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CONSISTENCY AND INCLUSION RESULTS FOR TOEPLITZ MATRICES OF BOUNDED LINEAR OPERATORS

by

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1. Introduction

Recently, Ramanujan [11] discussed some notions of classical summability in a Banach space setting. Summability in certain linear topological spaces has also been considered in [1], [6], [9], [10], [12], [14], [15], and [17]. Ramanujan's discussion depends on duality theory of vector sequence spaces rather than on regularity conditions of the matrices involved. In this paper we show that the regularity conditions and classical techniques of [4], [5], [7], [8], [13] and [16] may be readily employed to prove consistency and inclusion results in a general F-space setting. In particular, our Theorems 4.1 and 4.4 answer a question raised in [11] and Theorem 5.2 extends a result of Cowling [5].

2. Notation and Definitions

Most of our notation corresponds to that of [11]. Given a vector space E over the scalar field K of complex numbers, a *vector sequence space* $S(E)$ over E is a set of sequences (X_n) of vectors X_n in E, which set also forms a linear space over K under the usual componentwise operations. Let $w(E)$ denote the vector sequence space of all sequences of vectors from E. When denoting scalar sequence spaces, we use the standard notation of S, instead of $S(K)$. In the sequel, let (E, p_i) and (F, q_j) be F-spaces, i.e., locally convex Hausdorff spaces which are metrisable and complete, whose topologies are generated, respectively, by the countable collections (p_i) and (q_j) of semi-norms. Let m, c, c_0 denote the scalar sequence spaces of bounded, convergent, null sequences, respectively, with the sup norm topology. Denote by $l_p (p > 0)$ the scalar sequence space of sequences $X = (X_n)$ such that $\sum |X_n| p^n < \infty$ and let l_∞ be the space of $X = (X_n)$ such that $\sum |X_n| p^n < \infty$ for all $p > 0$. The space l_p has the norm topology given by $||X|| = \sum |X_n| p^n$, while l_∞ has the F-topology given by the family $\{h_n : n = 1, 2, \cdots\}$ of semi-norms, with $h_n(X) = \max \{|\sum X_k z^k| : |z| = n\}$.

We shall be concerned with the following vector sequence spaces:

$$m(E) = \{X \in w(E) : \sup_n p_i(X_n) < \infty, i = 1, 2, \cdots\};$$

$$c(E) = \{X \in w(E) : X \text{ converges in } (E, p_i)\};$$

$$c_0(E) = \{X \in w(E) : X \text{ converges to } \theta \text{ in } (E, p_i)\};$$

$$l_p(E) = \{X \in w(E) : \sum_{n=0}^{\infty} p_i(X_n)p^n < \infty, i = 1, 2, \cdots\};$$

$$l_\infty(E) = \{X \in w(E) : \sum_{n=0}^{\infty} p_i(X_n)p^n < \infty, p > 0, i = 1, 2, \cdots\}.$$

Let $A = (A_{nk})$ denote an infinite matrix of bounded linear operators of E into F. Given X in $w(E)$, formally define $y = (y_n)$ by

$$y_n = \sum_{k=0}^{\infty} A_{nk}(X_k), \; n = 0, 1, 2, \cdots,$$

and write $y = AX$. If $\alpha(E)$, $\beta(F)$ are two vector sequence spaces, we say that $A \in \Gamma(\alpha(E), \beta(F))$ if $X \in \alpha(E)$ implies $y \in \beta(F)$, Call A *reversible* if the equation $y = AX$ has a unique solution X in $\alpha(E)$ for each y in $\beta(F)$. Denote by $D_A(E, F)$, the *domain of* A, the set $\{X = (X_n) \in w(E) : y = AX$ exists in $w(F)\}$. The *β-summability field of* A, denoted by $\beta_A(E, F)$ is the set $\{X = (X_n) \in w(E) : y = AX$ exists in $w(F)$ and $y \in \beta(F)\}$.

3. Preliminaries

In this section we shall state a number of results for the subsequent consistency and inclusion theorems.

PROPOSITION 3.1 [10]. *If E, F are locally convex spaces and E is quasi-complete (closed and bounded sets of E are complete) then any collection of continuous linear operators from E into F which is simply bounded is bounded for the topology of uniform convergence on bounded sets.*

Note that if E is an F-space it is quasi-complete and also barrelled.

PROPOSITION 3.2 [10]. *Let (T_n) be a sequence of continuous linear operators on E into F, where E, F are F-spaces. If $\lim_n T_n(X)$ exists for each X in a fundamental set of E and if for each X in E, $(T_n(X))$ is bounded in F then $T(X) = \lim_n T_n(X)$ exists for all X in E, and T is a continuous linear operator of E into F.*

The next two propositions are easily proved using the techniques of [10], [14] and [15].

PROPOSITION 3.3 *The matrix* $A = (A_{nk}) \in \Gamma(c_0(E), c(F))$ *if and only if*

(i) *for each bounded set* M_α *in* E *and for each fixed* j,

$$q_j(\sum_{k=0}^{m} A_{nk} X_k) \leqq K_{\alpha, j} \text{ for } n, m = 0, 1, \cdots$$

and $X_k \in M_\alpha$, $k = 0, 1, \cdots$; and

(ii) *for each* $X \in E$ *and each fixed* $k = 0, 1, \cdots$, $\lim_n A_{nk}X$ *exists.*

Also, $\lim_A X \equiv \lim_n \sum_k A_{nk}X_k = \theta$ *whenever* $\lim X = \theta$ *if and only if* (i) *holds and* (ii) *is specialized to*

(ii)′ $\lim_n A_{nk}X = \theta$ *for each* $X \in E$ *and each* $k = 0, 1, \cdots$

PROPOSITION 3.4. *The matrix* $A = (A_{nk}) \in \Gamma(l_1(E), c(F))$ *if and only if*

(i) *for each bounded set* M_α *in* E *and for each fixed* j,

$$q_j(A_{nk}X_k) \leqq K_{\alpha, j} \text{ for } n, k = 0, 1, \cdots$$

and $X_k \in M_\alpha$, $k = 0, 1, \cdots$; *and*

(ii) *for each* $X \in E$ *and each fixed* $k = 0, 1, \cdots$, $\lim_n A_{nk}X$ *exists. If* (i) *and* (ii) *hold then* $\lim_A X = \sum_k \lim_n A_{nk}X_k$ *for* $X \in l_1(E)$.

PROPOSITION 3.5. *Let* p *and* q *be positive numbers.*

(i) $A = (A_{nk}) \in \Gamma(l_p(E), l_q(F))$ *if and only if for each bounded set* M_α *in* E *and for each fixed* j,

$$\sum_{n=0}^{m} q^n p^{-k} q_j(A_{nk} X_k) \leqq K_{\alpha, j, p, q} \text{ for } m, k = 0, 1, \cdots$$

and $X_k \in M_\alpha$, $k = 0, 1, \cdots$.

(ii) *If, for each positive integer* r *there exists an* $s = s(r) \geqq r$ *such that for each bounded set* M_α *in* E *and for each fixed* $j = 1, 2, \cdots$,

$$\sum_{n=0}^{m} r^n s^{-k} q_j(A_{nk} X_k) \leqq K_{\alpha, j, r, s} \text{ for } m, k = 0, 1, \cdots$$

and $X_k \in M_\alpha$, $k = 0, 1, \cdots$, *then* $A = (A_{nk}) \in \Gamma(l_\infty(E), l_\infty(F))$. *The converse holds if the topology of* F *is given by the finite collection* (q_i), $i = 1, \cdots, l$.

(iii) *If there exists an* $s = s(q) > 0$ *such that for each bounded set* M_α *in* E *and for each fixed* $j = 1, 2, \cdots$,

$$\sum_{n=0}^{m} q^n s^{-k} q_j(A_{nk} X_k) \leqq K_{\alpha, j, s, q} \text{ for } m, k = 0, 1, \cdots$$

and $X_k \in M_\alpha$, $k = 0, 1, \cdots$, *then* $A = (A_{nk}) \in \Gamma(l_\infty(E), l_q(F))$. *The converse holds if the topology of* F *is given by the finite collection* (q_i), $i = 1, \cdots, l$.

(iv) $A = (A_{nk}) \in \Gamma(l_p(E), l_\infty(F))$ *if and only if for all* $r > 0$ *and for each bounded set* M_α *in* E *and for each fixed* $j = 1, 2, \cdots$,

$$\sum_{n=0}^{m} r^n p^{-k} q_j(A_{nk} X_k) \leqq K_{\alpha, j, p, r} \text{ for } m, k = 0, 1, \cdots$$

and $X_k \in M_\alpha, k = 0, 1, \cdots$.

PROOF. Part (i) follows from [15] and the fact that the maps $(X_k) \leftrightarrow (X_k p^k)$ and $(y_n) \leftrightarrow (y_n p^n)$ are one-to-one correspondences between $l_p(E)$ and $l_1(E)$ and between $l_p(F)$ and $l_1(F)$, respectively.

We prove (ii) as follows. Suppose that for each positive integer r there exists an $s = s(r) \geqq r$ such that $A \in \Gamma(l_s(E), l_r(F))$, i.e. the condition of (ii) holds. We note that $l_v(E) \subseteq l_w(E)$ if $0 < w < v$ and $l_\infty(E) = \bigcap_{v=1}^{\infty} l_v(E)$. Let $X = (X_n) \in l_\infty(E)$ and let a positive integer r be given. Therefore, there exists an $s = s(r) \geqq r$ such that $A \in \Gamma(l_s(E), l_r(F))$. Since $X \in l_s(E)$, $A(X) \in l_r(F)$. This shows that $A \in \Gamma(l_\infty(E), l_\infty(F))$. Conversely, assume the topology of F is given by the finite collection $(q_i)_{i=1}^{l}$ and let $A \in \Gamma(l_\infty(E), l_\infty(F))$. For each $X \in l_p(E)$, let $\sigma_p^i(X) = \sum_{n=0}^{\infty} p_i(X_n)p^n$, $p > 0$, $i = 1, 2, \cdots$. Then σ_p^i is a seminorm on $l_p(E)$. Let the topology of $l_\infty(E)$be given by the sup of $\Phi(v\Phi)$ where $\Phi = \{\sigma_p^i : p = 1, 2, \cdots; i = 1, 2, \cdots\}$. Then $(l_\infty(E), v\Phi)$ is a locally convex complete space. Let q be a given positive integer .Then $A \in \Gamma(l_\infty(E), l_q(F))$ and for each $y = (y_n) \in l_q(F)$ we let $u_q(y) = \sum_{n=0}^{\infty} t(y_n)q^n$, where $t = \sum_{i=1}^{l} q_i$. Then $(l_q(F), u_q)$ is a locally convex complete space and A is continuous as a map from $(l_\infty(E), v\Phi)$ to $(l_q(F), u_q)$. Thus there exists a number M and a finite collection, say $\sigma_1^1, \cdots, \sigma_p^i$, of seminorms such that for each $X \in l_\infty(E)$

$$u_q(AX) \leqq M(\sigma_1^1(X) + \cdots + \sigma_p^i(X)).$$

Since, for each j, $\sigma_r^j(X) \leqq \sigma_s^j(X)$ if $r \leqq s$, we may choose p such that $p \geqq q$ and

$$u_q(AX) \leqq M(\sigma_p^1(X) + \cdots + \sigma_p^i(X)).$$

Let $\alpha(X) = \sigma_p^1(X) + \cdots + \sigma_p^i(X)$ and $(l_p(E), \alpha)$ is a locally convex complete space. Since $l_1(E) \supseteq l_2(E) \supseteq \cdots \supseteq l_p(E)$ it follows that A is continuous as a map from $l_\infty(E)$ with the α-topology of $l_p(E)$ to $(l_q(F), u_q)$. For vectors $t \in E$ define $\delta^n t = (\theta, \theta, \cdots, t, \theta, \cdots)$ (θ is the zero vector). Then $\delta^n t \in l_\infty(E)$, $n = 0, 1, \cdots$, and the series $\sum_{n=0}^{\infty} \delta^n X_n$ converges to $X = (X_n)$ in the α-topology for each $X \in l_p(E)$. Therefore $l_\infty(E)$ is dense in $(l_p(E), \alpha)$ and we may extend A to a continuous linear operator T from $(l_p(E), \alpha)$ to $(l_q(F), u_q)$. But for $X = (X_k) \in l_p(E)$,

$$T(x) = \sum_{k=0}^{\infty} T(\delta^k X_k)$$

and so for each $n = 0, 1, \cdots$,

$$(T(X))_n = \sum_{k=0}^{\infty} A_{nk} X_k = (A(X))_n,$$

which shows that T is still given by the matrix A. Therefore, to each $q = 1, 2, \cdots$, there exists a $p = p(q) \geqq q$ such that $A \in \Gamma(l_p(E), l_q(F))$.

The proof of (iii) is similar to that of (ii) and (iv) follows from the fact that $l_\infty(F) = \bigcap_{r>0} l_r(F)$.

Let $w(E)$, $w(F)$ be given the weakest linear topologies such that the coordinates are continuous, i.e., $w(E)$ has the F-topology given by the family of semi-norms $\pi_{ki}(X) = p_i(X_k)$, $k = 0, 1, \cdots, i = 1, 2, \cdots$ and $w(F)$ has the F-topology given by the collection $\sigma_{kj}(y) = q_j(y_k)$ $k = 0, 1, \cdots, j = 1, 2, \cdots$. If E is an F-space and $H = w(E)$ has its topology given as above, then the corresponding FH-spaces (see [13]) will be called FK-spaces over E.

Proposition 3.6. *Let λ be any one of c, c_0, l_p, l_∞. Then $D_A(E, F)$ and $\lambda_A(E, F)$ are FK-spaces over E, with semi-norms $\{\pi_{ki}, \beta_{kj} : k = 0, 1, \cdots, i, j, = 1, 2, \cdots\}$ and $\{\pi_{ki}, \beta_{kj}, \uparrow_n^A : k, n = 0, 1, \cdots, i, j = 1,,2, \cdots\}$, respectively, where, for $X = (X_n) \in D_A(E, F)$*

$$\beta_{kj}(X) = \sup \left[q_j\{\sum_{u=0}^{m} A_{ku} X_u\} : m = 0, 1, \cdots\right],$$
$$k = 0, 1, \cdots, j = 1, 2, \cdots,$$

and, for $X \in \lambda_A(E, F)$, $\uparrow_n^A(X) = \uparrow_n(AX)$ where $\uparrow_n$ are the semi-norms generating the F-topology of $\lambda(F)$.

If A is row-finite β_{kj} may be omitted; if A is reversible, π_{ki}, β_{kj} may be omitted.

Proof. The result follows from Proposition 3.2 and standard techniques found in [13, p. 227].

It is easy to show that various choices of λ lead to the following generating families for the F-topology of $\lambda(F)$:

$$\lambda = c \text{ or } c_0 \cdots \sup\{q_j(y_n) : n = 0, 1, \cdots\}, \quad j = 1, 2, \cdots;$$

$$\lambda = l_p \qquad \cdots \sum_{k=0}^{\infty} q_j(y_k)p^k, \qquad j = 1, 2, \cdots;$$

$$\lambda = l_\infty \qquad \cdots \sum_{k=0}^{\infty} q_j(y_k)p^k, \qquad j, p = 1, 2, \cdots.$$

4. Consistency Theorems

Foı convenience we shall call the pair (α, β) *admissible* if α is one of l_p, l_∞ where $p \geqq 1$, and β is one of l_q, l_∞ where $q \geqq 1$. If α is any one of c, c_0, l_p, l_∞ and β is any one of c, c_0, l_q, l_∞ and $A \in \Gamma(\alpha(E), \beta(F))$, we say that A is *perfect* if $\alpha(E)$ is dense in $\beta_A(E, F)$, where $\beta_A(E, F)$ has the F-topology of Proposition 3.6. Let (α, β) be admissible and A, B be in $\Gamma(\alpha(E), \beta(F))$. For $X \in \beta_A(E, F)$ define $\sigma \circ A(X) = \sum\sum_{nk} A_{nk} X_k$. We write $B \sim A$ if $\sigma \circ A(X) = \sigma \circ B(X)$ for $X \in \alpha(E)$. If $\sigma \circ A(X) = \sigma \circ B(X)$ for $X \in \beta_B(E, F) \cap \beta_A(E, F)$, we say that *$B$ is σ-consistent with A.*

THEOREM 4.1. *Let (α, β) be admissible and $A \in \Gamma(\alpha(E), \beta(F))$. If $\alpha = \beta = l_\infty$ or $\alpha = l_\infty$ we assume that the topology of F is given by the q_i, $i = 1 \cdots l$. Then $\sigma \circ A \in L(\beta_A(E, F), F)$ (continuous linear operators on $\beta_A(E, F)$ to F) and A is σ-consistent with every $B \in \Gamma(\alpha(E), \beta(F))$ such that $B \sim A$ and $\beta_A(E, F) \subseteq \beta_B(E, F)$ if and only if A is perfect.*

We shall give the proof for the $\alpha = \beta = l_\infty$ case. The other cases will be seen to be similar. First we require two lemmas on the representation of continuous linear functionals.

LEMMA 4.2. *If $\rho \in [l_{\infty_A}(E, F)]'$ (topological dual of $l_{\infty_A}(E, F)$) then there exist $g \in [D_A(E, F)]'$, $h \in [l_\infty(F)]'$ such that, for $X \in l_{\infty_A}(E, F)$,*

$$\rho(X) = \sum_{k=0}^{\infty} g \circ \mu_k^{-1}(X_k) + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} h \circ \pi_n^{-1} \circ A_{nk}(X_k).$$

The inverse projections $\mu_k^{-1} : E \to D_A(E, F)$ and $\pi_n^{-1} : F \to l_\infty(F)$ are defined by $\mu_k^{-1}(t) = \delta^k t$, $\pi_n^{-1}(t) = \delta^n t$, where $\delta^n t = (\theta, \theta, \theta, \cdots, t, \theta, \cdots)$. We have $g \circ \mu_k^{-1} \in E'$ and *$h \circ \pi_n^{-1} \circ A_{nk} \in E'$, n, $k = 0, 1, \cdots$.*

PROOF. If $\rho \in [l_{\infty_A}(E, F)]' = (A^{-1}[l_\infty(F)])'$, a version of Theorem 5 of [13, p. 230] for vector sequence spaces implies there exist $g \in [D_A(E, F)]'$ and $h \in [l_\infty(F)]'$ such that for all $X \in l_{\infty_A}(E, F)$, $\rho(X) = g(X) + h \circ A(X)$. If $X = (X_k) \in D_A(E, F)$ then $\delta^k X_k = (\theta, \theta, \cdots, X_k, \theta, \cdots)$ and clearly $\sum_{k=0}^{m} \delta^k X_k \to X$ in the $D_A(E, F)$-topology. Therefore $g(X) = \sum_{k=0}^{\infty} g(\delta^k X_k)$, the series converging in K. Similarly, if $y = (y_n) \in l_\infty(F)$ then $\sum_{k=0}^{m} \delta^k y_k \to y$ in the topology of $l_\infty(F)$ and we obtain $h(y) = \sum_{k=0}^{\infty} h(\delta^k y_k)$ where the series of scalars converges. If μ_k^{-1}, π_n^{-1} are defined as in the statement of the lemma, then Proposition 3.6 shows that $\mu_k^{-1} \in L(E, D_A(E, F))$ and $\pi_n^{-1} \in L(F, l_\infty(F))$. Thus, for $X = (X_k) \in l_{\infty_A}(E, F)$ and $y = (y_n) = Ax$,

$$\begin{aligned}\rho(X) = g(X) + h \circ A(X) &= \sum_{k=0}^{\infty} g(\delta^k X_k) + \sum_{n=0}^{\infty} h(\delta^n y_n) \\ &= \sum_{k=0}^{\infty} g \circ \mu_k^{-1}(X_k) + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} h \circ \pi_n^{-1} \circ A_{nk}(X_k).\end{aligned}$$

LEMMA 4.3. *If $t \in (F, q_i, i = 1, \cdots, l)$ and $\rho \in [l_{\infty_A}(E, F)]'$ then there exists a method $B = B(t)$ such that $B \in \Gamma(l_\infty(E), l_\infty(F))$, $l_{\infty_A}(E, F) \subseteq l_{\infty_B}(E, F)$ and $\sigma \circ B(X) = \rho(X)t$* for all *$X \in l_{\infty_A}(E, F)$.*

PROOF. In the notation of Lemma 4.2, define $B = (B_{nk})$ by the following equations. For $X \in E$, let

$$B_{0k}(X) = [g \circ \mu_k^{-1}(X) + h \circ \pi_0^{-1} \circ A_{0k}(X)]t, \quad k = 0, 1, \cdots,$$
$$B_{nk}(X) = [h \circ \pi_n^{-1} \circ A_{nk}(X)]t, \quad n = 1, 2, \cdots, k = 0, 1, \cdots.$$

Clearly, $B_{nk} \in L(E, F)$ and $\sigma \circ B(X) = \rho(X)t$ for $X \in l_{\infty_A}(E, F)$. We apply Proposition 3.5 (ii) to show that $B \in \Gamma(l_\infty(E), l_\infty(F))$.

Fix j and a bounded set M_α in E. Let $X = (X_k)$ where $X_k \in M_\alpha$ $(k = 0, 1, \cdots)$ and let q be a positive integer. We must show the existence of $p_0 = p_0(q) \geqq q$ and a number $K = K_{\alpha, j, p_0, g}$ such that

$$\sum_{n=0}^{m} q^n p_0^{-k} q_j(B_{nk} X_k) \leqq K_{\alpha, j, p_0, q} \text{ for } m, k = 0, 1, \cdots. \tag{4.1}$$

Let k be a given positive integer. Then, for any positive integer p,

$$\sum_{n=0}^{m} q^n p^{-k} q_j(B_{nk} X_k) \leqq q_j \left(\frac{g \circ \mu_k^{-1}(X_k)t}{p^k} \right) + \sum_{n=0}^{m} q^n p^{-k} q_j(h \circ \pi_n^{-1} \circ A_{nk}(X_k)t)$$

We have

$$q_j \left(\frac{g(\delta^k X_k)t}{p^k} \right) = \left| \frac{g(\delta^k X_k)}{p^k} \right| q_j(t)$$

and we shall show the existence of a positive integer p_1 and a number $R = R(\alpha, p_1)$ such that

$$\left| \frac{g(\delta^k X_k)}{p_1^k} \right| \leqq R, \; k = 0, 1, \ldots$$

Let z, n, i, μ be given positive integers and suppose $p_i(X_k) \leqq l$, $k = 0, 1, \cdots$ It follows that

$$\pi_{n,i}(\delta^k(X_k/z^k)) = p_i[(\delta^k(X_k/z^k))_n] \leqq \frac{l}{z^k}, \; k = 0, 1, \cdots,$$

and

$$\beta_{n,\mu}(\delta^k(X_k/z^k)) = \sup \left[q_\mu\{ \sum_{u=0}^{m} A_{nu}[\delta^k(X_k/z^k)]_u \} : m = 0, 1, \ldots \right]$$
$$= \frac{q_\mu(A_{nk} X_k)}{z^k}, \; k = 0, 1, \cdots.$$

Since $A \in \Gamma(l_\infty(E), l_\infty(F))$, for each $\mu = 1, 2, \cdots$ there exists a positive integer $z = z(\mu)$ and a number $R' = R'(\mu)$ such that

$$\frac{q_\mu(A_{nk} X_k)}{z^k} \leqq R', \; k = 0, 1, \cdots.$$

Since $g \in [D_A(E, F)]'$, there exists a number $M \geqq 0$ and positive integers a, b, c, d such that, for any positive integer p,

$$\left|\frac{g(\delta^k X_k)}{p^k}\right| \leqq M\,[\sum_{i=1}^{a}\sum_{n=1}^{b}\pi_{n,i}(\delta^k(X_k/p^k))+\sum_{m=1}^{c}\sum_{\mu=1}^{d}\beta_{m,\mu}(\delta^k(X_k/p^k))]$$

for $k = 0, 1, \cdots$. It follows that there exists a positive integer p_1 and number $R = R(\alpha, p_1)$ such that

$$\left|\frac{g(\delta^k X_k)}{p_1^k}\right| \leqq R,\ k = 0, 1, \cdots.$$

Next, since $A \in \Gamma(l_\infty(E), l_\infty(F))$, Proposition 3.5 (ii) yields for each positive integer w, an $s = s(w) \geqq w$ and a number $T = T(\alpha, \mu, s, w)$ such that

$$\sum_{n=0}^{m} w^n s^{-k} q_\mu(A_{nk}X_k) \leqq T,\ m, k = 0, 1, \cdots.$$

We have $\pi_n^{-1} \circ A_{nk}(X_k) = \delta^n A_{nk}(X_k) = (\theta, \theta, \cdots, A_{nk}, \theta, \cdots)$ with $A_{nk}X_k$ appearing in the n-th position. Let $y^k = (y_u^k)$ with $y_u^k = \theta(u \neq n)$, $y_n^k = A_{nk}X_k$, $k = 0, 1, \cdots$. The remark following Proposition 3.6 says that the toplogy of $l_\infty(F)$ is given by the collection $\{Q_{\mu,r} : \mu, r = 1, 2, \cdots\}$ with

$$Q_{\mu,r}(y^k) = \sum_{u=0}^{\infty} q_\mu(y_u^k) r^u = q_\mu(A_{nk}X_k) r^n.$$

Since $h \in [l_\infty(F)]'$, there exists a number $N \geqq 0$ and positive integers e, f such that

$$\begin{aligned} |h(y^k)| &\leqq N\sum_{\mu=1}^{e}\sum_{r=1}^{f} Q_{\mu,r}(y^k) \\ &= N\sum_{\mu=1}^{e}\sum_{r=1}^{f} q_\mu(A_{nk}X_k) r^n,\ k = 0, 1, \cdots. \end{aligned}$$

It follows that, for positive integers p,

$$\sum_{n=0}^{m} q^n p^{-k} q_j(h \circ \pi_n^{-1} \circ A_{nk}(X_k)t) \leqq N\sum_{\mu=1}^{e}\sum_{r=1}^{f}\sum_{n=0}^{m}(rq)^n p^{-k} q_\mu(A_{nk}X_k) q_j(t),$$
$$k = 0, 1, \cdots.$$

Thus there exists an $s_1 = s_1(q) \geqq q$ and a number $T_1 = T_1(\alpha, j, s_1, q)$ such that

$$\sum_{n=0}^{m} q^n s_1^{-k} q_j(h \circ \pi_n^{-1} \circ A_{nk}(X_k)t) \leqq T_1,\ k, m = 0, 1, \cdots.$$

Finally, combining the above we obtain a $p_0 = p_0(q) \geqq q$ and a number

$K = K(\alpha, j, p_0, q)$ such that (4.1) holds. The proof that $l_{\infty_A}(E, F) \subseteq l_{\infty_B}(E, F)$ is straightforward and Lemma 4.3 is proved.

PROOF OF THEOREM 4.1. Repeated applications of Proposition 3.2 show that $\sigma \circ A \in L(l_{\infty_A}(E, F), F)$ and, if $l_{\infty_A}(E, F) \subseteq l_{\infty_B}(E, F)$, $\sigma \circ B \in L(l_{\infty_A}(E, F), F)$ also. Utilizing Lemma 4.3 above, the proof now goes through just like that of [4, Theorem 3].

Let $A, B \in \Gamma(c_0(E), c_0(F))$. If $\lim_B X = \theta$ for $X \in c_{0_A}(E, F) \cap c_B(E, F)$ we say that *A is consistent with B*. Thus $c_{0_A}(E, F) \subseteq c_B(E, F)$ and A consistent with B imply $c_{0_A}(E, F) \subseteq c_{0_B}(E, F)$.

THEOREM 4.4 *Let* $A \in \Gamma(c_0(E), c_0(F))$.

(i) *If* X *is in the* $c_{0_A}(E, F)$*-closure of* $c_0(E)$ *then* $\lim_B X = \theta$ *for every* $B \in \Gamma(c_0(E), c_0(F))$ such that $c_{0_A}(E, F) \subseteq c_B(E, F)$.

(ii) *If* $X \in c_{0_A}(E, F)$ *is not in the* $c_{0_A}(E, F)$*-closure of* $c_0(E)$ *then, for any scalar* μ *and non-zero* $t \in F$, *there exists a* $B \in \Gamma(c_0(E), c_0(F))$ *such that* $c_{0_A}(E, F) \subseteq c_B(E, F)$ and $\lim_B X = \mu t$.

(iii) *Method* A *is consistent with every* $B \in \Gamma(c_0(E), c_0(F))$ *such that* $c_{0_A}(E, F) \subseteq c_B(E, F)$ *if and only if* A *is perfect.*

PROOF. Clearly, (iii) follows from (i) and (ii). Part (i) is obvious since $c_{0_A}(E, F) \subseteq c_B(E, F)$ implies (by Proposition 3.2) that $\lim_B$ is continuous on $c_{0_A}(E, F)$ in the topology of that space. Under the hypotheses of (ii) we choose $f \in [c_{0_A}(E, F)]'$ such that $f(X) = \mu$ and f vanishes on $c_0(E)$. Then we construct a $B \in \Gamma(c_0(E), c_0(F))$ such that $c_{0_A}(E, F) \subseteq c_B(E, F)$ and $\lim_B y = f(y)t$ for all $y \in c_{0A}(E, F)$. First, since $\sum_{k=0}^{m} \delta^k z_k \to z$ in the topology of $c_0(F)$ if $z \in c_0(F)$ we have (in a manner similar to the proof of Lemma 4.2)

$$f(y)t = \sum_{k=0}^{\infty} g \circ \mu_k^{-1}(y_k)t + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} h \circ \pi_n^{-1} \circ A_{nk}(y_k)t$$

for $y = (y_k) \in c_{0_A}(E, F)$, where μ_k^{-1}, π_n^{-1} are as in Lemma 4.2 with $l_\infty(F)$ replaced by $c_0(F)$, $g \in [D_A(E, F)]'$, and $h \in [c_0(F)]'$. Define $B = (B_{mk})$by the equations

$$B_{mk}(s) = [g \circ \mu_k^{-1}(s) + \sum_{n=0}^{m} h \circ \pi_n^{-1} \circ A_{nk}(s)]t, \; s \in E, \; m, k = 0, 1, \cdots.$$

We have

$$\sum_{n=0}^{m} \sum_{k=0}^{\infty} h \circ \pi_n^{-1} \circ A_{nk}(y_k) = \sum_{k=0}^{\infty} \sum_{n=0}^{m} h \circ \pi_n^{-1} \circ A_{nk}(y_k)$$

for $y = (y_k) \in c_{0_A}(E, F)$ and $m = 0, 1, 2, \cdots$, since, for each n, $\sum_{k=0}^{\infty} h \circ \pi^{-1} \circ A_{nk}(y_k)$ is a convergent series of scalars. Therefore,

$$\begin{aligned}\lim_B y &= \lim_m \sum_{k=0}^{\infty} B_{mk}(y_k)\\ &= \sum_{k=0}^{\infty} g \circ \mu_k^{-1}(y_k)t + \lim_m \sum_{k=0}^{\infty} \sum_{n=0}^{m} h \circ \pi_n^{-1} \circ A_{nk}(y_k)t\\ &= \sum_{k=0}^{\infty} g \circ \mu_k^{-1}(y_k)t + \lim_m \sum_{n=0}^{m} \sum_{k=0}^{\infty} h \circ \pi_n^{-1} \circ A_{nk}(y_k)t = f(y)t.\end{aligned}$$

It follows that $B \in \Gamma(c_0(E), c_0(F))$ and $c_{0_A}(E, F) \subseteq c_B(E, F)$. This completes the proof of Theorem 4.4.

A sequence (f_n) of continuous linear functionals on an F-space F is said to satisfy property $P(A, \beta)$ if it is bounded for the topology of uniform convergence on bounded sets and $\sum\sum_{kn} f_n(A_{nk}X_k)$ converges in K for every $X = (X_k) \in \beta_A(E, F)$ where $A = (A_{nk}) \in \Gamma(\alpha(E), \beta(F))$ and (α, β) is admissible.

THEOREM 4.5. *Let (α, β) be admissible and $A \in \Gamma(\alpha(E), \beta(F))$. If $\alpha = \beta = l_\infty$ or $\alpha = l_\infty$ let the topology of F be given by q_i, $i = 1, \cdots, l$. Then A is perfect if and only if for each (f_n) satisfying $P(A, \beta)$ we have*

$$\sum_n \sum_k f_n(A_{nk} X_k) = \sum_k \sum_n f_n(A_{nk} X_k)$$

for all $X = (X_k) \in \beta_A(E, F)$.

PROOF. We give the proof for the $\alpha = \beta = l_\infty$ case. It follows from Lemma 4.2 and Proposition 3.1 that $f \in [l_{\infty_A}(E, F)]'$ implies there exist uniformly bounded sequences (g_n), (h_n) with $g_n \in E'$, $h_n \in F'$ such that

$$f(X) = \sum_n g_n(X_n) + \sum_n h_n(\sum_k A_{nk} X_k)$$

for all $X = (X_k) \in l_{\infty_A}(E, F)$. The proof now follows the lines of [4, Theorem 1].

In [2] Brown gives an example of a perfect non-reversible $l_1 - l_1$ matrix $A = (a_{nk})$ of scalars for which

$$\sum_k \sum_n f_n(a_{nk} X_k)$$

diverges for a certain $X \in l_{1_A}$ and uniformly bounded sequence (f_n) of continuous linear functionals. In his example $f_n(X) = X$ for all $X \in K$, $n = 0, 1, \cdots$.

Let (α, β) be admissible and $A = (A_{nk}) \in \Gamma(\alpha(E), \beta(F))$. We say that A is *type M^** if, for every sequence (f_n) of continuous linear functionals on F which is bounded for the topology of uniform convergence on bounded sets, the conditions

$$\sum_n f_n(A_{nk}(X)) = 0, \ k = 0, 1, \cdots, X \in E,$$

imply $f_n \equiv 0$, $n = 0, 1, \cdots$.

THEOREM 4.6. *Let (α, β) be admissible and $A \in \Gamma(\alpha(E), \beta(F))$ be reversible. If $\alpha = \beta = l_\infty$ or $\alpha = l_\infty$ let $(q_i) = (q_i, i = 1, \cdots l)$. Then A is perfect if and only if A is type M^*.*

PROOF. Using the representation of $[\beta_A(E, F)]'$ given in the proof of Theorem 4.5, the proof becomes a repetition of that of [4, Theorem 2].

We may now state the following corollary which extends a result of Macphail [7]. The case $\alpha = \beta = l_\infty$ is not considered in [7].

COROLLARY 4.7. *Let (α, β) be admissible and $A \in \Gamma(\alpha(E), \beta(F))$ be reversible. If $\alpha = \beta = l_\infty$ or $\alpha = l_\infty$ let $(q_i) = (q_i, i = 1 \cdots l)$. Then A is σ-consistent with every $B \in \Gamma(\alpha(E), \beta(F))$ such that $B \sim A$ and $\beta_A(E, F) \subseteq \beta_B(E, F)$ if and only if A is type M^*.*

Let (α, β) be admissible and $A = (A_{nk}) \in \Gamma(\alpha(E), \beta(F))$. An $X = (X_k) \in \beta_A(E, F)$ is said to be *perfect* if

$$\sum_n f_n(\sum_k A_{nk} X_k) = \sum_k \sum_n f_n(A_{nk} X_k)$$

for all sequences (f_n) of continuous linear functionals on F which satisfy $P(A, \beta)$. An $X = (X_k) \in \beta_A(E, F)$ is said to be *type N* if

$$\sum_n f_n(\sum_k A_{nk} X_k) = 0$$

for all (f_n) satisfying $P(A, \beta)$.

PROPOSITION 4.8. *Let (α, β) be admissible and $A \in \Gamma(\alpha(E), \beta(F))$. If $\alpha = \beta = l_\infty$ or $\alpha = l_\infty$* let $(q_i) = (q_i, i = 1, \cdots, l)$.

(i) *$X \in \beta_A(E, F)$ is perfect if and only if $X \in \beta_A(E.F)$-closure of $\alpha(E)$;*

(ii) *if $X \in \beta_A(E, F)$ and A is reversible then X is perfect if X is type N.*

PROOF. The proof of (i) follows the lines of [2, Lemma 2].

(ii) Let $X \in \beta_A(E, F)$ be type N and $f \in [\beta_A(E, F)]'$, $f(y) = 0$ for $y \in \alpha(E)$. It follows from the representation of $[\beta_A(E, F)]'$ that

$$f(y) = \sum_n h_n(\sum_k A_{nk} y_k) - \sum_k \sum_n h_n(A_{nk} y_k), \ y \in \beta_A(E, F),$$

where (h_n) satisfies $P(A, \beta)$. Since A is reversible, Proposition 3.6 says that the topology of $\beta_A(E, F)$ is given by $(\uparrow_n^A)$ where $\uparrow_n^A(y) = \uparrow_n(Ay)$ and $(\uparrow_n)$ generates the topology of $\beta(F)$. It follows easily that

$$f(y) = \sum_n h_n(\sum_k A_{nk} y_k), \ y \in \beta_A(E, F)$$

where (h_n) satisfies $P(A, \beta)$. Therefore $f(X) = 0$ and X is perfect.

The following example, suggested by the referee, shows that the converse of 4.8 (ii) is false. Take $\alpha = \beta = l_1$, $E = F = K$, and $A =$ identity matrix. Then $A \in \Gamma(l_1, l_1)$, each $X \in l_{1_A}$ is perfect, A is reversible, yet every $X \in l_{1_A}$ is not type N.

Let (α, β) be admissible and $A \in \Gamma(\alpha(E), \beta(F))$ be reversible. If $\alpha = \beta = l_\infty$ or $\alpha = l_\infty$ let $(q_i) = (q_i, i = 1, \cdots l)$. Consider the following:

(i) A is type M^*;

(ii) A is perfect;

(iii) all $X \in \beta_A(E, F)$ are perfect;

(iv) all $X \in \beta_A(E, F)$ are type N.

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv).

The next result gives the usual characterization of type M^* for our setting.

THEOREM 4.9. *Let* (α, β) *be admissible and* $A \in \Gamma(\alpha(E), \beta(F))$. *If* $\alpha = \beta = l_\infty$ *or* $\alpha = l_\infty$ *let* $(q_i) = (q_i, i = 1, \cdots l)$. *The following are equivalent*:

(i) *A is type M^**;

(ii) *The collection of sequences* $(A_{nk}(X))$, $k = 0, 1, \cdots, X \in E$, *is fundamental in* $\beta(F)$;

(iii) $A[\alpha(E)]$ *is fundamental in* $\beta(F)$.

PROOF. The proof employs known procedures (see [11]) and the following facts.

Let (f_n) be a uniformly bounded sequence of continuous linear functionals on F and define $f(y) = \sum_n f_n(y_n)$, $y \in \beta(F)$. We claim that $f \in (\beta(F))'$. (The proof is given for $\beta = l_1$; the other cases are similar). Let $y = (y_n) \in l_1(F)$ and $r_n(X) = |f_n(X)|$ for $X \in F$. Then r_n is a semi-norm on F and, since (f_n) is bounded for the topology of uniform convergence on bounded sets, there exists a number $M = M(j) \geqq 0$, independent of n, such that $r_n(X) \leqq Mq_j(X)$, $j = 1, 2, \cdots, n = 0, 1, \cdots$. (See, e.g., [14] or [15]). Therefore

$$|f(y)| \leqq M(j) \sum_{n=0}^{\infty} q_j(y_n)$$

and $f \in (l_1(F))'$. Now let $z \in A[\alpha(E)]$ and write $z = A(\mu)$, $\mu \in \alpha(E)$. We claim that

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f_n(A_{nk}\mu_k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} f_n(A_{nk}\mu_k).$$

We verify this as follows for the $\alpha = \beta = l_1$ case. If $A = (A_{nk}) \in \Gamma(l_1(E), l_1(F))$ and (f_n) is bounded for the topology of uniform convergence on bounded sets, $f_n \in F'$, we claim that $B = (f_n \circ A_{nk}) \in \Gamma(l_1(E), l_1)$. Clearly $f_n \circ A_{nk} \in E'$ for all n, k and we use Proposition 3.5 (i) to show that $B \in \Gamma(l_1(E), l_1)$. Let $X_u \in M_q$, a bounded set in E, for $u = 0, 1, \cdots$. Since (f_n) is bounded for the topology of uniform convergence on bounded sets, just as before we have, for $j = 1, 2, \cdots$, a number $M(j) \geqq 0$ such that

$$|f_n(A_{nu} X_u)| \leqq M(j) q_j(A_{nu} X_u), \; n, u = 0, 1, \cdots.$$

Proposition 3.5 (i) implies that $B \in \Gamma(l_1(E), l_1)$. The required interchange of summation now follows from observations in [15].

5. Inclusion Theorems

Suppose α, β are any of c, c_0, l_p, l_∞ and $A = (A_{nk}) \in \Gamma(\alpha(E), \beta(F))$ is reversible. Fix $X \in F$ and let the sequence $\xi^p = \xi^p(X)$ correspond to $\delta^p X$ under A, i.e.,

$$\delta_n^p = (\delta^p X)_n = \sum_{k=0}^{\infty} A_{nk} \xi_k^p, \; n, p = 0, 1, \cdots.$$

Define maps $f_{p,n}$ from F to E by the equations $f_{p,n}(X) = \xi_n^p$, $n, p = 0, 1, \cdots$. Clearly, $f_{p,n} \in L(F, E)$ for all n, p.

THEOREM 5.1. *Let $A \in \Gamma(c_0(E), c_0(F))$ be reversible and let $f_{p,n}$, $n, p = 0, 1, \cdots$ be the maps defined above. Let $B = (B_{nk})$ be a row finite matrix of continuous linear operators on E to F. Then $c_{0_A}(E, F) \subseteq c_B(E, F)$ if and only if*

(i) $D = (D_{pn}) \in \Gamma(c_0(F), c(F))$, *where*

$$D_{pn} = \sum_{k=0}^{\infty} B_{pk} \circ f_{kn}.$$

Also, $c_{0_A}(E, F) \subseteq c_{0_B}(E, F)$ if and only if $D \in \Gamma(c_0(F), c_0(F))$.

PROOF. If $X = (X_k) \in c_{0_A}(E, F)$ and $y = AX$ we define $\rho_k(y) = X_k$. Using the fact that A is reversible it is easy to show that $\rho_k \in L(c_0(F), E)$, $k = 0, 1, \cdots$. It follows that

$$\rho_k(y) = \sum_{n=0}^{\infty} \rho_{nk}(y_n), \; k = 0, 1, \cdots,$$

where $\rho_{nk} \in L(F, E)$. We can easily show that $\rho_{nk} \equiv f_{kn}$ for all n, k so that

$$X_k = \sum_{n=0}^{\infty} f_{kn}(y_n), \; k = 0, 1, \cdots.$$

Then

$$\sum_{k=0}^{q} B_{pk} X_k = \sum_{n=0}^{\infty} \sum_{k=0}^{q} B_{pk} \circ f_{kn}(y_n)$$

and the series on the left converges in F whenever $(X_k) \in c_{0_A}(E, F)$ since B is row finite. Using the fact that B is row finite, we then have

$$\sum_{k=0}^{\infty} B_{pk} X_k = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_{pk} \circ f_{kn}(y_n), \ p = 0, 1, \cdots.$$

Finally, the left-hand sequence is in $c(F)$ if and only if (i) holds.

We have an analogous result for absolute summability (see also [5]).

THEOREM 5.2. *Let $A \in \Gamma(l_1(E), l_1(F))$ be reversible and let $B = (B_{nk})$ be a matrix of continuous linear operators on E to F. Then $l_{1_A}(E, F) \subseteq l_{1_B}(E, F)$ if and only if*

(i) *for each $p = 0, 1, \cdots, C^p = (C^p_{qn}) \in \Gamma(l_1(F), c(F))$ where $C^p_{qn} = \sum_{k=0}^{q} B_{pk} \circ f_{kn}$; and*

(ii) $D = (D_{pn}) \in \Gamma(l_1(F), l_1(F))$.

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