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ON THE STRUCTURE OF HILBERT CUBE MANIFOLDS

by

T. A. Chapman

1. Introduction

Let s denote the countable infinite product of open intervals and let I^∞ denote the Hilbert cube, i.e. the countable infinite product of closed intervals. A *Fréchet manifold* (or *F-manifold*) is a separable metric space having an open cover by sets each homeomorphic to an open subset of s. A *Hilbert cube manifold* (or *Q-manifold*) is a separable metric space having an open cover by sets each homeomorphic to an open subset of I^∞.

In [2] it is shown that real Hilbert space l_2 is homeomorphic to s and indeed it is known that all separable infinite-dimensional Fréchet spaces are homeomorphic (see [2] for references). Thus F-manifolds can be viewed as separable metric manifolds modeled on any separable infinite-dimensional Fréchet space. Using linear space apparatus and a number of earlier results, Henderson [9] has obtained embedding, characterization, and representation theorems concerning F manifolds (see [10] for generalizations to manifolds modeled on more general infinite-dimensional linear spaces).

In [6] a number of results similar in nature to those of [9] were obtained concerning certain incomplete, sigma-compact countably infinite-dimensional manifolds. Some results were also established in [6] concerning the relationship of such incomplete manifolds to Q-manifolds. Since the nature of these results is such that a good bit of information about Q-manifolds can be obtained from the 'related' incomplete manifolds, we thus have a device for attacking Q-manifold problems.

It is the purpose of this paper to use 'related' incomplete manifolds to establish for Q-manifolds some more results similar to those of [9]. We list the main results of this paper in section 2.

Unfortunately we leave important questions concerning Q-manifolds unanswered. We call particular attention to the paper *Hilbert cube manifolds* [Bull. Amer. Math. Soc. 76 (1970), 1326–1330], in which the author gives an extensive list of open questions concerning Q-manifolds.

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2. Statements of results

A (*topological*) *polyhedron* is a space homeomorphic ($\cong$) to $|K|$, where K is a *complex* (i.e. a countable locally-finite simplicial complex). Unless otherwise specified all polyhedra will be topological polyhedra. West [15] has shown that $P\times s$ is an F-manifold and $P\times I^{\infty}$ is a Q-manifold, for any polyhedron P.

A closed set F in a space X in said to be a *Z-set* in X provided that for each non-null homotopically trivial (i.e. all homotopy groups are trivial) open subset U of X, $U\diagdown F$ is non-null and homotopically trivial. We use the representation $s = \Pi_{i=1}^{\infty} I_i^0$ and $I^{\infty} = \Pi_{i=1}^{\infty} I_i$, where for each $i > 0$ I_i^0 is the open interval $(-1, 1)$ and I_i is the closed interval $[-1, 1]$.

In Theorem 1 we show how to 'fatten-up' a polyhedron which is a Z-set in a Q-manifold to a 'nice' neighborhood of the polyhedron. This will be useful in the sequel.

THEOREM 1. *Let X be a Q-manifold and let P be a polyhedron which is also a Z-set in X. If $q \in I^{\infty}\diagdown\{(0, 0, \cdots)\}$, then there is an open embedding $h : P\times (I^{\infty}\diagdown\{q\}) \to X$ such that $h(x, (0, 0, \cdots)) = x$, for all $x\in P$.*

In [9] the following results are established.

(1) Every F-manifold can be embedded as an open subset of l_2.

(2) If X and Y are F-manifolds having the same homotopy type (i.e. $X \sim Y$), then $X \cong Y$.

(3) If X is any F-manifold, then there is a polyhedron P for which $X \cong P\times l_2$.

If J is a simple closed curve, then $J\times I^{\infty}$ is a Q-manifold which cannot be embedded as an open subset of I^{∞}. Also, I^{∞} and $I^{\infty}\diagdown\{\text{point}\}$ are Q-manifolds of the same homotopy type which are not homeomorphic. Thus the obvious straightforward analogues of (1) and (2) for Q-manifolds are not valid. Most of the results that follow are concerned with obtaining partial analogues of (1), (2), and (3) for Q-manifolds.

THEOREM 2. *Let X be a Q-manifold and let P be any polyhedron such that $X \sim P$. Then there is a Z-set $F \subset X$ such that $X\diagdown F \cong P\times (I^{\infty}\diagdown \{\text{point}\})$.*

Each Q-manifold is an ANR and it follows from [11] that each separable metric ANR has the homotopy type of some polyhedron. Thus each Q-manifold has the homotopy type of some polyhedron.

THEOREM 3. *Let X be any Q-manifold and let P be any polyhedron such that $X \sim P$. Then $X\times [0,1) \cong P\times (I^{\infty}\diagdown\{\text{point}\})$.*

COROLLARY 1. *If X is any Q-manifold, then there is a polyhedron P such that $X\times [0, 1) \cong P\times I^{\infty}$.*

COROLLARY 2. *If X and Y are Q-manifolds such that $X \sim Y$, then* $X \times [0, 1) \cong Y \times [0,1)$.

COROLLARY 3. *If P and R are polyhedra such that $P \sim R$, then* $P \times (I^\infty \setminus \{\text{point}\}) \cong R \times (I^\infty \setminus \{\text{point}\})$.

In a sense Corollary 3 is analogous to a result of West [15]. It is shown there that if a polyhedron P is a *formal deformation* of a polyhedron R (in the sense of Whitehead [16]), then $P \times I^\infty \cong R \times I^\infty$.

THEOREM 4. *If X is a Q-manifold, then $X \times [0, 1)$ can be embedded as an open subset of I^∞.*

COROLLARY 4. *If X is a Q-manifold, then $X = U \cup V$, where U and V are open subsets of X which are homeomorphic to open subsets of I^∞.*

If X is any Q-manifold, then it is shown in [5] that $X \cong X \times I^\infty$ (and therefore $X \cong X \times [0, 1]$). Thus the above results offer some information about the internal structure of Q-manifolds.

In [10] it is shown that if X and Y are F-manifolds and $f : X \to Y$ is a homotopy equivalence, then f is homotopic to a homeomorphism of X onto Y. We obtain a corresponding property for Q-manifolds which strengthens Corollary 2.

THEOREM 5. *Let X, Y be Q-manifolds and let $f : X \to Y$ be a homotopy equivalence. Then there is a homeomorphism of $X \times [0, 1)$ onto $Y \times [0, 1)$ which is homotopic to $f \times \text{id} : X \times [0, 1) \to Y \times [0, 1)$.*

The following results are some partial answers to questions concerning compact Q-manifolds.

THEOREM 6. *Let X be a compact Q-manifold and assume that $X \sim P$, where P is a compact polyhedron. Then there is a copy P' of P in X such that P' is a Z-set in X and $X \setminus P' \cong P \times (I^\infty \setminus \{\text{point}\})$.*

COROLLARY 5. *If X is a compact homotopically trivial Q-manifold, then $X \cong I^\infty$.*

THEOREM 7. *Let X be a compact Q-manifold and assume that $X \sim P$, where P is a compact polyhedron. Then there is an embedding $h : X \to I^\infty$ such that $\text{Bd}(h(X)) \cong P \times I^\infty$ and $Cl(I^\infty \setminus h(X)) \cong I^\infty$.*

In regard to Theorem 7 we remark that in [8] a similar, and somewhat stronger, result is established for F-manifolds.

We show that if X is an open subset of I^∞, then the factor $[0,1)$ of Corollary 1 can be omitted.

THEOREM 8. *If X is an open subset of I^∞, then there is a polyhedron P such that $X \cong P \times I^\infty$.*

We remark that the proof of this result is quite different from the proof of the corresponding property for open subsets of l_2 (see [8]).

We also establish a Schoenflies-type result for Q-manifolds.

THEOREM 9. *Let X and Y be Q-manifolds and let $f, g : X \to Y$ be closed embeddings which are homotopy equivalences and such that $f(X)$, $g(X)$ are bicollared in Y ('bicollared' is defined in Section 3). Then the homeomorphism $g \circ f^{-1} \times \mathrm{id} : f(X) \times [0,1) \to g(X) \times [0, 1)$ can be extended to a homeomorphism of $Y \times [0, 1)$ onto itself.*

We remark that in the case $X = Y = I^\infty$, the factor $[0,1)$ can be omitted in the statement of Theorem 9. The proof of this follows routinely from [17].

The proof of Theorem 9 applies to give us a corresponding result for F-manifolds.

THEOREM 10. *Let X and Y be F-manifolds and let $f, g : X \to Y$ be closed embeddings which are homotopy equivalences and such that $f(X)$, $g(X)$ are bicollared in Y. Then the homeomorphism $g \circ f^{-1} : f(X) \to g(X)$ can be extended to a homeomorphism of Y onto itself.*

In case $X = Y = l_2$, Theorem 10 follows routinely from the Schoenflies result of [13].

3. Preliminaries

In this section we describe some of the apparatus that will be used in the succeeding sections.

For spaces X and Y, a continuous function $f: X \to Y$ is said to be *proper* provided that the inverse image of each compact subset of Y is compact. Then a *proper homotopy* is a homotopy $F: X \times I \to Y$ which is a proper map (we let $I = [0, 1]$).

For each integer $n > 0$ let $W_n^+ = \{(x_i) \in I^\infty | x_n = 1\}$ and $W_n^- = \{(x_i) \in I^\infty | x_n = -1\}$. We call W_n^+ and W_n^- *endslices* of I^∞. For each integer $n > 0$ we let $\pi_n : I^\infty \to \Pi_{i=1}^n I_i$ be the natural projection and put $B(I^\infty) = I^\infty \setminus s$.

A subset of I^∞ of the form $\Pi_{i=1}^\infty J_i$ is called a *basic closed set* in I^∞ provided that J_i is a closed subinterval of I_i for each $i > 0$, and $J_i = I_i$ for all but finitely many i. Note that any basic closed subset of I^∞ may be viewed as a Hilbert cube, with its topological boundary being a finite union of endslices.

Let X and Y be spaces and $\mathfrak{U}$ be an open cover of Y. Then functions $f, g : X \to Y$ are said to be $\mathfrak{U}$-*close* provided that for each $x \in X$, $f(x)$ and $g(x)$ lie in some element of $\mathfrak{U}$. A function $f: Y \to Y$ is said to be *limited by*

$\mathfrak{U}$ provided that f and id_Y (the identity function on Y) are $\mathfrak{U}$-close. A function $f: X \times I \to Y$ is said to be limited by $\mathfrak{U}$ provided that for each $x \in X$, $f(\{x\} \times I)$ lies in a member of $\mathfrak{U}$.

Following Anderson [1] we say that a subset M of a metric space X has the *compact absorption property* in X (or M is a *cap-set* for X) if

(1) $M = \bigcup_{n=1}^{\infty} M_n$, where each M_n is a compact Z-set in X such that $M_n \subset M_{n+1}$, and

(2) for each $\varepsilon > 0$, each integer $m > 0$, and each compact subset F of X, there is an integer $n > 0$ and an embedding $h: F \to M_n$ such that $h|F \cap M_m = \mathrm{id}$ and $d(h, \mathrm{id}) < \varepsilon$.

For each integer $n > 0$ let $\Sigma_n = \Pi_{i=1}^{\infty}[-n/(n+1), n/(n+1)]$ and $\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n$. In [1] it is shown that Σ and $B(I^\infty)$ are cap-sets for I^∞.

We will need the following properties of cap-sets in Q-manifolds. All of these can be found in [6]. We let X represent a Q-manifold.

LEMMA 3.1. *Cap-sets exist in Q-manifolds, and any cap-set for X is of the form $P \times \Sigma$, for any polyhedron P satisfying $P \sim X$.*

LEMMA 3.2. *If M is a cap-set for X and $F \subset X$ is a Z-set, then $M \cup F$ and $M \setminus F$ are cap-sets for X.*

LEMMA 3.3. *If M and N are cap-sets for X and $\mathfrak{u}$ is an open cover of X, then there is a homeomorphism of X onto itself which takes M onto N and which is limited by $\mathfrak{u}$.*

LEMMA 3.4. *If M is a cap-set for X and $F \subset X$ is a closed set satisfying $F \cap M = \emptyset$, then F is a Z-set in X.*

LEMMA 3.5. *If P is a polyhedron, then $P \times \Sigma_n$ is a Z-set in $P \times \Sigma$. If M is a cap-set for X and $F \subset M$ is a Z-set in M, then $Cl_X(F)$ (the closure of F in X) is a Z-set in X.*

LEMMA 3.6. *If M is a cap-set for X, then $X \setminus M$ is an F-manifold satisfying $X \setminus M \sim X$. In fact, $M \cong X \times B(I^\infty)$, which is a cap-set for $X \times I^\infty$. If $F \subset X \setminus M$ is a Z-set in $X \setminus M$, then $Cl_X(F)$ is a Z-set in X.*

Let X be a space and let $\mathfrak{u}$ be any open cover of X. Then define $\mathrm{St}^0(\mathfrak{u}) = \mathfrak{u}$ and for each $n > 0$ define $\mathrm{St}^n(\mathfrak{u})$ to consist of all sets of the form $A \cup (\cup\{U \in \mathfrak{u} | U \cap A \neq \emptyset\})$, where $A \in \mathrm{St}^{n-1}(\mathfrak{u})$.

The following result on extensions of homeomorphisms in Q-manifolds is established in [3].

LEMMA 3.7. *Let X be a Q-manifold, $\mathfrak{u}$ be an open cover of X, F_1 and F_2 be Z-sets in X, and let $h: F_1 \to F_2$ be a homeomorphism. If there is a proper homotopy $H: F_1 \times I \to X$ such that $H_0 = \mathrm{id}$, $H_1 = h$, and H*

is limited by $\mathfrak{U}$, *then* h *can be extended to a homeomorphism of* X *onto itself which is limited by* $\mathrm{St}^4(\mathfrak{U})$.

The following characterization of Z-sets in Q-manifolds is established in [6].

LEMMA 3.8. *Let* X *be a* Q*-manifold and let* $F \subset X$ *be a closed set. Then* F *is a* Z*-set in* X *if and only if there is a homeomorphism of* X *onto* $X \times I^\infty$ *taking* F *into* $X \times \{(0, 0, \cdots)\}$.

It is shown in [3] that for any Z-set F in a Q-manifold X, there is a homeomorphism of X onto $X \times I^\infty$ such that x is taken to $(x, (0, 0, \cdots)$, for all $x \in F$. It is shown in [7] that a corresponding property for F-manifolds is also true.

We say that a subset A of a space X is *bicollared* provided that there exists an open embedding $h : A \times (-1, 1) \to X$ satisfying $h(x, 0) = x$, for all $x \in A$.

Let X be a metric space and A be a closed subset of X. An open cover $\mathfrak{U}$ of $X \setminus A$ is said to be *normal with respect to* A provided that for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $U \in \mathfrak{U}$ and $d(A, U) < \delta$, then $\mathrm{diam}(U) < \varepsilon$. Under these circumstances it is easy to see that any homeomorphism $h : X \setminus A \to X \setminus A$ which is limited by $\mathfrak{U}$ has an extension to a homeomorphism $\tilde{h} : X \to X$ which satisfies $\tilde{h}|A = \mathrm{id}$.

4. Proof of Theorem 1

For any complex K, we use $K^{(n)}$ to denote the n^{th} barycentric subdivision of K and K_n to denote the n-skeleton of K. For any subset C of $|K|$ and integers $m, n > 0$, we let $\mathrm{St}(C, K_n^{(m)})$ denote the subset of $|K|$ consisting of the union of the closed simplexes of $K_n^{(m)}$ which intersect C, where $K_n^{(m)}$ will always mean the m^{th} barycentric subdivision of K_n.

We now present a sequence of lemmas that will lead up to a proof of Theorem 1. The proof we give uses an induction on the n-skeletons of a triangulation of the polyhedron P. The fourth lemma we establish is the actual inductive step, and the first three are technical results that we need there.

LEMMA 4.1. *Let* K *be a complex,* $n > 0$ *be an integer,* C *be a compact subset of* $|K|$ *such that* $\mathrm{St}(C, K_{n+1}) \subset |K_n|$, *and let* $L = \mathrm{St}(|K_n|, K_{n+1}^{(2)})$. *Then there is a homeomorphism* $h : L \times I^\infty \to |K_n| \times I^\infty$ *such that* $h|C \times I^\infty = \mathrm{id}$, $h(L \times W_1^+) = |K_n| \times W_1^+$, *and* $h(x, (0, 0, \cdots)) = (x, (0, 0, \cdots))$, *for all* $x \in |K_n|$.

PROOF. Let $Q = \Pi_{i=2}^\infty I_i$. It follows from Theorem 4.2 of [15] that there is a homeomorphism $h' : L \times Q \to |K_n| \times Q$. Since the collapse (see

[15] for definitions) from L to $|K_n|$ takes place in $|K|\setminus C$, an open set missing C, the proof given there immediately implies that we may additionally require that $h'|C\times Q = \text{id}$. Although the condition $h'(x, (0, 0 \cdots)) = (x, (0, 0, \cdots))$, for all $x \in |K_n|$, is not mentioned in [15], it can easily be obtained from the apparatus given there. All one has to do is follow the steps in the proof of Theorem 4.2 of [15], correcting at each stage of the collapse to achieve our required condition.

Now define $h : L\times I^\infty \to |K_n|\times I^\infty$ so that $h(x, (x_1, x_2, \cdots)) = (y, (x_1, y_2, y_3, \cdots))$, for all $x \in L$ and $(x_1, x_2, \cdots) \in I^\infty$, where $h'(x, (x_2, x_3, \cdots)) = (y, (y_2, y_3, \cdots))$. Then h obviously fulfills our requirements.

Let B_r^n be the n-dimensional ball of radius r $(0 < r \leqq 1)$ and S_r^{n-1} the boundary of B_r^n. For convenience we will assume that

$$B_r^n = \{(x_i) \in I^\infty | \sum_{i=1}^n x_i^2 \leqq r^2 \quad \text{and} \quad x_i = 0 \text{ for } i > n\},$$

$$S_r^{n-1} = \{(x_i) \in I^\infty | \sum_{i=1}^n x_i^2 = r^2 \quad \text{and} \quad x_i = 0 \text{ for } i > n\}.$$

LEMMA 4.2. *Let X be a Q-manifold, $F \subset X$ be a closed set, and let $f : B_1^n \to X$ be an embedding such that $f(B_1^n)$ is a Z-set and $f(B_1^n) \cap F \subset f(S_1^{n-1})$. For any $r \in (0,1)$ there is an embedding $h : B_r^n \times I^\infty \to X$ satisfying the following properties.*

(1) $h(x, (0, 0, \cdots) = f(x)$, *for all* $x \in B_r^n$,

(2) $\text{Bd}\,(h(B_r^n \times I^\infty)) = h(B_r^n \times W_1^+) \cup h(S_r^{n-1} \times I^\infty)$,

(3) $\text{Bd}\,(h(B_r^n \times I^\infty))$ *is bicollared,*

(4) $h(B_r^n \times I^\infty) \cap (F \cup f(B_1^n)) = f(B_r^n)$.

PROOF. It is clear that there is an embedding $g_1 : I^\infty \to X$ and a finite union W of endslices of I^∞ such that $f((0, 0, \cdots)) \,\varepsilon\, g_1(I^\infty \setminus W)$ and $\text{Bd}(g_1(I^\infty)) = g_1(W)$. Choose $\varepsilon > 0$ so that $f(B_\varepsilon^n) \subset g_1(I^\infty \setminus W)$ and use Lemma 3.7 to get a homeomorphism $g_2 : X \to X$ satisfying $g_2 \circ f(B_\varepsilon^n) = f(B_1^n)$. Then $(g_2 \circ g_1)^{-1} \circ f(B_1^n)$ is a Z-set in I^∞ missing W.

Applying Lemma 3.7 to I^∞ there is a homomorphism $g_3 : I^\infty \to I^\infty$ satisfying $g_3(W) = W$ and $g_3 \circ (g_2 \circ g_1)^{-1} \circ f(x) = x$, for all $x \in B_{r_1}^n$, where $r < r_1 < 1$. Choose $m > n$ and $\delta \in (0, 1)$ such that $K \cap W = \emptyset$ and $K \cap g_3 \circ (g_2 \circ g_1)^{-1}(f(B_1^n) \cup F) = B_r^n$, where

$$K = \pi_n(B_r^n) \times \prod_{i=n+1}^{m} [-\delta, \delta] \times \prod_{i=m+1}^{\infty} I_i.$$

Then put

$$Q = \pi_n(B_r^n) \times \prod_{i=n+1}^{m} [-\delta, \delta] \times [-\tfrac{1}{2}, 1] \times \prod_{i=m+2}^{\infty} I_i.$$

It is obvious that there is a homeomorphism $g_4 : B_r^n \times I^\infty \to Q$ satisfying

$$g_4(B_r^n \times W_1^+) = \pi_n(B_r^n) \times \prod_{i=n+1}^{m} [-\delta, \delta] \times \{-\tfrac{1}{2}\} \times \prod_{i=m+2}^{\infty} I_i,$$

$$g_4(S_r^{n-1} \times I^\infty) = \pi_n(S_r^{n-1}) \times \prod_{i=n+1}^{m} [-\delta, \delta] \times [-\tfrac{1}{2}, 1] \times \prod_{i=m+2}^{\infty} I_i,$$

and $g_4(x, (0, 0, \cdots)) = x$, for all $x \in B_r^n$. Then $h = g_2 \circ g_1 \circ g_3^{-1} \circ g_4$ is our required embedding.

LEMMA 4.3. *Let K be a complex, $n > 0$ be an integer, C be a compact subset of $|K|$ satisfying $\mathrm{St}(C, K_{n+1}) \subset |K_n|$, and let $L = \mathrm{St}(|K_n|, K_{n+1}^{(2)})$. Let X be a Q-manifold and let $h : L \times I^\infty \to X$ be a closed embedding such that $\mathrm{Bd}(h(L \times I^\infty)) = h(L \times W_1^+)$ and it is bicollared. Let $F \subset X$ be a Z-set such that*

$$F \cap [h(L \times \{(0, 0, \cdots)\}) \cup h(C \times I^\infty) \cup h(\mathrm{Bd}(L) \times (I^\infty \setminus W_1^+))] = \emptyset,$$

where $\mathrm{Bd}(L)$ is the topological boundary of L in $|K_{n+1}|$. Then there exists a homeomorphism $f : X \to X$ such that

$$f|h(L \times \{(0, 0, \cdots)\}) \cup h(\mathrm{Bd}(L) \times I^\infty) \cup h(C \times I^\infty) = \mathrm{id}$$

and $f(F) \cap h(L \times I^\infty) \subset h(\mathrm{Bd}(L) \times W_1^+)$.

PROOF. Let $A = h(L \times [-1,0] \times \{(0, 0, \cdots)\}) \cup h(L \times W_1^-)$ which is a Z-set in X, and let $B = h(C \times I^\infty) \cup h(L \times \{(0, 0, \cdots)\}) \cup h(\mathrm{Bd}(L) \times I^\infty)$, which is closed in X. Let $X' = X \setminus B$, $A' = A \cap X'$, and $F' = F \cap X'$. Since A' and F' are intersections of Z-sets in X with the open subset X' of X, it follows that A' and F' are Z-sets in X'. Now choose an open cover $\mathfrak{U}$ of X' which is normal with respect to B.

Using Lemma 3.8 there is a homeomorphism $f_1 : X' \to X' \times I^\infty$ such that $f_1(A' \cup F') \subset X' \times \{(0, 0, \cdots)\}$. We can obviously obtain a homeomorphism $f_2 : X' \times I^\infty \to X' \times I^\infty$ such that $f_2 \circ f_1(F') \cap f_1(A') = \emptyset$ and f_2 is limited by $f_1(\mathfrak{U})$. Then $f_1^{-1} \circ f_2 \circ f_1 : X' \to X'$ is a homeomorphism limited by $\mathfrak{U}$ and satisfying $f_1^{-1} \circ f_2 \circ f_1(F') \cap A' = \emptyset$. From Section 3 it follows that $f_1^{-1} \circ f_2 \circ f_1$ extends to a homeomorphism $g : X \to X$ such that $g|B = \mathrm{id}$ and $g(F) \cap A \cup B \subset h(\mathrm{Bd}(L) \times W_1^+)$.

We can use a motion in $L \times I^\infty$ in only the I_1-direction and transfer it back to X by means of h to obtain a homeomorphism $g_1 : X \to X$ such that $g_1|B = \mathrm{id}$ and $g_1 \circ g(F) \cap h(L \times [-1, \tfrac{1}{2}] \times \Pi_{i=2}^\infty I_i) = \emptyset$. The problem is now to move $g_1 \circ g(F) \setminus (h(\mathrm{Bd}(L) \times W_1^+)$ the rest of the way out of $h(L \times I^\infty)$, with no motion taking place on B. Because $\mathrm{Bd}(h(L \times I^\infty))$ is bicollared, we can easily find a homeomorphism $g_2 : X \to X$ satisfying $g_2|B = \mathrm{id}$ and $g_2 \circ g_1 \circ g(F) \cap h(L \times I^\infty) \subset h(\mathrm{Bd}(L) \times W_1^+)$. Then put $f = g_2 \circ g_1 \circ g$ to satisfy our requirements.

We now combine these results to obtain the inductive step in the proof of Theorem 1.

LEMMA 4.4 *Let K be a complex, let $n > 0$ be an integer, and let C be a compact subset of $|K|$ such that $\mathrm{St}(C, K_{n+1}) \subset |K_n|$. Let X be a Q-manifold and let $\varphi : |K| \to X$ be an embedding such that $\varphi(|K|)$ is a Z-set. Let $h_n : |K_n| \times I^\infty \to X$ be a closed embedding such that $\mathrm{Bd}(h_n(|K_n| \times I^\infty)) = h_n(|K_n| \times W_1^+)$ and it is bicollared, $h_n(|K_n| \times I^\infty) \cap \varphi(|K|) \subset \varphi(\mathrm{St}(|K_n|, K^{(3)}))$, and $h_n(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_n|$. Then there exists a closed embedding $h_{n+1} : |K_{n+1}| \times I^\infty \to X$ such that $\mathrm{Bd}(h_{n+1}(|K_{n+1}| \times I^\infty)) = h_{n+1}(|K_{n+1}| \times W_1^+)$ and it is bicollared, $h_{n+1}(|K_{n+1}| \times I^\infty) \cap \varphi(|K|) \subset \varphi(\mathrm{St}(|K_{n+1}|, K^{(3)}))$, $h_{n+1}|C \times I^\infty = h_n|C \times I^\infty$, and $h_{n+1}(x, (0, 0, \cdots) = \varphi(x)$, for all $x \in |K_{n+1}|$.*

PROOF. Let $L = \mathrm{St}(|K_n|, K_{n+1}^{(2)})$ and let $\mathrm{Bd}(L)$ represent the boundary of L in $|K_{n+1}|$. Let $\{\sigma_i\}_{i=1}^\infty$ be the collection of $(n+1)$-simplexes of K and note that $\sigma_i' = \mathrm{Cl}(\sigma_i \setminus L)$ is an $(n+1)$-cell contained in the combinatorial interior of σ_i. For each i let $\mathrm{Bd}(\sigma_i')$ denote the combinatorial boundary of σ_i'. (We are assuming that if $i \neq j$, then $\sigma_i \neq \sigma_j$. If the collection of $(n+1)$-simplexes of K is finite, then the argument is similar). It follows from the given conditions that $\varphi(\bigcup_{i=1}^\infty \sigma_i') \cap h_n(|K_n| \times I^\infty) = \emptyset$.

Using Lemma 4.2 there is a closed embedding f : $(\bigcup_{i=1}^\infty \sigma_i') \times I^\infty \to X$ such that the following properties are satisfied.

(1) $f((\bigcup_{i=1}^\infty \sigma_i') \times I^\infty) \cap h_n(|K_n| \times I^\infty) = \emptyset,$

(2) $f(x, (0, 0, \cdots)) = \varphi(x), \qquad \text{for all } x \in \bigcup_{i=1}^\infty \sigma_i',$

(3) $f((\bigcup_{i=1}^\infty \sigma_i') \times I^\infty) \cap \varphi(|K|) = \varphi(\bigcup_{i=1}^\infty \sigma_i'), \qquad \text{and}$

(4) $\mathrm{Bd}\,(f((\bigcup_{i=1}^\infty \sigma_i') \times I^\infty)) = f((\bigcup_{i=1}^\infty \sigma_i') \times W_1^+) \cup f((\bigcup_{i=1}^\infty \mathrm{Bd}\,(\sigma_i')) \times I^\infty)$

and it is bicollared.

For each i let $\mathrm{Int}(\sigma_i') = \sigma_i' \setminus \mathrm{Bd}(\sigma_i')$ and put

$$X' = X \setminus f((\bigcup_{i=1}^\infty \mathrm{Int}(\sigma_i')) \times (I^\infty \setminus W_1^+)),$$

which is a Q-manifold containing

$$f((\bigcup_{i=1}^\infty \sigma_i') \times W_1^+) \cup f((\bigcup_{i=1}^\infty \mathrm{Bd}(\sigma_i')) \times I^\infty)$$

as a Z-set. (This last assertion easily follows since $\mathrm{Bd}(f((\bigcup_{i=1}^\infty \sigma_i') \times I^\infty))$ is bicollared). Using Lemma 4.1 there is a homeomorphism $\theta : L \times I^\infty \to |K_n| \times I^\infty$ such that $\theta(x, (0, 0, \cdots)) = (x, (0, 0, \cdots))$, for all $x \in |K_n|$,

$\theta|C\times I^\infty = \mathrm{id}$, and $\theta(L\times W_1^+) = |K_n|\times W_1^+$. Then $\tilde{h}_n = h_n\circ\theta : L\times I^\infty \to X'$ is a closed embedding such that $\tilde{h}_n(x, (0, 0, \cdots)) = \varphi(x)$, for all $x\in |K_n|$, $\mathrm{Bd}(\tilde{h}_n(L\times I^\infty)) = \tilde{h}_n(L\times W_1^+)$ and it is bicollared, and $\tilde{h}_n|C\times I^\infty = h_n|C\times I^\infty$.

Let us consider the two sets $\tilde{h}_n(L\times\{(0, 0, \cdots)\}) \cup \tilde{h}_n(\mathrm{Bd}(L)\times I^\infty)$ and $f((\bigcup_{i=1}^\infty \mathrm{Bd}(\sigma_i'))\times I^\infty) \cup \varphi(L)$, which are Z-sets in X'. Define a homeomorphism α of the former onto the latter such that $\alpha\circ\tilde{h}_n(x, (0, 0, \cdots)) = \varphi(x)$, for all $x\in L$, and $\alpha\circ\tilde{h}_n(x, t) = f(x, t)$, for all $x\in\mathrm{Bd}(L)$ and $t\in I^\infty$. Using the fact that $\varphi(x) = \tilde{h}_n(x, (0, 0, \cdots))$, for all $x\in|K_n|$, and the fact that $f(x, (0, 0, \cdots)) = \varphi(x)$, for all $x\in\bigcup_{i=1}^\infty\sigma_i'$, it is clear that α is properly homotopic to the identity in X'. In fact, there is an open cover $\mathfrak{U}$ of $X'\setminus h_n(C\times I^\infty)$ which is normal with respect to $h_n(C\times I^\infty)$ and for which there is a proper homotopy

$$H : [(\tilde{h}_n(L\times\{(0, 0, \cdots)\}) \cup \tilde{h}_n(\mathrm{Bd}(L)\times I^\infty))\setminus h_n(C\times I^\infty)] \times I \to X'\setminus h_n(C\times I^\infty)$$

satisfying $H_0 = \mathrm{id}$,

$$H_1 = \alpha|[\tilde{h}_n(L\times\{(0, 0, \cdots)\}) \cup \tilde{h}_n(\mathrm{Bd}(L)\times I^\infty)]\setminus h_n(C\times I^\infty),$$

and H is limited by $\mathfrak{U}$. Using Lemma 3.7 we can extend α to a homeomorphism $\tilde{\alpha} : X' \to X'$ satisfying $\tilde{\alpha}|h_n(C\times I^\infty) = \mathrm{id}$. Then

$$\tilde{\alpha}\circ\tilde{h}_n : L\times I^\infty \to X'$$

is a closed embedding which satisfies $\mathrm{Bd}(\tilde{\alpha}\circ\tilde{h}_n(L\times I^\infty)) = \tilde{\alpha}\circ\tilde{h}_n(L\times W_1^+)$ and it is bicollared,

$$\tilde{\alpha}\circ\tilde{h}_n|C\times I^\infty = h_n|C\times I^\infty,\ \tilde{\alpha}\circ\tilde{h}_n|\mathrm{Bd}(L)\times I^\infty = f|\mathrm{Bd}(L)\times I^\infty,$$

and $\tilde{\alpha}\circ\tilde{h}_n(x, (0, 0, \cdots)) = \varphi(x)$, for all $x\in L$.

Now let $F = f((\bigcup_{i=1}^\infty\sigma_i')\times W_1^+)$, which is a Z-set in X' satisfying $F\cap[\tilde{\alpha}\circ\tilde{h}_n(L\times\{(0, 0, \cdots)\}) \cup \tilde{\alpha}\circ\tilde{h}_n(\mathrm{Bd}(L\times(I^\infty\setminus W_1^+))] = \emptyset$. Using Lemma 4.3 there is a homeomorphism $\beta : X' \to X'$ satisfying

$$\beta(F)\cap\tilde{\alpha}\circ\tilde{h}_n(L\times I^\infty) \subset \tilde{\alpha}\circ\tilde{h}_n(\mathrm{Bd}(L)\times W_1^+)$$

and

$$\beta|\tilde{\alpha}\circ\tilde{h}_n(L\times\{(0, 0, \cdots)\}) \cup \tilde{\alpha}\circ\tilde{h}_n(\mathrm{Bd}(L)\times I^\infty) \cup \tilde{\alpha}\circ\tilde{h}_n(C\times I^\infty) = \mathrm{id}.$$

Thus $f : (\bigcup_{i=1}^\infty\sigma_i')\times I^\infty \to X$ and $\beta^{-1}\circ\tilde{\alpha}\circ\tilde{h}_n : L\times I^\infty \to X$ are closed embeddings which are compatible, i.e. we can patch them together to obtain a closed embedding $h_{n+1}' : |K_{n+1}|\times I^\infty \to X$ which satisfies $\mathrm{Bd}(h_{n+1}'(|K_{n+1}|\times I^\infty)) = h_{n+1}'(|K_{n+1}|\times W_1^+)$, $h_{n+1}'|C\times I^\infty = h_n|C\times I^\infty$, and $h_{n+1}'(x, (0, 0, \cdots)) = \varphi(x)$, for all $x\in|K_{n+1}|$.

Of course we have made no provision to require that

$$\mathrm{Bd}(h'_{n+1}(|K_{n+1}| \times I^\infty))$$

be bicollared, but this presents no problem since

$$\mathrm{Bd}(h'_{n+1}(|K_{n+1}| \times [-1, \tfrac{1}{2}] \times \Pi_{i=2}^\infty I_i))$$

is bicollared. It is also true that we might not have

$$h'_{n+1}(|K_{n+1}| \times I^\infty) \cap \varphi(|K|) \subset \varphi(\mathrm{St}(|K_{n+1}|, K^{(3)})),$$

but this can be clearly achieved by 'squeezing'

$$h'_{n+1}(|K_{n+1}| \times I^\infty) \text{ close to } \varphi(|K_{n+1}|).$$

Thus we can modify h'_{n+1} to obtain our required h_{n+1}.

Proof of Theorem 1.

Write $X = \bigcup_{n=1}^\infty X_n$, where each X_n is a compact set contained in the interior of X_{n+1}. Let K be a complex and let $\varphi : |K| \to P$ be a homeomorphism. Let H_1 be a finite subcomplex of K such that $P \cap X_1 \subset \varphi(|H_1|)$ and choose n_1 large enough so that

$$\mathrm{St}(|H_1|, K_{n_1+1}) \subset |K_{n_1}|.$$

One can clearly construct a closed embedding $h_0 : |K_0| \times I^\infty \to X$ which satisfies $h_0(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_0|$, and

$$\mathrm{Bd}(h_0(|K_0| \times I^\infty)) = h_0(|K_0| \times W_1^+)$$

and it is bicollared. Then using Lemma 4.4 and an obvious inductive argument we can obtain a closed embedding $h_{n_1} : |K_{n_1}| \times I^\infty \to X$ satisfying $h_{n_1}(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_{n_1}|$, and

$$\mathrm{Bd}(h_{n_1}(|K_{n_1}| \times I^\infty)) = h_{n_1}(|K_{n_1} \times W_1^+)$$

and it is bicollared.

Now let H_2 be a finite subcomplex of K so that $|H_1| \subset \mathrm{Int}(|H_2|)$ and $\varphi(|K|) \cap X_2 \subset \varphi(|H_2|)$. Choose $n_2 > n_1$ such that

$$\mathrm{St}(|H_2|, K_{n_2+1}) \subset |K_{n_2}|.$$

Using Lemma 4.4 and an inductive argument we can find a closed embedding $h_{n_2} : |K_{n_2}| \times I^\infty \to X$ such that $h_{n_2}(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_{n_2}|$, $\mathrm{Bd}(h_{n_2}(|K_{n_2}| \times I^\infty)) = h_{n_2}(|K_{n_2}| \times W_1^+)$ and it is bicollared, and $h_{n_2}|\, |H_1| \times I^\infty = h_{n_1}|\, |H_1| \times I^\infty$.

In general let $\{H_i\}_{i=1}^\infty$ be a collection of finite subcomplexes of K so that for each i, $|H_i| \subset \mathrm{Int}(|H_{i+1}|)$ and $\varphi(|K|) \cap X_i \subset \varphi(|H_i|)$. Choose integers $\{n_i\}_{i=1}^\infty$ such that for each i, $n_i < n_{i+1}$ and

$$\mathrm{St}(|H_i|, K_{n_i+1}) \subset |K_{n_i}|.$$

Using the above techniques we find that for each $i > 0$ there is a closed embedding $h_{n_i} : |K_{n_i}| \times I^\infty \to X$ such that $h_{n_i}(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_{n_i}|$, $\mathrm{Bd}(h_{n_i}(|K_{n_i}| \times I^\infty)) = h_{n_i}(|K_{n_i}| \times W_1^+)$ and it is bicollared, and $h_{n_{i+1}}| \, |H_i| \times I^\infty = h_{n_i}| \, |H_i| \times I^\infty$. For each $x \in |H_i| \times (I^\infty \setminus W_1^+)$ define $h'(x) = h_{n_i}(x)$. It is clear that $h' : |K| \times (I^\infty \setminus W_1^+) \to X$ is an open embedding satisfying $h'(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K|$. Since $I^\infty \setminus W_1^+ \cong I^\infty \setminus \{\text{point}\}$ we can clearly modify h' to obtain our required open embedding h.

5. Proof of Theorem 2

We will first establish two technical results concerning cap-sets in Q-manifolds. These are used only in the proof of Theorem 2.

LEMMA 5.1. *Let X be a Q-manifold, P be a polyhedron, $\varphi : P \times \Sigma \to X$ be an embedding such that $\varphi(P \times \Sigma)$ is a cap-set for X and $\varphi(P \times \Sigma_1)$ is closed in X, and let F be a compact Z-set in X. Then there is a homeomorphism $h : X \to X$ such that $h(F) \subset \varphi(P \times \Sigma_2)$ and $h|\varphi(P \times \Sigma_1) = \mathrm{id}$.*

PROOF. By Lemma 3.5. it follows that $\varphi(P \times \Sigma_1)$ is a Z-set in X. Let $X' = X \setminus \varphi(P \times \Sigma_1)$, $F' = F \cap X'$, and $M = \varphi(P \times \Sigma) \setminus \varphi(P \times \Sigma_1)$. Then X' is a Q-manifold, F' is a Z-set in X', and M is a cap-set for X'. Choose an open cover $\mathfrak{U}$ of X' which is normal with respect to $\varphi(P \times \Sigma_1)$.

Lemma 3.2. implies that $M \cup F'$ is a cap-set for X'. Using Lemma 3.3 there is a homeomorphism $f : X' \to X'$ such that $f(M \cup F') = M$ and f is limited by $\mathfrak{U}$. Then f clearly extends to a homeomorphism $\tilde{f} : X \to X$ satisfying $\tilde{f}|\varphi(P \times \Sigma_1) = \mathrm{id}$ and $\tilde{f}(F) \subset \varphi(P \times \Sigma)$.

Put $F^* = \pi_\Sigma \circ \varphi^{-1} \circ \tilde{f}(F)$, which is a compact set in Σ. Clearly there is a proper isotopy $g_t : F^* \cup \Sigma_1 \to \Sigma$ such that $g_0 = \mathrm{id}$, $g_1(F^*) \subset \Sigma_2$ and $g_t|\Sigma_1 = \mathrm{id}$ for all t. Now define an isotopy

$$h_t : \tilde{f}(F) \cup \varphi(P \times \Sigma_1) \to \varphi(P \times \Sigma) \text{ by } h_t \circ \varphi(x, y) = \varphi(x, g_t(y)),$$

for all $(x, y) \in P \times \Sigma$ that satisfy $\varphi(x, y) \in \tilde{f}(F) \cup \varphi(P \times \Sigma_1)$. Note that $h_1(\tilde{f}(F) \cup \varphi(P \times \Sigma_1))$ is a Z-set in X and h_t is a proper isotopy. Using Lemma 3.7 we can extend h_1 to a homeomorphism $g : X \to X$. Then $h = g \circ \tilde{f}$ fulfills our requirements.

LEMMA 5.2. *Let X be a Q-manifold, P be a polyhedron, and let $\varphi : P \times \Sigma \to X$ be an embedding such that $\varphi(P \times \Sigma)$ is a cap-set for X and $\varphi(P \times \Sigma_2)$ is closed in X. Let $h : P \times I^\infty \to X$ be a closed embedding so that $h(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots))$, for all $x \in P$, and $\mathrm{Bd}(h(P \times I^\infty) = h(P \times W_1^+)$. If $F \subset X$ is a compact Z-set, then there is a homeomorphism $f : X \to X$ such that $f(F) \subset h(P \times I^\infty)$ and $f|h(P \times W_1^-) = \mathrm{id}$.*

PROOF. Let Let $\theta : \varphi(P\times\Sigma_2) \to h(P\times\Sigma_2)$ be the homeomorphism defined by $\theta \circ \varphi(x, y) = h(x, y)$, for all $(x, y) \in P\times\Sigma_2$. It is clear that θ is properly homeotopic to the identity. Let φ_1 be an extension of θ to a homeomorphism of X onto itself. Then $\varphi_1 \circ \varphi : P\times\Sigma \to X$ is an embedding such that $\varphi_1 \circ \varphi(P\times\Sigma)$ is a cap-set for X, $\varphi_1 \circ \varphi(P\times\Sigma_1) = h(P\times\Sigma_1)$, $\varphi_1 \circ \varphi(P\times\Sigma_2) = h(P\times\Sigma_2)$, and $\varphi_1 \circ \varphi(x, (0, 0, \cdots)) = h(x, (0, 0, \cdots))$, for all $x \in P$.

It is clear that there exists a homeomorphism $\alpha : h(P\times\Sigma_1) \to h(P\times W_1^-)$ such that $\alpha \circ h(x, (0, 0, \cdots)) = h(x, (-1, 0, 0, \cdots))$ for all $x \in P$, and for which α is properly homotopic to the identity, with the homotopy taking place inside $h(P\times I^\infty)$. By choosing covers appropriately and using Lemma 3.7 we can extend α to a homeomorphism $\varphi_2 : X \to X$ which satisfies $\varphi_2|X\setminus h(P\times I^\infty) = \text{id}$. It is clear now that $\tilde{\varphi} = \varphi_2 \circ \varphi_1 \circ \varphi : P\times\Sigma \to X$ is an embedding such that $\tilde{\varphi}(P\times\Sigma)$ is a cap-set for X and $\tilde{\varphi}(P\times\Sigma_2)$ is a Z-set in X.

Using Lemma 5.1 there is a homeomorphism $f : X \to X$ such that $f(F) \subset \tilde{\varphi}(P\times\Sigma_2)$ and $f|\tilde{\varphi}(P\times\Sigma_1) = \text{id}$. This implies that $f|h(P\times W_1^-) = \text{id}$. Note that $\varphi_1 \circ \varphi(P\times\Sigma_2) = h(P\times\Sigma_2)$ and

$$\varphi_2 \circ \varphi_1 \circ \varphi(P\times\Sigma_2) = \varphi_2 \circ h(P\times\Sigma_2) \subset h(P\times I^\infty),$$

which implies that $f(F) \subset h(P\times I^\infty)$.

PROOF OF THEOREM 2.

Roughly the idea of the proof is to find a copy of P in X which is a Z-set, use Theorem 1 to build a 'nice' open set around this polyhedron, and use Lemma 5.2 to 'blow up' this open set to engulf a cap-set. The part of X that this open set misses is the Z-set F which we are looking for.

Using Lemma 3.1 let $\varphi : P\times\Sigma \to X$ be an embedding such that $\varphi(P\times\Sigma)$ is a cap-set for X. A routine argument proves that if A is any locally compact subset of X, then $Cl(A)\setminus A$ is a closed subset of X. Thus, $F_1 = Cl(\varphi(P\times\Sigma_2))\setminus\varphi(P\times\Sigma_2)$ is a closed subset of X missing $\varphi(P\times\Sigma)$. It follows from Lemma 3.4 that F_1 is a Z-set in X. Put $X' = X\setminus F_1$ and note that $\varphi(P\times\Sigma)$ is a cap-set for X'. But we now have $\varphi(P\times\Sigma_2)$ a Z-set in X', because it is closed.

Using Theorem 1 there is a closed embedding $h : P\times I^\infty \to X'$ such that $h(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots))$, for all $x \in P$, and

$$\text{Bd}(h(P\times I^\infty)) = h(P\times W_1^+).$$

Write $\varphi(P\times\Sigma) = \bigcup_{n=1}^\infty M_n$, a tower of compact Z-sets. Using Lemma 5.2 there is a homeomorphism $f_1 : X' \to X'$ such that

$$f_1(M_1) \subset h(P\times[-1, \tfrac{1}{2}]\times\Pi_{i=2}^\infty I_i).$$

Then put $g_1 = f_1^{-1}$ to complete the first step of our construction.

Now let $X'' = X' \setminus g_1 \circ h\ (P \times [-1, \frac{1}{2}) \times \Pi_{i=2}^{\infty} I_i)$, which is obviously a Q-manifold containing $g_1 \circ h(P \times \{\frac{1}{2}\} \times \Pi_{i=2}^{\infty} I_i)$ as a Z-set. Put $M_2' = M_2 \cap X''$, which is clearly a compact Z-set in X''. One can obviously construct a homeomorphism $\alpha : X' \to X''$ such that

$$\alpha \circ g_1 \circ h(x, (0, 0, \cdots)) = g_1 \circ h(x, (\tfrac{2}{3}, 0, 0, \cdots)),$$

for all $x \in P$. Then $\varphi' = \alpha \circ g_1 \circ \varphi : P \times \Sigma \to X''$ is an embedding such that $\varphi'(P \times \Sigma)$ is a cap-set for X'' and

$$\varphi'(x, (0, 0, \cdots)) = g_1 \circ h(x, (\tfrac{2}{3}, 0, 0, \cdots)),$$

for all $x \in P$. Also $g_1 \circ h : P \times [\frac{1}{2}, 1] \times \Pi_{i=2}^{\infty} I_i \to X''$ is a closed embedding satisfying $\mathrm{Bd}_{X''}\ (g_1 \circ h(P \times [\frac{1}{2}, 1] \times \Pi_{i=2}^{\infty} I_i)) = g_1 \circ h(P \times W_1^+)$.

Once more applying Lemma 5.2 there is a homeomorphism $f_2 : X'' \to X''$ such that $f_2 | g_1 \circ h(P \times \{\frac{1}{2}\} \times \Pi_{i=2}^{\infty} I_i) = \mathrm{id}$ and

$$f_2(M_2') \subset g_1 \circ h\ (P \times [\tfrac{1}{2}, \tfrac{3}{4}] \times \Pi_{i=2}^{\infty} I_i).$$

Then let $\tilde{f}_2$ be the extension of f_2 to all of X' such that

$$\tilde{f}_2 | g_1 \circ h\ (P \times [-1, \tfrac{1}{2}] \times \Pi_{i=2}^{\infty} I_i) = \mathrm{id}.$$

Now put $g_2 = \tilde{f}_2^{-1}$, which is a homeomorphism of X' onto itself satisfying $g_2 | g_1 \circ h(P \times [-1, \frac{1}{2}] \times \Pi_{i=2}^{\infty} I_i) = \mathrm{id}$ and

$$M_2 \subset g_2 \circ g_1 \circ h(P \times [-1, \tfrac{3}{4}] \times \Pi_{i=2}^{\infty} I_i).$$

It is then clear that we can obtain a sequence $\{g_i\}_{i=1}^{\infty}$ of homeomorphisms of X' onto itself such that

$$M_n \subset g_n \circ g_{n-1} \circ \cdots \circ g_1 \circ h \left(P \times \left[-1, 1 - \frac{1}{2^n} \right] \times \prod_{i=2}^{\infty} I_i \right)$$

and

$$g_n | g_{n-1} \circ \cdots \ g_1 \circ h \left(P \times \left[-1, 1 - \frac{1}{2^{n-1}} \right] \times \prod_{i=2}^{\infty} I_i \right) = \mathrm{id},$$

for all $n > 1$. Then let $g(x) = \lim_{n \to \infty} g_n \circ \cdots \circ g_1(x)$ for all

$$x \in h(P \times (I^{\infty} \setminus W_1^+).$$

It is clear that $g : h(P \times (I^{\infty} \setminus W_1^+)) \to X'$ is an open embedding such that $g \circ h(P \times (I^{\infty} \setminus W_1^+))$ contains $\varphi(P \times \Sigma)$. Thus

$$F_2 = X' \setminus g \circ h(P \times (I^{\infty} \setminus W_1^+))$$

is a Z-set in X' and therefore $F = F_1 \cup F_2$ is a Z-set in X such that $X \setminus F \cong P \times (I^{\infty} \setminus W_1^+)$.

6. Proofs of Theorems 3, 4, 5 and their Corollaries

The following result will be used in the proof of Theorem 3.

LEMMA 6.1. *Let X be a Q-manifold and let $F \subset X$ be a Z-set. Then $(X \setminus F) \times [0, 1) \cong X \times [0, 1)$, where the homeomorphism can be chosen to be homotopic to the inclusion of $(X \setminus F) \times [0, 1)$ in $X \times [0, 1)$.*

PROOF. If X_1 is any Q-manifold and $C \subset X_1$ is any Z-set, then $C \times [0, 1]$ is a Z-set in $X_1 \times [0, 1]$. In order to see this let us take a homeomorphism h_1 of X_1 onto $X_1 \times I^\infty$ taking C into $X_1 \times \{(0, 0, \cdots)\}$. Then $h_1 \times \mathrm{id} : X_1 \times [0, 1] \to X_1 \times I^\infty \times [0, 1]$ is a homeomorphism which takes $C \times [0, 1]$ into $X_1 \times \{(0, 0, \cdots)\} \times [0, 1]$. Let

$$h_2 : X_1 \times I^\infty \times [0,1] \to X_1 \times I^\infty$$

be a homeomorphism in which [0,1] is factored back into X_1 Then $h_2 \circ (h_1 \times \mathrm{id}) : X_1 \times [0,1] \to X_1 \times I^\infty$ is a homeomorphism taking $C \times [0, 1]$ into $X_1 \times \{(0, 0, \cdots)\}$, and by Lemma 3.8 it follows that

$$h_2 \circ (h_1 \times \mathrm{id})(C \times [0, 1])$$

is a Z-set in $X_1 \times I^\infty$. Thus $C \times [0, 1]$ is a Z-set in $X_1 \times [0,1]$.

Let $A = (X \times \{1\}) \cup (F \times [0, 1])$ and $B = (X \times \{1\}) \cup (F \times [\frac{1}{2}, 1])$ be subsets of $X \times [0, 1]$. Since A and B are Z-sets in $X \times [0, 1]$ we can use Lemma 3.7 to get a homeomorphism $f : X \times [0, 1] \to X \times [0, 1]$ satisfying $f(A) = B$ and $f | X \times \{1\} = \mathrm{id}$. It follows from [3] that we can additionally choose f to be isotopic to $\mathrm{id}_{X \times [0, 1]}$ (with each level fixed on $X \times \{1\}$). Therefore $f | X \times [0, 1)$ gives a homeomorphism of $X \times [0, 1)$ onto itself which is homotopic (in $X \times [0, 1)$) to $\mathrm{id}_{X \times [0, 1)}$.

Let $h_t : [0,1] \to [0,1]$ be a homotopy which satisfies the following properties:

(1) $h_0 = \mathrm{id}$,
(2) $h_1([\frac{1}{2}, 1]) = \{1\}$,
(3) $h_1 | [0, \frac{1}{2}]$ is a homeomorphism of $[0, \frac{1}{2}]$ onto $[0, 1]$,
(4) $h_t : [0, 1] \to [0, 1]$ is a homeomorphism for all $t \neq 1$.

Define a continuous function $g : X \times [0, 1] \to X \times [0, 1]$ as follows: for each $x \in X$ and $y \in [0, 1]$, let $g(x, y) = (x, h_t(y))$, where $t = 1/(1+d(x, F))$. Clearly $g | (X \times [0, 1]) \setminus B$ gives a homeomorphism of $(X \times [0, 1]) \setminus B$ onto $X \times [0, 1)$ which is homotopic to the inclusion of $(X \times [0, 1]) \setminus B$ in $X \times [0, 1)$. Then $g \circ f | (X \setminus F) \times [0, 1)$ gives a homeomorphism of $(X \setminus F) \times [0, 1)$ onto $X \times [0,1)$ which is homotopic to the inclusion of $(X \setminus F) \times [0, 1)$ in $X \times [0, 1)$.

We will also need the following result.

LEMMA 6.2. *Let X be a Q-manifold, P be a polyhedron, and let $f: P\times(I^\infty\setminus W_1^+)\to X$ be a homotopy equivalence. Then there exists an open embedding $g: P\times(I^\infty\setminus W_1^+)\to X$ such that g is homotopic to f and $X\setminus g(P\times(I^\infty\setminus W_1^+))$ is a Z-set in X.*

PROOF. It follows routinely from the coordinate structure of I^∞ that there is a homeomorphism of $I^\infty\times I^\infty$ onto I^∞ which is homotopic to the projection of $I^\infty\times I^\infty$ onto the first factor. Since $X\times I^\infty\cong X$, it follows that there is a homeomorphism $\beta: X\times I^\infty\to X$ which is homotopic to π_X, the projection of $X\times I^\infty$ onto X. Define $f': P\to X$ by $f'(x)=f(x,(0,0.\cdots))$, for all $x\in P$. Then $f'$is also a homotopy equivalence.

It follows from [15] that $P\times s$ is an F-manifold and it follows routinely from the definition that $X\times s$ is an F-manifold. Note that

$$f'\times \mathrm{id}_s: P\times s\to X\times s$$

is a homotopy equivalence. Thus $f'\times\mathrm{id}_s$ is homotopic to a homeomorphism $\alpha: P\times s\to X\times s$ (see [10]).

Now $P\times\Sigma$ is a cap-set for $P\times s$ (see [6]) and therefore $\alpha(P\times\Sigma)$ is a cap-set for $X\times I^\infty$ (since $X\times I^\infty$ can be deformed into $X\times s$ with 'small' motions). Hence $\beta\circ\alpha(P\times\Sigma)$ is a cap-set for X. As in the proof of Theorem 2 let $F_1 = Cl(\varphi(P\times\Sigma_2))\setminus\varphi(P\times\Sigma_2)$, where $\varphi=\beta\circ\alpha|P\times\Sigma$, and let $h: P\times I^\infty\to X\setminus F_1$ be a closed embedding such that

$$h(x,(0,0,\cdots))=\varphi(x,(0,0,\cdots)),$$

for all $x\in P$, and $\mathrm{Bd}(h(P\times I^\infty)) = h(P\times W_1^+)$. In the proof of Theorem 2 a homeomorphism $g': h(P\times(I^\infty\setminus W_1^+))\to X\setminus F$ was constructed, where F is a Z-set in X containing F_1. Moreover it is clear from the construction given there that g' is homotopic to the inclusion of $h(P\times(I^\infty\setminus W_1^+))$ in X. Thus $g=g'\circ h|P\times(I^\infty\setminus W_1^+)$ gives an open embedding of $P\times(I^\infty\setminus W_1^+)$ in X whose complement is a Z-set in X. Moreover g is homotopic to $h' = h|P\times(I^\infty\setminus W_1^+)$. All that is left to do is prove that h' is homotopic to f.

To this end let $r: P\times(I^\infty\setminus W_1^+)\to P\times\{(0,0,\cdots)\}$ be given by $r(x,t)=(x,(0,0,\cdots))$, for all $x\in P$ and $t\in I^\infty\setminus W_1^+$. It is clear that h' is homotopic to $h'\circ r$ and $h'\circ r=\beta\circ\alpha\circ r$. Since α is homotopic to $f'\times\mathrm{id}_s$ it follows that $\beta\circ\alpha\circ r$ is homotopic to $\beta\circ(f'\times\mathrm{id}_s)\circ r$. But $\beta\circ(f'\times\mathrm{id}_s)\circ r$ is homotopic to $\pi_X\circ(f'\times\mathrm{id}_s)\circ r$. But $\pi_X\circ(f'\times\mathrm{id}_s)\circ r = f\circ r$, and since r is homotopic to $\mathrm{id}_{P\times(I^\infty\setminus W_1{}^+)}$ it follows that $f\circ r$ is homotopic to f.

PROOFS OF THEOREMS 3 AND 5.

Let $f: X\to Y$ be a homotopy equivalence, where X and Y are Q-

manifolds. Let P be a polyhedron for which there exists a homotopy equivalence $g: P\times(I^{\infty}\setminus W_1^+) \to X$. Using Lemma 6.2 we see that g is homotopic to a homeomorphism $\alpha: P\times(I^{\infty}\setminus W_1^+) \to X\setminus F_1$, where $F_1 \subset X$ is a Z-set. Also $f\circ g$ is homotopic to a homeomorphism $\beta: P\times(I^{\infty}\setminus W_1^+) \to Y\setminus F_2$, where $F_2 \subset Y$ is a Z-set. Using Lemma 6.1 it follows that $\alpha\times \mathrm{id}: (P\times(I^{\infty}\setminus W_1^+))\times[0,1) \to (X|F_1)\times[0,1)$ is homotopic to a homeomorphism $\gamma: (P\times(I^{\infty}\setminus W_1^+))\times[0,1)\to X\times[0,1)$, with the homotopy taking place in $X\times[0,1)$. Similarly $\beta\times\mathrm{id}$ is homotopic to a homeomorphism $\delta: (P\times(I^{\infty}\setminus W_1^+))\times[0,1) \to Y\times[0,1)$, with the homotopy taking place in $Y\times[0,1)$.

In order to see that $X\times[0,1)\cong P\times(I^{\infty}\setminus\{\text{point}\})$ note that γ^{-1} gives a homeomorphism of $X\times[0,1)$ onto $P\times(I^{\infty}\setminus W_1^+)\times[0,1)$. Since $I^{\infty}\setminus W_1^+ = [-1,1)\times\Pi_{i=2}^{\infty}\ I_i$ and since $[-1,1)\times[0,1)$ is obviously homeomorphic to $[-1,1)\times[0,1]$, we have $X\times[0,1)\cong P\times(I^{\infty}\setminus W_1^+)$. To finish the proof of Theorem 3 all we need do is note that $I^{\infty}\setminus W_1^+ \cong I^{\infty}\setminus\{\text{point}\}$.

For the proof of Theorem 5 note that $\delta\circ\gamma^{-1}: X\times[0,1)\to Y\times[0,1)$ is a homeomorphism. All that remains to be done is prove that $\delta\circ\gamma^{-1}$ is homotopic to $f\times\mathrm{id}$, or equivalently, to prove that δ is homotopic to $(f\times\mathrm{id})\circ\gamma$. But δ is homotopic to $\beta\times\mathrm{id}$, which in turn is homotopic to $(f\circ g)\times\mathrm{id} = (f\times\mathrm{id})\circ(g\times\mathrm{id})$. Since $g\times\mathrm{id}$ is homotopic to $\alpha\times\mathrm{id}$, and $\alpha\times\mathrm{id}$ is homotopic to γ, we are done.

PROOF OF COROLLARY 1.

Choose any polyhedron P for which $P\sim X$ and use Theorem 3 to get $X\times[0,1)\cong P\times(I^{\infty}\setminus\{\text{point}\})$. Now $I^{\infty}\setminus\{\text{point}\}\cong I^{\infty}\times[0,1)$, hence $P\times(I^{\infty}\setminus\{\text{point}\})\cong(P\times[0,1))\times I^{\infty}$. But $P\times[0,1)$ can obviously be triangulated by a complex.

PROOF OF COROLLARY 2.

Apply Theorem 3.

PROOF OF COROLLARY 3.

Apply Theorem 3.

PROOF OF THEOREM 4.

Let $Y = X\times s$, which is obviously an F-manifold satisfying $Y\sim X$. Using Henderson's open embedding theorem let $g: Y\to s$ be an open embedding. Let U be an open subset of I^{∞} for which $U\cap s = g(Y)$. Then U is a Q-manifold, and as $U\cap B(I^{\infty})$ is obviously a cap-set for U, it follows from Lemma 3.6 that $U\sim g(Y)$. Thus $X\sim U$. Using Corollary 2 we have $X\times[0,1)\cong U\times[0,1)$, and using the fact that $U\times[0,1]\cong U$ we have $U\times[0,1)\cong U\setminus F$, for some closed subset F of U. Thus $X\times[0,1)\cong U\setminus F$, which is open in I^{∞}.

PROOF OF COROLLARY 4.

Let $f: X \to X \times [0, 1]$ be a homeomorphism and put

$$U = f^{-1}(X \times [0, 1)),\ V = f^{-1}(X \times (0, 1]).$$

7. Proofs of Theorem 6, its Corollary, and Theorem 7

The following result will be used in the proof of Theorem 6.

LEMMA 7.1. *Let X be a compact Q-manifold and assume that $X \sim P$, for some compact polyhedron P. Then there is a copy P' of P in X which is a Z-set and a pseudo-isotopy $h_t : X \to X$ which satisfies the following properties.*

(1) $h_0 = \mathrm{id}$,
(2) $h_1(X) = P'$,
(3) $h_t|P' = \mathrm{id}$ *for all t, and*
(4) $h_t : X \to X$ is an embedding for all $t \neq 1$.

PROOF. Let $f: X \to X \times I^\infty$ be a homeomorphism. Since $X \times s$ is an F-manifold and $X \times s \sim P$, it follows that there is a homeomorphism $\varphi : P \times s \to X \times s$. Using the fact that $\varphi(P \times \{(0, 0, \cdots)\})$ is a compact subset of $X \times s$, it is clear that there is an isotopy $f_t : X \times I^\infty \to X \times I^\infty$ such that $f_0 = \mathrm{id}$, $f_1(X \times I^\infty) \subset X \times s$, and $f_t|\varphi(P \times \{(0, 0, \cdots)\}) = \mathrm{id}$, for all t.

One can obviously get a pseudo-isotopy $g_t : \varphi(P \times s) \to \varphi(P \times s)$ such that $g_0 = \mathrm{id}$, $g_1 \circ \varphi(P \times s) = \varphi(P \times \{(0, 0, \cdots)\})$, g_t is an embedding for all $t \neq 1$, and $g_t|\varphi(P \times \{(0, 0, \cdots)\}) = \mathrm{id}$, for all t. Then let $h'_t : X \times I^\infty \to X \times I^\infty$ be defined by

$$h'_t(x) = \begin{cases} f_{2t}(x), & \text{for } 0 \leqq t \leqq \frac{1}{2} \\ g_{2t-1} \circ f_1(x), & \text{for } \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

Obviously h'_t is a pseudo-isotopy satisfying

$$h'_0 = \mathrm{id},\ \ h'_1(X \times I^\infty) = \varphi(P \times \{(0, 0, \cdots)\}),\ \ h'_t|\varphi(P \times \{(0, 0, \cdots)\}) = \mathrm{id}$$

for all t, and h'_t is an embedding for all $t \neq 1$. Then let

$$P' = f^{-1} \circ \varphi(P \times \{(0, 0, \cdots)\})$$

and let $h_t : X \to X$ be defined by $h_t(x) = f^{-1} \circ h'_t \circ f(x)$.

PROOF OF THEOREM 6.

Using Theorem 3 and the fact that $X \cong X \times [0, 1]$, there is a copy X' of X in X which is a Z-set and there is a homeomorphism

$$f : P \times (I^\infty \setminus W_1^+) \to X \setminus X'.$$

Using Lemma 7.1 let P' be a copy of P in X' and let $h_t : X' \to X'$ be a pseudo-isotopy satisfying $h_0 = \mathrm{id}$, $h_1(X') = P'$, h_t is an embedding for all $t \neq 1$, and $h_t|P' = \mathrm{id}$ for all t. Since $P' \subset X'$ it easily follows that P' is a Z-set in X.

Let $\{U_i\}_{i=1}^{\infty}$ be any collection of open subsets of X such that $\bigcap_{i=1}^{\infty} U_i = P'$ and $X' \subset U_1$. Using the compactness of P and X we can find a number $t_1 \in (-1, 1)$ such that $f(P \times [t_1, 1) \times \Pi_{i=2}^{\infty} I_i) \subset U_1$. Let $V_1 = X \setminus f(P \times [-1, t_1] \times \Pi_{i=2}^{\infty} I_i)$, which is an open set containing X'. By choosing $t \in (0, 1)$ sufficiently close to 1 we have an embedding $h_t : X' \to X' \cap U_2$ which is properly homotopic to the identity, where the image of the proper homotopy is entirely contained in X'. Moreover this proper homotopy is limited by some open cover of V_1 which is normal with respect to $X \setminus V_1$. Thus we can apply Lemma 3.7 to extend h_t to a homeomorphism $g_1 : X \to X$ which satisfies

$$g_1 | f(P \times [-1, t_1] \times \Pi_{i=2}^{\infty} I_i) = \mathrm{id},$$

$g_1|P' = \mathrm{id}$, and $g_1(X') \subset U_2$.

Now choose $t_2 \in (t_1, 1)$ such that $g_1 \circ f(P \times [t_2, 1) \times \Pi_{i=2}^{\infty} I_i) \subset U_2$ and use the above techniques to construct a homeomorphism $g_2 : X \to X$ satisfying $g_2 | g_1 \circ f(P \times [-1, t_2] \times \Pi_{i=2}^{\infty} I_i) = \mathrm{id}$, $g_2|P' = \mathrm{id}$, and

$$g_2 \circ g_1(X') \subset U_3.$$

It is clear that we can continue this process to obtain homeomorphisms $\{g_i\}_{i=1}^{\infty}$ of X onto itself and numbers $t_1 < t_2 < \cdots < 1$ limiting to 1 such that

$$g_{i+1} | g_i \circ \cdots \circ g_1 \circ f(P \times [-1, t_{i+1}] \times \Pi_{i=2}^{\infty} I_i) = \mathrm{id},$$

$g_i \circ \cdots \circ g_1(X') \subset U_{i+1}$, and $g_i|P' = \mathrm{id}$, for all i. Then define $g : P \times (I^{\infty} \setminus W_1^+) \to X \setminus P'$ by $g(x) = \lim g_i \circ \cdots \circ g_1 \circ f(x)$. Clearly g is a homeomorphism which is what we wanted.

Proof of Corollary 5.

It follows from [12] that any homotopically trivial metric ANR is contractible. Thus X must be a compact contractible Q-manifold, hence it has the homotopy type of a point. It follows from Theorem 6 that $X \setminus \{\text{point}\} \cong I^{\infty} \setminus \{\text{point}\}$, thus $X \cong I^{\infty}$.

We will need the following result for the proof of Theorem 7.

Lemma 7.2. *Let X be a compact Q-manifold for which $X \sim P$, for some compact polyhedron P. Then there is an embedding $h : P \times I^{\infty} \to X$ such that* $\mathrm{Bd}(h(P \times I^{\infty})) = h(P \times W_1^+)$ *and there is a strong deformation retraction of X onto $h(P \times W_1^-)$.*

Proof. Let $\varphi : P \times s \to X \times s$ be a homeomorphism and let

$$h' : P \times I^\infty \to X \times I^\infty$$

be an embedding such that $h'(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots))$, for all $x \in P$, and $\mathrm{Bd}(h'(P \times I^\infty)) = h'(P \times W_1^+)$. Now $h'(P \times W_1^-)$ is a Z-set in $X \times I^\infty$, thus Lemma 3.7 implies that there is a homeomorphism $f : X \times I^\infty \to X \times I^\infty$ for which $f \circ h'(P \times W_1^-) = \varphi(P \times \Sigma_1)$.

Using an argument similar to that used in the proof of Lemma 7.1, there is a strong deformation retraction h_t of $X \times I^\infty$ onto $\varphi(P \times \Sigma_1)$. Thus $f^{-1} \circ h_t \circ f$ gives a strong deformation retraction of $X \times I^\infty$ onto $h'(P \times W_1^-)$. Using the fact that $X \cong X \times I^\infty$ we can easily transfer this information back to X.

PROOF OF THEOREM 7.

The procedure will be to attach a copy of I^∞ to X so that the resulting space is a compact contractible Q-manifold.

Assume that $\dim(P) \leqq n$ and consider P as linearly embedded in the $(2n+1)$-cell $\Pi_{i=1}^{2n+1} I_i$. Let $f : P \times [1, 2] \times I^\infty \to X$ be an embedding such that $\mathrm{Bd}(f(P \times [1, 2] \times I^\infty)) = f(P \times \{2\} \times I^\infty)$, where we consider

$$P \times [1, 2] \subset E^{2n+2}$$

($(2n+2)$-dimensional Euclidean space), and for which there is a strong deformation retraction of X onto $f(P \times \{1\} \times I^\infty)$.

Let X^* be the space constructed by attaching $(\Pi_{i=1}^{2n+2} I_i) \times I^\infty$ to X, with the attaching map being $f|P \times \{1\} \times I^\infty$. To show that X^* is a Q-manifold all we have to do is check at $f(P \times \{1\} \times I^\infty)$. We know from [15] that the product of any polyhedron with I^∞ gives a Q-manifold. Since there is obviously a neighborhood of $f(P \times \{1\} \times I^\infty)$ in X^* which is homeomorphic to $[(\Pi_{i=1}^{2n+2} I_i) \cup (P \times [1,2])] \times I^\infty$, we conclude that X^* is a compact Q-manifold.

To see that X^* is contractible we note that there is a strong deformation retraction of X^* onto the attached copy of $(\Pi_{i=1}^{2n+2} I_i) \times I^\infty$ in X^*. Thus it follows that X^* is contractible, hence $X^* \cong I^\infty$ by Corollary 5. The proof of the theorem is now complete.

8. Proof of Theorem 8

We will need the following preliminary result. A proof can easily be constructed using techniques similar to those used to establish Lemma 3.1 of [4]. For this reason we do not give a proof.

LEMMA 8.1. *Let J^∞ be a copy of I^∞. There is a continuous function $g : I^\infty \times [1, \infty) \to I^\infty \times J^\infty$ which satisfies the following properties.*

(1) *for n an integer and* $n \leqq u < n+1$, g_u *is a homeomorphism of* I^∞ *onto* $(I_1 \times \cdots \times I_n \times [n-u, u-n] \times \{(0, 0, \cdots)\}) \times J^\infty$, *where* g_u *is defined by* $g_u(x) = g(x, u)$, *for all* $x \in I^\infty$, *and*

(2) *for* $u \in [1, \infty)$ *and* $n \leqq u$ (*n an integer*),

$$\pi_n \circ \pi_{I^\infty} \circ g_u((x_i)) = (x_1, \cdots, x_n),$$

for all $(x_i) \in I^\infty$.

We will need one more preliminary result before we establish Theorem 8. We will need a definition first.

Let G be an open subset of I^∞. A continuous function $\varphi : G \to [1, \infty)$ is said to have the *local product property* with respect to G provided that for each $x \in G$ there is an integer $m(x) \leqq \varphi(x)$ such that the following properties are satisfied.

(1) for all $x = (x_i) \in G$, $\{(x_1, \cdots, x_{m(x)})\} \times \prod_{i=m(x)+1}^{\infty} I_i \subset G$

(2) for all $x = (x_i) \in G$ and $(y_{m(x)+1}, y_{m(x)+2}, \cdots) \in \prod_{i=m(x)+1}^{\infty} I_i$, $\varphi((x_i)) = \varphi(x_1, \cdots, x_{m(x)}, y_{m(x)+1}, y_{m(x)+2}, \cdots)$, and

(3) φ is unbounded near $I^\infty \setminus G$, i.e. for each $x \in \mathrm{Bd}(G)$ and each integer $n > 0$, there is an open set U containing x such that $\varphi(G \cap U) \subset [n, \infty)$.

LEMMA 8.2. *Let G be an open subset of* I^∞ *and assume that there is a continuous function* $\varphi : G \to [1, \infty)$ *which has the local product property with respect to G. Let* $\alpha : E^1 \to E^1$ (*where* E^1 *is the real line*) *be defined by* $\alpha(x) = x$, *for* $x \geqq 0$, *and* $\alpha(x) = 0$, for $x \leqq 0$. *Then* $G \cong G(\varphi) \times J^\infty$, *where*

$$G(\varphi) = \{(x_i) \in G |\, |x_i| \leqq \alpha(\varphi(x)-(i-1)), \textit{for all } i \geqq 1\}.$$

PROOF. Let $g : I^\infty \times [1, \infty) \to I^\infty \times J^\infty$ be the continuous function of Lemma 8.1. For each $x \in G$ let $h(x) = g(x, \varphi(x))$, which gives a homeomorphism of G onto $G(\varphi) \times J^\infty$. The details of the argument are elementary.

PROOF OF THEOREM 8.

Using a standard technique (for example see Lemma 6.1 of [6]) there is a countable star-finite collection $\mathfrak{U}$ of basic open subsets of I^∞ such that $G = \bigcup\{U | U \in \mathfrak{U}\}$ and $Cl(U) \subset G$, for all $U \in \mathfrak{U}$. (An open subset of I^∞ is basic provided that its closure is a basic closed set). It is clear that by subdividing $\{Cl(U) | U \in \mathfrak{U}\}$ we can get a countable star-finite collection

$\mathfrak{F}$ of basic closed subsets of I^∞ such that (1) $G = \bigcup\{F|F \in \mathfrak{F}\}$, (2) for each $F \in \mathfrak{F}$, Int (F) is a non-null basic open subset of I^∞, and (3) if $F_1, F_2 \in \mathfrak{F}$ and $F_1 \neq F_2$, then $F_1 \cap F_2$ lies in an endslice of each.

Without loss of generality we may assume that G is connected. Thus we can order $\mathfrak{F}$ as $\{F_i\}_{i=1}^\infty$ so that

$$\begin{aligned} \mathrm{St}(F_1, \mathfrak{F}) &= F_1 \cup F_2 \cup \cdots \cup F_{i(1)} \\ \mathrm{St}^2(F_1, \mathfrak{F}) &= F_1 \cup F_2 \cup \cdots \cup F_{i(1)} \cup F_{i(1)+1} \cup \cdots \cup F_{i(2)} \\ &\vdots \end{aligned}$$

where $1 = i(0) < i(1) < \cdots$ and $\mathrm{St}^n(F_1, \mathfrak{F})$ has the usual meaning.

For each $j > 0$ let $m(j)$ denote a positive integer such that $F_j = A_j \times \Pi_{i=m(j)+1}^\infty I_i$, where A_j is a basic closed subset of $\Pi_{i=1}^{m(j)} I_i$. By subdividing $\{F_i\}_{i=1}^\infty$ sufficiently (if necessary) we can choose $\{m(j)\}_{j=1}^\infty$ so that $m(j) = m(i(k))+1$, for all j satisfying $i(k)+1 \leqq j \leqq i(k+1)$.

For each $j > 0$ let $R_j = (A_j \times I_{m(j)+1}) \times \{(0, 0, \cdots)\}$.

Then $\{R_j\}_{j=1}^\infty$ is a locally-finite collection of finite-dimensional cells in G. It is clear that we can define a piecewise linear function $\varphi' : \bigcup_{j=1}^\infty R_j \to [1, \infty)$ which satisfies

(1) $\varphi'(x) = m(1)+2$, for all $x \in R_1$,

(2) $m(1)+j+1 < \varphi'(x) \leqq m(1)+j+2$, for all integers $j \geqq 1$ and $x \in (\bigcup_{i=i(j-1)+1}^{i(j)} R_i) \setminus \bigcup_{i=1}^{i(j-1)} R_i)$, and

(3) $\varphi'(x) = m(1)+j+2$, for all $x \in (\bigcup_{i=i(j-1)+1}^{i(j)} R_i) \cap (\bigcup_{i=i(j)+1}^\infty R_i)$.

Then extend φ' to a continuous function $\varphi : G \to [1, \infty)$ by defining $\varphi((x_i)) = \varphi'(x_1, \cdots, x_{m(j)+1}, 0, 0, \cdots)$, for all $(x_i) \in F_j$. It is clear that φ has the local product property with respect to G. Using Lemma 8.2 we find that $G \cong G(\varphi) \times J^\infty$. If we can prove that $G(\varphi)$ can be triangulated by a complex, then we will be done.

We have chosen $\{F_i\}_{i=1}^\infty$ so that for the corresponding $\{R_i\}_{i=1}^\infty$, $R_i \cap R_j$ lies in a face of each, for $i \neq j$. It is obvious that we could have chosen $\{F_i\}_{i=1}^\infty$ so that if $i > j$, then $R_i \cap R_j$ is exactly a face of R_i. This will aid in an inductive triangulation of $G(\varphi)$. The details of the triangulation are tedious, but elementary. Accordingly we only sketch the details.

There is obviously a triangulation Δ_1' of R_1 such that for each i, with $1 < i \leqq i(1)$, $R_i \cap R_1$ is triangulated by a subcomplex of Δ_1'. We can extend Δ_1' to a triangulation Δ_1 of

$$B_1 = \{(x_i) \in F_1 |\ |x_i| \leqq \alpha(\varphi((x_i))-(i-1)), \text{ for all } i \geqq 1\}$$

so that for $1 < i \leqq i(1)$, $R_i \cap B_1$ is triangulated by a subcomplex of Δ_1.

We have chosen $\{R_i\}_{i=1}^\infty$ so that for each $i > 0$, $R_{i+1} \cap (R_1 \cup \cdots \cup R_i)$

is a union of faces of R_{i+1}. Using this fact and an inductive procedure on $\{R_2, \cdots, R_{i(1)}\}$ we can extend Δ_1 to a triangulation Δ_2' of

$$B_1 \cup (R_2 \cup \cdots \cup R_{i(1)})$$

so that if $i(1) < i \leqq i(2)$, then $R_i \cap (B_1 \cup (R_2 \cup \cdots \cup R_{i(1)}))$ is triangulated by a subcomplex of Δ_2'. Put

$$B_2 = \{(x_i) \in F_1 \cup \cdots \cup F_{i(1)} | \, |x_i| \leqq \alpha(\varphi((x_i)) - (i-1)), \text{ for all } i \geqq 1\}$$

and extend Δ_2' to a triangulation Δ_2 of B_2 so that for $i(1) < i \leqq i(2)$, $R_i \cap B_2$ is triangulated by a subcomplex of Δ_2. It is clear that we can inductively continue this process to obtain our desired triangulation.

9. Proofs of Theorems 9 and 10

The following lemma is a basic separation result which will be needed in the proofs of Theorems 9 and 10.

LEMMA 9.1. *Let X be a metric* ANR, *A be a closed subset of X which is an* ANR *and for which the inclusion map $i : A \to X$ is a homotopy equivalence, and let $h : A \times (-1, 1) \to X$ be an open embedding such that $h(x, 0) = x$, for all $x \in A$. Then we can write $X \setminus A = U \cup V$, where U and V are disjoint open subsets of X satisfying $h(A \times (0, 1)) \subset U$ and $h(A \times (-1, 0)) \subset V$. Moreover, there are strong deformation retractions of $Cl(U)$ and $Cl(V)$ onto A.*

PROOF. The proof of the existence of disjoint open subsets U, V of X satisfying $X \setminus A = U \cup V$, $h(A \times (0, 1)) \subset U$, and $h(A \times (-1, 0)) \subset V$ is straightforward. We merely remark that in the case A is connected the desired separation follows immediately from the reduced Mayer-Vietoris sequence of the excisive couple $\{h(A \times (-1, 1)), X \setminus A\}$. In case A is not connected one can do a standard argument on the components of A.

The inclusion map $i : A \to X$ being a homotopy equivalence means that A is a weak deformation retract of X. Since A and X are ANR's it follows that A is a strong deformation retract of X (see [14], page 31). Let $f_t : X \to X$ be a strong deformation retraction of X onto A, where $f_0 = \text{id}$ and f_1 is a retraction of X onto A.

Let $g : X \to X$ be defined by

$$g(x) = \begin{cases} x, & \text{for } x \in Cl(U) \\ f_1(x), & \text{for } x \in Cl(V), \end{cases}$$

which is clearly continuous. Define $h_t = g \circ f_t$, for all $r \in [0,1]$. It is clear that $h_t(Cl(U)) \subset Cl(U)$, for all t. Thus $h_t|Cl(U)$ defines a strong deformation retraction of $Cl(U)$ onto A. Similarly A is a strong deformation retract of $Cl(V)$.

We will now give a proof of Theorem 9. For its proof we will use Lemma 9.1 and some of the results that have been established for Q-manifolds in this paper. We will not prove Theorem 10, since similar results for F-manifolds that have been established elsewhere will permit a proof similar to that given for Theorem 9.

PROOF OF THEOREM 9.

Note that X and Y are metric ANR's and the inclusion maps $i: f(X) \to Y$, $j: g(X) \to Y$ are obviously homotopy equivalences. Thus we can apply Lemma 9.1 to obtain disjoint pairs U_1, U_2 and V_1, V_2 of open subsets of Y such that the following properties are satisfied.

(1) $Y \setminus f(X) = U_1 \cup U_2$ and $Y \setminus g(X) = V_1 \cup V_2$,

(2) $f(X) = Cl(U_1) \cap Cl(U_2)$ and $g(X) = Cl(V_1) \cap Cl(V_2)$,

(3) $f(X)$ is collared in each of $Cl(U_1)$, $Cl(U_2)$, and $g(X)$ is collared in each of $Cl(V_1)$, $Cl(V_2)$,

(4) $f(X)$ is a strong deformation retract of each of $Cl(U_1)$, $Cl(U_2)$, and $g(X)$ is a strong deformation retract of each of $Cl(V_1)$, $Cl(V_2)$.

From (3) it easily follows that $Cl(U_1)$ and $Cl(V_1)$ are Q-manifolds. Let $r: Cl(U_1) \to f(X)$ be a retraction homotopic to id and note that the map $g \circ f^{-1} \circ r: Cl(U_1) \to Cl(V_1)$ is a homotopy equivalence. Using Theorem 6 we know that $(g \circ f^{-1} \circ r) \times \mathrm{id}: Cl(U_1) \times [0, 1) \to Cl(V_1) \times [0, 1)$ is homotopic to a homeomorphism $h_1: Cl(U_1) \times [0, 1) \to Cl(V_1) \times [0, 1)$.

Now $g \times \mathrm{id}: X \times [0, 1) \to Cl(V_1) \times [0, 1)$ and $h_1 \circ (f \times \mathrm{id}): X \times [0, 1) \to Cl(V_1) \times [0, 1)$ are homotopic embeddings. It is easy to see that $(g \times \mathrm{id})(X \times [0, 1))$ and $h_1 \circ (f \times \mathrm{id})(X \times [0, 1))$ are Z-sets in $Cl(V_1) \times [0, 1)$. Using Corollary 6.1 of [3] there is a homeomorphism

$$h_2 : Cl(V_1) \times [0, 1) \to Cl(V_1) \times [0, 1)$$

which satisfies $h_2 \circ h_1 \circ (f \times \mathrm{id}) = g \times \mathrm{id}$. Put $h' = h_2 \circ h_1$, which is a homeomorphism of $Cl(U_1) \times [0, 1)$ onto $Cl(V_1) \times [0, 1)$ which satisfies $h' \circ (f \times \mathrm{id}) = g \times \mathrm{id}$.

Similarly we can obtain a homeomorphism

$$h'' : Cl(U_2) \times [0, 1) \to Cl(V_2) \times [0, 1)$$

which satisfies $h'' \circ (f \times \mathrm{id}) = g \times \mathrm{id}$. Then define $h : Y \times [0, 1) \to Y \times [0, 1)$ by $h|Cl(U_1) \times [0, 1) = h'$ and $h|Cl(U_2) \times [0, 1) = h''$.

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