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ANOTHER PROPERTY OF THE SORGENFREY LINE

by

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1. Introduction

The Sorgenfrey line S, one of the most important counterexamples in general topology, is obtained by retopologizing the set of real numbers, taking half-open intervals of the form $[a, b[$ to be a base for S. Recently, R. W. Heath and E. Michael proved that $S^{\aleph_0}$, the product of countably many copies of S, is perfect (= closed sets are G_δ's) [3]. At the Washington State Topology Conference in March, 1970, the question of whether $S^{\aleph_0}$ is subparacompact was raised. In this paper, we give an affirmative answer to this question and, in the process, give some positive results concerning products of more general subparacompact spaces.

2. Definitions and preliminary results

(2.1) Definition. A space X is *subparacompact* if every open cover of X has a σ-locally finite closed refinement.

The definitive study of subparacompact spaces is [2].

(2.2) Definition [3]. A space X is *perfect* if every closed subset of X is a G_δ in X. A space is *perfectly subparacompact* if it is perfect and subparacompact.

It is clear that if $\mathscr{G}$ is any collection of open subsets of a perfectly subparacompact space X, then there is a collection $\mathscr{F}$ of closed subsets of X which is σ-locally finite in X, refines $\mathscr{G}$ and has $\cup \mathscr{F} = \cup \mathscr{G}$.

(2.3) Definition (Morita, [4]). X is a P-space if for each open cover $\{U(\alpha_1, \cdots, \alpha_n) : \alpha_j \in A,\ n \geqq 1\}$ of X which satisfies $U(\alpha_1, \cdots, \alpha_n) \subseteq U(\alpha_1, \cdots, \alpha_n, \alpha_{n+1})$ whenever $\alpha_1, \cdots, \alpha_n, \alpha_{n+1} \in A$ there is a closed cover $\{H(\alpha_1, \cdots, \alpha_n) : \alpha_j \in A, n \geqq 1\}$ of X which satisfies:

(i) $H(\alpha_1, \cdots, \alpha_n) \subseteq U(\alpha_1, \cdots, \alpha_n)$ for each n-tuple $(\alpha_1, \cdots, \alpha_n)$ of elements of A;

(ii) if $\langle \alpha_n \rangle$ is a sequence of elements of A such that $\bigcup_{n=1}^{\infty} U(\alpha_1, \cdots, \alpha_n) = X$, then $\bigcup_{n=1}^{\infty} H(\alpha_1, \cdots, \alpha_n) = X$.

It is easily seen that if X is perfect, then X is a P-space.
The following definition is equivalent to the definition in [5].

(2.4) DEFINITION. A space Y is a *Σ-space* if there is a sequence $\langle \mathscr{F}(n) \rangle$ of locally finite closed covers of Y such that

(i) each $\mathscr{F}(n)$ is closed under finite intersections;

(ii) $\mathscr{F}(n) = \{F(\alpha_1, \cdots, \alpha_n) : \alpha_j \in A\}$;

(iii) each $F(\alpha_1, \cdots, \alpha_n)$ is the union of all $F(\alpha_1, \cdots, \alpha_n, \beta)$, $\beta \in A$;

(iv) for each $y \in Y$, the set $C(y) = \bigcap_{n=1}^{\infty} [\cap \{F \in \mathscr{F}(n) : y \in F\}]$ is countably compact;

(v) for each $y \in Y$, there is a sequence $\langle \alpha_n \rangle$ of elements of A such that $\{F(\alpha_1, \cdots, \alpha_n) : n \geqq 1\}$ is an *outer network for* $C(y)$ in Y, i.e., if V is open in Y and $C(y) \subseteq V$, then for some $n \geqq 1$, $C(y) \subseteq F(\alpha_1, \cdots, \alpha_n) \subseteq V$.

The sequence $\langle \mathscr{F}(n) \rangle$ is called a *spectral Σ-network for Y.*

It is clear that if the space Y in (2.4) is subparacompact, then each of the sets $C(y)$ will be compact. That the converse is also true is an unpublished result of E. Michael. Thus, any Σ-space Y in which closed, countably compact sets are compact is subparacompact. This is the case, for example, if Y is metacompact or θ-refinable[6].

(2.5) PROPOSITION. Suppose that X and Y are regular subparacompact spaces. If X is a P-space and Y is a Σ-space, then $X \times Y$ is subparacompact.

REMARK. The proof of (2.5) closely parallels the proof of Theorem 4.1 of [5]; we present only an outline.

PROOF. Since $X \times Y$ is regular, it is enough to show that if $\mathscr{W}$ is an open cover of $X \times Y$ which is closed under finite unions, then $\mathscr{W}$ has a σ-locally finite refinement. Let $\mathscr{W}$ be such a cover. Let $\langle \mathscr{F}(n) \rangle$ be a spectral Σ-network for Y, as in (2.4). For each $n \geqq 1$ and for each n-tuple $(\alpha_1, \cdots, \alpha_n)$ of elements of A, let $\mathscr{U}(\alpha_1, \cdots, \alpha_n) = \{R \subseteq X : R$ is open and $R \times F(\alpha_1, \cdots, \alpha_n)$ is contained in some member of $\mathscr{W}\}$. Let $U(\alpha_1, \cdots, \alpha_n) = \cup \mathscr{U}(\alpha_1, \cdots, \alpha_n)$ and let $\mathscr{U} = \{U(\alpha_1, \cdots, \alpha_n) : n \geqq 1$ and $\alpha_i \in A\}$. Then $\mathscr{U}$ is an open cover of X as in (2.3) so $\mathscr{U}$ has a closed refinement $\mathscr{H} = \{H(\alpha_1, \cdots, \alpha_n) : n \geqq 1$ and $\alpha_i \in A\}$ as in (2.3). Each $H(\alpha_1, \cdots, \alpha_n)$ is subparacompact, being closed in X, and is covered by $\mathscr{U}(\alpha_1, \cdots, \alpha_n)$. Hence there are collections $\mathscr{J}(\alpha_1, \cdots, \alpha_n) = \bigcup_{m=1}^{\infty} \mathscr{J}_m(\alpha_1, \cdots, \alpha_n)$ of closed subsets of $H(\alpha_1, \cdots, \alpha_n)$ which cover the sets $H(\alpha_1, \cdots, \alpha_n)$ and which are σ-locally finite in X. Let $\mathscr{S}(m, n) = \{J \times F(\alpha_1, \cdots, \alpha_n) : J \in \mathscr{J}_m(\alpha_1, \cdots, \alpha_n)$ and $F(\alpha_1, \cdots, \alpha_n) \in \mathscr{F}(n)\}$ and let $\mathscr{S} = \bigcup_{m,n \geqq 1} \mathscr{S}(m, n)$. Then $\mathscr{S}$ is σ-locally finite in $X \times Y$

and refines $\mathscr{W}$. To show that $\mathscr{S}$ covers $X \times Y$, let $(x, y) \in X \times Y$ and let $\langle \alpha_n \rangle$ be a sequence of elements of A such that $\{F(\alpha_1, \cdots, \alpha_n) : n \geqq 1\}$ is an outer network for $C(y)$ in Y. Then $\bigcup_{n=1}^{\infty} U(\alpha_1, \cdots, \alpha_n) = X$, so $\bigcup_{n=1}^{\infty} H(\alpha_1, \cdots, \alpha_n) = X$. Choose n such that $x \in H(\alpha_1, \cdots, \alpha_n)$ and m such that some $J \in \mathscr{J}_m(\alpha_1, \cdots, \alpha_n)$ has $x \in J$. Then $(x, y) \in J \times F(\alpha_1, \cdots, \alpha_n) \in \mathscr{S}(m, n) \subseteq \mathscr{S}$.

(2.6) COROLLARY. If X is regular and perfectly subparacompact and if Y is metrizable, then $X \times Y$ is subparacompact.

PROOF. Any perfect space is a P-space and any metrizable space is a subparacompact Σ-space.

Our proof that $S^{\aleph_0}$ is subparacompact proceeds inductively to show that S^n is perfectly subparacompact for each $n \geqq 1$ and then invokes the following proposition.

(2.7) PROPOSITION. Suppose $\{X(k) : k \geqq 1\}$ is a collection of spaces such that each finite product $\prod\{X(k) : 1 \leqq k \leqq n\}$ is perfectly subparacompact. Then so is $\prod\{X(k) : k \geqq 1\}$.

PROOF. Let $Y = \prod\{X(k) : k \geqq 1\}$ and let $Y(n) = \prod\{X(k) : 1 \leqq k \leqq n\}$. Let $\mathscr{W} = \{W(\alpha) : \alpha \in A\}$ be a basic open cover of Y. Thus, for each $\alpha \in A$, there is an integer $N(\alpha)$ and open subsets $U(k, \alpha)$ of $X(k)$ for $1 \leqq k \leqq N(\alpha)$ such that, if $\pi_k : Y \to X(k)$ denotes the usual projection, then $W(\alpha) = \cap \{\pi_k^{-1}[U(k, \alpha)] : 1 \leqq k \leqq N(\alpha)\}$. Let $A(n) = \{\alpha \in A : N(\alpha) = n\}$ and let

$$\mathscr{G}(n) = \{U(1, \alpha) \times \cdots \times U(n, \alpha) : \alpha \in A(n)\}.$$

Since $Y(n)$ is perfectly subparacompact, there is a σ-locally finite collection $\mathscr{F}(n) = \bigcup_{m=1}^{\infty} \mathscr{F}(n, m)$ of closed subsets of $Y(n)$ which refines $\mathscr{G}(n)$ and covers $\cup \mathscr{G}(n)$. Let

$$\mathscr{H}(n, m) = \{F \times \prod_{k=n+1}^{\infty} X(k) : F \in \mathscr{F}(n, m)\},$$

Then $\mathscr{H} = \bigcup_{m,n \geqq 1} \mathscr{H}(n, m)$ is a σ-locally finite closed cover of Y which refines $\mathscr{W}$. Hence Y is subparacompact. That Y is also perfect follows from Proposition 2.1 of [3].

The following concept is the key to our proof that S^n is perfectly subparacompact for each finite $n \geqq 1$.

(2.8) DEFINITION. A space X is *weakly θ-refinable* if for each open cover $\mathscr{U}$ of X, there is an open cover $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}(n)$ of X which refines $\mathscr{U}$ and which has the property that if $p \in X$, then there is an n such that p belongs to exactly k members of $\mathscr{V}(n)$, where k is some finite, positive number. The collection $\mathscr{V}$ is called a *weak θ-refinement of $\mathscr{U}$*.

REMARK. In the language of [6], the refinement $\mathscr{V}$ in (2.8) is σ-distributively point finite. As defined above, weak θ-refinability is not the same as θ-refinability as defined in [6]. Weak θ-refinability and certain related properties will be studied in [1]; here we only summarize the properties which we will need in § 3.

(2.9) PROPOSITION. a) Any subparacompact space is weakly θ-refinable; b) if X is perfect and weakly θ-refinable, then X is subparacompact; c) if $\mathscr{G}$ is any collection of open subsets of a perfect, weakly θ-refinable space X, then there is an open cover $\mathscr{H}$ of $\cup(\mathscr{G})$ which is a weak θ-refinement of $\mathscr{G}$.

To conclude this section, we restate the recent result of Heath and Michael which was mentioned in the Introduction.

(2.10) THEOREM [3]. $S^{\aleph_0}$ is perfect. Hence so is S^n for each finite n.

3. The Sorgenfrey line

(3.1) PROPOSITION. For any $n \geqq 1$, S^n is perfectly subparacompact.

PROOF. We argue inductively. The result is certainly true for $n = 1$, since $S^1 = S$ is even perfectly paracompact. Let us assume, therefore, that S^n is known to be perfectly subparacompact and prove that S^{n+1} must also be perfectly subparacompact. Since S^{n+1} is perfect by (2.10), it suffices, by (2.9)b, to show that S^{n+1} is weakly θ-refinable. Let $Y = S^{n+1}$.

Let $\mathscr{W} = \{W(\alpha) : \alpha \in A\}$ be a basic open cover of the space $Y = S^{n+1}$, where $W(\alpha) = U(1, \alpha) \times \cdots \times U(n+1, \alpha)$ with $U(k, \alpha) = [a(k, \alpha), b(k, \alpha)[$ for each $\alpha \in A$ and $k \leqq n+1$. Let $\hat{U}(k, \alpha) =]a(k, \alpha), b(k, \alpha)[$ and let $W(k, \alpha)$ be the set $U(1, \alpha) \times \cdots \times \hat{U}(k, \alpha) \times \cdots \times U(n+1, \alpha)$. Observe that each $W(k, \alpha)$ is open in the space

$$Y(k) = S \times \cdots \times S \times R \times S \times \cdots \times S$$

where R, the real numbers with the usual topology, replaces S in the k^{th} coordinate. Furthermore, since $Y(k)$ is homeomorphic to $S^n \times R$, our induction hypothesis together with (2.6) and the fact that the product of a perfect space with a metric space is again perfect, guarantees that $Y(k)$ is perfectly subparacompact. Hence by (2.9)$^{\text{c}}$ the collection $\mathscr{G}(k) = \{W(k, \alpha) : \alpha \in A\}$ has a weak θ-refinement $\mathscr{H}(k)$ which covers $\cup \mathscr{G}(k)$ and which consists of open subsets of $Y(k)$. But then $\mathscr{H}(k)$ is also a collection of open subsets of S^{n+1}.

Let

$$Z = \{x \in Y : \text{if } x \in W(\alpha) \in \mathscr{W}, \text{ then } x = (a(1, \alpha), \cdots, a(n+1, \alpha))\}.$$

Then

$$Z = Y \setminus \cup \{[\cup \mathscr{H}(k)] : 1 \leqq k \leqq n+1\}.$$

For each $x \in Z$, choose $\alpha(x) \in A$ such that $x \in W(\alpha(x))$ and observe that if x and y are distinct elements of Z, then $y \notin W(\alpha(x))$. Letting

$$\mathscr{H}(0) = \{W(\alpha(x)) : x \in Z\},$$

we obtain a collection of open subsets of Y which covers Z in such a way that each point of Z belongs to exactly one member of $\mathscr{H}(0)$. Therefore, $\mathscr{H} = \cup \{\mathscr{H}(k) : k \geqq 0\}$ is a weak θ-refinement of $\mathscr{W}$, as required to show that $Y = S^{n+1}$ is weakly θ-refinable.

(3.2) THEOREM. $S^{\aleph_0}$ is perfectly subparacompact.

PROOF. Apply (3.1), (2.10) and (2.7).

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