COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 24, nº 3 (1972), p. 355-358
<http://www.numdam.org/item?id=CM_1972__24_3_355_0>

COMPOSITIO MATHEMATICA, Vol. 24, Fasc. 3, 1972, pag. 355–358
Wolters-Noordhoff Publishing
Printed in the Netherlands

ON VALUES OMITTED BY UNIVALENT FUNCTIONS WITH TWO PRE-ASSIGNED VALUES

by

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1. Introduction

Let $\mathfrak{M}$ denote the set of all functions $f(Z)$ which are analytic and univalent in the unit disc Δ and which are normalized by the conditions $f(0) = 0$ and $f(Z_0) = Z_0$; here Z_0 is a fixed point in Δ, $Z_0 \neq 0$. For the class $\mathfrak{M}$, the following result was established recently.

LEMMA. *If* $f \in \mathfrak{M}$, *then the image domain* $f(\Delta)$ *contains the disc* $[W \,|\, |W| < \frac{1}{4}(1-|Z_0|)^2]$. *The constant* $\frac{1}{4}(1-|Z_0|)^2 \equiv R_0$ *is the best possible one* [2,3].

Let C_R denote the circle $[W \,|\, |W| = R]$, let $f \in \mathfrak{M}$. and let $m(R,f) \equiv m[C_R \diagdown f(\Delta) \cap C_R]$ be the Lebesgue measure of the set of values on C_R not taken on by $f(z)$. It follows from the Lemma that $m(R,f) = 0$ for $0 \leqq R < R_0$ and $m(R,f) = 2\pi R$ for $R > 1$.
In this note we shall evaluate the expression

$$m(R) \equiv \sup\, [m(R,f) \,|\, f \in \mathfrak{M}] \tag{1}$$

for R fixed, $R_0 \leqq R < 1$. The result we obtain is analogous to one obtained by Jenkins [1]: he considered the class S of univalent functions $f(z)$ subject to the usual normalization $f(0) = 0$ and $f'(0) = 1$. Our result reduces to that of Jenkins if we allow $Z_0 \to 0$.

2. The principal result

If Ω is a simply-connected domain in the complex domain S^2, and if a and b are two points in Ω, then $\rho(a, b, \Omega)$ will denote the hyperbolic distance between the points a and b with respect to Ω.

If Ω is a simply-connected domain in the complex domain, and if Ω contains the points a and b, then Ω^* denotes the domain obtained from

[1] The first author acknowledges support received under National Science Foundation Grant GP-11158.
[2] The second author acknowledges support received from I.R.E.X.

Ω by circular symmetrization with respect to the half-line $[a, b, \infty]$, which has its finite end-point at a and passes through the point b.

The symbol $J(R, t)$ denotes the 'fork-domain' defined by

$$J(R, t) \equiv S^2 \setminus [[-\infty, -R) \cup \{w | w = \mathrm{Re}^{i\varphi}, \quad t \leqq \varphi \leqq 2\pi - t\}].$$

Here R is fixed, $R_0 \leqq R < 1$, and t is fixed, $0 \leqq t < \pi$.

Our principal result is the following one.

THEOREM. *The bound in* (1) *is given by the formula*

$$m(R) = 2R \arccos (1 - 2D^2), \tag{2}$$

where

$$D = \frac{2(1-d)(R-d) + 8d\sqrt{R}}{(1+d)^2\sqrt{R}} - 1, \qquad d \equiv |z_0|, \tag{3}$$

for R fixed, $\frac{1}{4}(|-1Z_0|)^2 \equiv R_0 \leqq R < 1$. The extremal function for this problem maps Δ onto a suitably-chosen fork-domain.

PROOF. Compactness considerations yield the result that there exists at least one extremal function; later considerations will show that there is only one extremal function.

Let $f(Z)$ be an extremal function for the bound (1). In view of the conformal invariance of the hyperbolic distance, we have

$$\operatorname{arctanh} |Z_0| = \rho(0, Z_0, \Delta) = \rho(0, Z_0, f(\Delta)). \tag{4}$$

If $f(\Delta)^*$ is the domain obtained from $f(\Delta)$ by circular symmetrization with respect to the half-line $[0, Z_0, \infty)$, then it is well-known that

$$\rho(0, Z_0, f(\Delta)) \geqq \rho(0, Z_0, f(\Delta)^*) \tag{5}$$

holds. Now it is geometrically clear that $f(\Delta)^*$ is contained in a fork-domain $J(R, t)$ for some t, and for this $J(R, t)$ we have

$$\rho(0, Z_0, f(\Delta)^*) \geqq \rho(0, Z_0, J(R, t)). \tag{6}$$

From (4), (5) and (6) we obtain the inequality

$$\rho(0, Z_0, \Delta) \leqq \rho(0, Z_0, J(R, t)), \tag{7}$$

which is our fundamental one. Since $\rho(0, Z_0, J(R, t))$ is an increasing function of t ,it follows from (7) that in order to determine $m(R)$ it is sufficient to determine t_0 so that

$$\operatorname{arc\,tanh} |Z_0| = f(0, Z_0, \Delta) = \rho(0, Z_0, J(R, t_0)) \tag{8}$$

holds. In order to do this, we shall find a function $f \in \mathfrak{M}$ that maps Δ onto the fork-domain $J(R, t_0)$. It is easy to show that the function is unique.

There is no loss in generality in taking $Z_0 = d > 0$. Now we obtain the function that maps Δ onto $J(R, t_0)$ as a composition $W(Z) = W(\zeta(Z))$, where $\zeta = \zeta(Z)$ and $W = W(\zeta)$ are determined by

$$\frac{(1+\zeta)^2}{\zeta} = \frac{(1+z)^2(1+D)^2}{4zD} \tag{9}$$

and

$$w = \frac{R\zeta(1-D\zeta)}{(D-\zeta)}, \tag{10}$$

respectively. Here D is a real constant, $0 < D < 1$, to be determined by the condition

$$W(d) = W(\zeta(d)) = d. \tag{11}$$

The function $\zeta = \zeta(Z)$ in (9) maps Δ onto the slit-disc

$$\Delta^* = [\zeta \,|\, |\zeta| < 1] \setminus [\zeta \,|\, D < \zeta < 1].$$

The function $W = W(\zeta)$ in (10) maps Δ^* onto the fork-domain $J(R, t_0)$ with $t_0 = 2D^2 - 1$. The requirement (11), with (9) and (10), now yields (2). This completes the proof.

COROLLARY. *If $f \in S$, then*

$$m(R) = 2R \arccos (8\sqrt{R} - 8R - 1), \qquad \tfrac{1}{4} \leqq R \leqq 1.$$

PROOF. If we take $Z_0 = 0$ in the preceding theorem, then $\mathfrak{M}$ becomes the well-known class S, while $R_0 = \frac{1}{4}$ and $D = 2\sqrt{R} - 1$. Thus we obtain the earlier result due to Jenkins, alluded to in a preceding paragraph, from the present one by a simple limiting process.

3. Final Remark

It is well-known that circular symmetrization preserves the starlikeness of a domain. Hence it is possible to try to obtain the analogue of our theorem for the class of starlike functions $\mathfrak{M}^*$, a subclass of $\mathfrak{M}$. However the calculations are so formidable, that we have not been able to complete them.

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(Oblatum 12–III–1971)

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