C^2$ vanishing on $C\mathscr{I}$ and such that

$$\int_{\mathscr{I}} U_0(v) dv \neq 0.$$

Letting $c_0 = a_k = b_k = c_k = d_k \equiv 0$ for $k = 1, 2, \cdots$ then defines the non-trivial rotation field

$$y = U_0(v)(\sin u \cdot e_1 - \cos u e_2),$$

where $P_2 \neq 0$ according to Proposition 2. Thus S has intermediate flexibility with respect to F. Q.E.D.

If the meridian M contains a single straight segment perpendicular to the axis, P_2 has the direction of the axis for every rotation field y. For M containing more than one segment, P_2 can assume any direction and magnitude by suitable choice of y.

COROLLARY. *A surface in class τ with an analytic meridian does not have intermediate flexibility.*

6. Application to Liebmann surfaces

DEFINITION 6. A Liebmann surface is a surface of revolution whose meridian contains an outer convex arc (i.e. convex away from the axis of revolution) with tangents perpendicular to the axis of revolution at the endpoints.

It is known [2] that every deformation field on a Liebman surface has the 'Euler form'

$$z = [a, x] + b,$$

with a and b constant vectors. Thus for *globally defined* deformation fields the corresponding rotation fields are trivial. Combining this result with our Theorem 3 we can drop the requirement that z be globally defined as follows:

THEOREM 4. *Let S be a Liebman surface of class τ whose meridian does not contain a segment perpendicular to the axis of revolution. Then S admits only trivial rotation fields.*

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