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CONTINUITY OF LOCAL TIMES FOR MARKOV PROCESSES

by

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1. Introduction

Local times have become a useful tool in the investigation of Markov processes (see for instance [2], [11], [12] and [18]). They have also been investigated for Gaussian, but not necessarily Markovian, processes (e.g. in [1]). In some sense the local time L_t^x at the point x for a Markov process $\{X_t\}_{t \geqq 0}$ measures the amount of time X spent at x during the time interval $[0, t]$. For standard Markov processes $\{X_t\}$, Blumenthal and Getoor [2], or [3], Ch. V.3 showed that $L_t^{x_0}$ exists if x_0 is regular for $\{x_0\}$. Thus, one can define L_t^x for each regular point x. If all points x are regular one would like to know whether L_t^x can be defined such that it has measurability and continuity properties in x and such that $L_t^{\cdot}$ is the density of the occupation time measure [2]

$$(1.1) \qquad \mu_t(B) = \int_0^t I_B(X_s)\, ds$$
$$= \text{measure of amount of time spent in } B \text{ by } \{X_s\}_{0 \leqq s \leqq t}.$$

Berman [1] showed some interesting consequences of the existence of such a continuous density of $\mu_t(\cdot)$. Trotter [19] was the first to construct a local time L_t^x for Brownian motion which is continuous in (x, t) and satisfies a.s. [2]

$$(1.2) \qquad \mu_t(B) = \int_B L_t^x\, dx,\ t > 0,\ B \text{ a Borel set in } \boldsymbol{R}.$$

Boylan [4] and later Meyer [14] gave a similar construction for more general processes (see also [3], V.3.22–31), whereas Blumenthal and Getoor [2], sect. 3) showed that (1.2) is satisfied under mild assumptions

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[2] $I_A(\cdot)$ is the indicator function of the set A. a.s. stands for almost surely (see [3], def. I.5.7). E^x denotes expectation w.r.t. P^x, the measure corresponding to a process starting at x [see [3], sect. I.3). Throughout we shall use the notation and terminology of [3].

(which, however, do not necessarily imply the existence of a continuous version of the function $x \to L_t^x$; see sect. 3 below).

The results of [4] and [14] concern a standard process X with state space E, an interval of the real line, and all points regular. As usual

$$(1.3) \qquad T_x = \inf\{t > 0 : X_t = x\} = \text{first hitting time of } x.$$

Assume that there exists a function $h : [0, \infty) \to [0, 1]$ such that $h(x) \downarrow 0$ as $x \downarrow 0$ and such that for all $M > 0$ there exists a $C = C(M) < \infty$ for which

$$(1.4) \qquad 1 - E^x e^{-T_y} \leqq Ch(|x-y|),\ x, y \in E \cap [-M, +M].$$

It is proved in [14] (see also [3], V.3.30; also [4], where the hypotheses have a slightly different form) that if in addition

$$(1.5) \qquad \sum n\{h(2^{-n})\}^{\frac{1}{2}} < \infty,$$

then the local time $L_t^x(\omega)$ can be chosen continuous in (x, t) for all ω. We shall show in sect. 2 that the factor n in the sum in (1.5) can be dropped, i.e., that

$$(1.6) \qquad \sum \{h(2^{-n})\}^{\frac{1}{2}} < \infty$$

is already sufficient for the existence of a continuous local time.

In sect. 3 we derive a necessary condition for the existence of a continuous local time. Roughly speaking if

$$\limsup_{\alpha \to \infty} (\log \alpha) E^x \int_0^\infty e^{-\alpha t} d_t L_t^x \geqq \delta > 0$$

in some uniform sense, then the local time L_t^x cannot be continuous in x.

The conditions of the theorems become particularly simple for processes with stationary independent increments. For such processes we know exactly when all points are regular, [6], and all conditions can be expressed in terms of the characteristic function (and Lévy measure of the process). The precise result is as follows:

THEOREM 4. *Let $\{X_t\}_{t \geqq 0}$ be a right continuous one-dimensional process with stationary independent increments and characteristic function*

$$Ee^{i\lambda X_t} = Ee^{i\lambda(X_{t+s} - X_s)} = e^{-t\psi(\lambda)}$$

where

$$\psi(\lambda) = -ia\lambda + \tfrac{1}{2}\sigma^2\lambda^2 - \int_{\boldsymbol{R}-\{0\}} \left[e^{i\lambda y} - 1 - \frac{i\lambda y}{1+y^2}\right] \nu(dy).$$

If $\sigma^2 > 0$ or $\nu(\boldsymbol{R} - \{0\}) = \infty$ and 0 is regular for $\{0\}$, then for $\alpha > 0$ there exists a bounded continuous density u^α for the potential kernel i.e.

$$(1.7) \qquad E^x \int_0^\infty \mathrm{e}^{-\alpha t} f(X_t)\, dt = \int_{\boldsymbol{R}} u^\alpha(y-x) f(y)\, dy$$

for non-negative measurable f. There also exists for each x a continuous additive functional l_t^x (a local time at x) such that

$$(1.8) \qquad E^x \int_0^\infty e^{-\alpha t} d_t l_t^y = u^\alpha(y-x),$$

and such that for $t \geqq 0$ the map $(x, \omega) \to l_t^x$ is $\mathscr{E} \times \mathscr{F}_t$ measurable [3] *and a.s.*

$$(1.9) \qquad \mu_t(B) = \int_B l_t^x\, dx,\ t \geqq 0,\ B \text{ Borel set in } \boldsymbol{R}.$$

The probability that l_t^x has a version which is a continuous function of (x, t) equals zero or one. If

$$(1.10) \qquad \delta(u) = \sup_{|x| \leqq u} \frac{1}{\pi} \int_{-\infty}^{+\infty} (1 - \cos x\lambda) \operatorname{Re} \frac{1}{1+\psi(\lambda)}\, d\lambda$$

satisfies

$$(1.11) \qquad \sum_{n=1}^{\infty} \{\delta(2^{-n})\}^{\frac{1}{2}} < \infty$$

then one can take $l_t^x(\omega)$ continuous in (x, t) for all ω. On the other hand, if

$$(1.12) \qquad \limsup_{\alpha \to \infty} (\log \alpha) \int_{-\infty}^{+\infty} \operatorname{Re} \frac{1}{\alpha + \psi(\lambda)}\, d\lambda > 0,$$

then no continuous version of $x \to l_t^x$ exists (see (4.11)*).*

Examples illustrate the gap between the sufficient condition (1.11) and the necessary condition

$$\lim_{\alpha \to \infty} (\log \alpha) \int_{-\infty}^{+\infty} \operatorname{Re} \frac{1}{\alpha + \psi(\lambda)}\, d\lambda = 0.$$

It would be interesting to close this gap and to find a necessary and sufficient condition for the existence of a continuous local time for processes with stationary independent increments. Another open problem is whether such processes can have a bounded version of the local time in the case where no continuous version exists.

[3] Here $\mathscr{E}$ is the σ-algebra of Borel sets of $\boldsymbol{R}$.

2. A sufficient condition for the existence of a continuous local time

Throughout, we use the terminology and notation of [3]. In particular, $X = \{X_t\}_{t \geqq 0}$ will always be a standard Markov process with state space $(E, \mathscr{E})$ and measure P^x for the process starting at x. $\mathscr{F}_t$ is a suitable completion of $\sigma\{X_s : s \leqq t\}$ (see Sect. I.3 and I.5 of [3]). For a positive $\mathscr{E}$ measurable function f

$$U^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dt.$$

For $A \in \mathscr{E}$, $T_A = \inf\{t > 0 : X_t \in A\}$ and $T_x = T_{\{x\}}$. We call x *regular* provided that x is regular for $\{x\}$; that is if

$$P^x(T_x = 0) = 1. \tag{2.1}$$

If y is regular, $L_t^y(\omega)$ is a local time at y, i.e., a continuous additive functional with support $\{y\}$; such a local time exists and is unique up to a multiplicative factor ([3], theorem V.3.13). Moreover, it has a bounded α-potential for any $\alpha > 0$ and we usually normalize it by

$$E^y \int_0^\infty e^{-t} dL_t^y = 1. \tag{2.2}$$

If y is regular we define for $\alpha > 0$

$$\psi^\alpha(x, y) = E^x(e^{-\alpha T_y}) \tag{2.3}$$

$$v^\alpha(x, y) = E^x \int_0^\infty e^{-\alpha t} dL_t^y. \tag{2.4}$$

We sometimes write $\psi_y^\alpha(x) = \psi^\alpha(x, y)$ when we want to regard $\psi^\alpha(x, y)$ as function of x depending on the parameter y. It follows easily from (2.2) and the fact that the support of L^y is $\{y\}$ that

$$v^1(x, y) = E^x \int_0^\infty e^{-t} dL_t^y = \psi^1(x, y). \tag{2.5}$$

A straightforward computation (see [2] lemma 1.1 and [3] proposition IV.2.3) yields

$$v^\alpha(x, y) = b_\alpha(y)\psi^\alpha(x, y) = \psi^1(x, y) - (\alpha - 1)U^\alpha \psi_y^1(x) \tag{2.6}$$

$$b_\alpha(y) = 1 - (\alpha - 1)U^\alpha \psi_y^1(y) > 0. \tag{2.7}$$

We now assume that *each x in E is regular.* Under mild assumptions one can choose a version of the local time which is a.s. the density of the occupation time measure. More precisely one has

THEOREM 1. *Assume that $\psi^1(x, y)$ is jointly Borel measurable, i.e., $\psi^1 \in \mathscr{E} \times \mathscr{E}$. Then one can choose the local time $L_t^x(\omega)$ to satisfy* (2.2) *and* *such that for each $t > 0$ the map $(s, x, \omega) \to L_s^x(\omega)$ from $[0, t] \times E \times \Omega \to [0, \infty)$ is $\mathscr{B}_t \times \mathscr{E} \times \mathscr{F}_t$ measurable where $\mathscr{B}_t$ is the σ-algebra of Borel subsets of $[0, t]$, and for each (x, ω), $t \to L_t^x(\omega)$ is left continuous and increasing. (Of course, for each x, L^x is a continuous additive functional.) Assume further that X has a reference measure ξ* [4]*. Then there exists a strictly positive finite Borel function g on $\mathscr{E}$ such that $l_t^x(\omega) \stackrel{\text{def}}{=} g(x)\, L_t^x(\omega)$ almost surely satisfies*

$$\mu_t(B, \omega) \stackrel{\text{def}}{=} \int_0^t I_B(X_s)\, ds = \int_B l_t^x(\omega) \xi(dx) \tag{2.8}$$

for all $t \geqq 0$ and $B \in \mathscr{E}$ simultaneously. Moreover, if for each $\alpha > 0$ we define $u^\alpha(x, y) = v^\alpha(x, y)\, g(y)$, then

$$U^\alpha f(x) = \int u^\alpha(x, y) f(y) \xi(dy) \tag{2.9}$$

for each $f \in \mathscr{E}^+$, $u^\alpha(\cdot, y)$ is uniformly α-excessive for each y, $u^1(y, y) = g(y)$ and

$$\psi^1(x, y) - (\alpha - 1) U^\alpha \psi_y^1(x) = \frac{u^\alpha(x, y)}{u^1(y, y)}. \tag{2.10}$$

Finally

$$E^x \int_0^\infty e^{-\alpha t} dl_t^y = u^\alpha(x, y). \tag{2.11}$$

PROOF. This theorem is due to Blumenthal and Getoor (theorem 3.2 and corollary 3.4 in [2], and V-3.41 of [3]). However, Blumenthal and Getoor make several extraneous assumptions and are not very explicit at several stages in their argument. Consequently we will make several comments on the proof.

Arguing more or less as in [2] ($g_n(x, y)$ in the proof of Theorem 3.2 should now be replaced by

$$n[\psi^1(x, y) - e^{-1/n} P_{1/n} \psi_y^1(x)];$$

compare also Theorem IV.3.8 in [3]) one constructs for each x a multiplicative function M^x such that for each t the map $(x, \omega) \to M_t^x(\omega)$ is $\mathscr{E} \times \mathscr{F}_t$ measurable and such that for each x and t, $M_t^x = \exp(-L_t^x)$ almost surely, where for each x, L^x is a local time at x satisfying (2.2). In particular $t \to E^\lambda(M_t^x)$ is continuous for any initial measure λ, and so if we define

$$\overline{M}_0^x(\omega) = M_0^x(\omega); \ \overline{M}_t^x(\omega) = \inf_{r<t} M_r^x(\omega); \ t > 0$$

where the infimum is over all rationals $r < t$, then $M_t^x = \overline{M}_t^x$ almost

[4] Essentially, ξ is a reference measure if it is equivalent to $U^\alpha(x, A) = U^\alpha I_A(x)$ (viewed as a measure in A) for all x. See [3], definition V.1.1.

surely for each x and t. However $\overline{M}_t^x(\omega)$ is clearly left continuous and decreasing in t for all (x, ω) and still $\mathscr{E} \times \mathscr{F}_t$ measurable in (x, ω) for each t. We let $\overline{L}_t^x(\omega) = -\log \overline{M}_t^x(\omega)$. This has the desired measurability properties and is left continuous. But for fixed x and t, $\overline{L}_t^x = L_t^x$ almost surely and since L^x is a continuous additive functional it follows that for each x, $t \to \overline{L}_t^x$ is almost surely continuous. Thus $\overline{L}_t^x(\omega)$ has all of the properties claimed for the local time in the second sentence of Theorem 1. We now drop the bar '$-$' from our notation and let $L_t^x(\omega)$ denote a version of the local time with these properties.

Now let ξ be a fixed reference measure for X. Let η be a finite measure equivalent to ξ. Define

$$L_t(\omega) = \int L_t^y(\omega)\eta(dy). \tag{2.12}$$

This exists for each t and the map $(s, \omega) \to L_s(\omega)$ from $[0, t] \times \Omega$ to $[0, \infty]$ is $\mathscr{B}_t \times \mathscr{F}_t$ measurable. Also, by (2.5),

$$E^x \int_0^\infty e^{-t} dL_t = \int \psi^1(x, y)\eta(dy) \leqq \eta(E) < \infty.$$

Therefore $t \to L_t$ is finite almost surely. A standard argument using Fubini's theorem now shows that L is a continuous additive functional.

Next let $f \in \mathscr{E}^+$ and suppose that $U_L^1 f = 0$ (see [3], IV.2.1 for notation). Then

$$0 = U_L^1 f(x) = \int \psi^1(x, y) f(y)\eta(dy),$$

and so for each x, $\psi^1(x, y) f(y) = 0$ a.e. in y. But ψ^1 is jointly measurable and so a.e. in y, $\psi^1(x, y) f(y) = 0$ a.e. in x. Suppose that $\{f > 0\}$ has positive measure. Then there exists a y such that $\psi^1(\cdot, y) = 0$ a.e., and hence everywhere because $\psi^1(\cdot, y)$ is 1-excessive (see [2] lemma 1.1 and [3], II.3.2). But $\psi^1(y, y) = 1$ because y is regular, and so $f = 0$ a.e. Therefore $U^1 f = 0$. Let $A_t = t \wedge \zeta$, where ζ is the 'lifetime' of X. It follows from (V.2.8) of [3] and the above calculation that $A = hL$ for some $h \in \mathscr{E}^+$. Consequently

$$\begin{aligned} A_t &= \int_0^t h(X_s) dL_s \\ &= \int \eta(dy) \int_0^t h(X_s) dL_s^y \\ &= \int h(y) L_t^y \eta(dy) = \int g(y) L_t^y \xi(dy) \end{aligned} \tag{2.13}$$

where $g \in \mathscr{E}^+$. This means that a.s., $t \to A_t$ and $t \to \int g(y)\, L_t^y\, \xi(dy)$ are identical functions of t. Now define $l_t^x(\omega) = g(x)\, L_t^x(\omega)$, and observe that by [3], V.3.9, one has for each y a.s. for all t

$$l_t^y = \int_0^t I_{\{y\}}(X_s)\, dl_s^y. \tag{2.14}$$

Thus, by virtue of (2.13) and (2.14) one has a.s. for all $B \in \mathscr{E}$ and all t

$$\mu_t(B, \omega) = \int_0^t I_B(X_s)\, dA_s = \int \xi(dy) \int_0^t I_B(X_s) dl_s^y$$
$$= \int \xi(dy) \int_0^t I_B(X_s) I_{\{y\}}(X_s)\, dl_s^y = \int_B \xi(dy) I_s(X_s)\, dl_s^y,$$

i.e. (2.8). It is also clear that $g > 0$ a.e., and hence we may assume that g is strictly positive. Using (2.13) again we see that for any $\alpha > 0$ and $f \in \mathscr{E}^+$

$$U^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dA_t = \int v^\alpha(x, y) f(y) g(y) \xi(dy),$$

and the remaining assertions in Theorem 1 are easy consequences of this, lemma 1.1 in [2], (2.5), and (2.6).

In the remainder of this section we assume that

(2.15) $(E, \mathscr{E})$ is an interval of the real line with its usual Borel structure.

We come now to the main result of this section.

THEOREM 2. *Let X be a standard process with state space $(E, \mathscr{E})$ of the form* (2.15) *and such that each x in $\mathscr{E}$ is regular. Define for $u > 0$*

$$p(u) = \sup_{\substack{x, y \in E \\ |x-y| \leq u}} [1 - \psi^1(x, y)\psi^1(y, x)]^{\frac{1}{2}}. \tag{2.16}$$

Assume that

$$\int_0^1 p(u) u^{-1}\, du < \infty. \tag{2.17}$$

Then the local time $L_t^x(\omega)$ may be chosen so that almost surely $(x, t) \to L_t^x(\omega)$ is continuous and increasing in t.

Before coming to the proof of Theorem 2 let us observe that $p(\cdot)$ is increasing so that (2.17) holds if and only if

$$\sum p(2^{-n}) < \infty, \tag{2.18}$$

while the condition in the Boylan theorem is essentially

$$\sum n p(2^{-n}) < \infty. \tag{2.19}$$

See page 225 of [3]. Also it will appear from the proof that if for each positive integer $M > 0$ one defines

$$p_M(u) = \sup_{\substack{|x-y| \leq u \\ x, y \in E \cap [-M, M]}} [1 - \psi^1(x, y)\psi^1(y, x)]^{\frac{1}{2}} \tag{2.20}$$

then it would suffice to assume that for each M the function p_M satisfies (2.17).

Our proof of theorem 2 is based on the following beautiful lemma which is due to Garsia, Rodemich, and Rumsey [9]. See [10] for an extremely simple proof.

LEMMA 1. *Let* $[a, b]$ *be a compact interval. Let* $p(u)$ *be an even function defined on* $[a-b, b-a]$ *that is increasing on* $[0, b-a]$ *and satisfies* $\lim_{u\to 0} p(u) = 0$. *Let* $\Psi(u)$ *be an even convex function defined on* $(-\infty, \infty)$ *which is increasing on* $[0, \infty]$ *and satisfies* $\lim_{u\to\infty} \Psi(u) = \infty$. *Let f be a measurable function on* $[a, b]$ *such that*

$$\int_a^b \int_a^b \Psi\left[\frac{f(x)-f(y)}{p(x-y)}\right] dx\,dy \leqq B < \infty. \tag{2.20}$$

Then for (Lebesgue) almost all (x, y) *in* $[a, b]\times[a, b]$ *one has*

$$|f(x)-f(y)| \leqq 8\int_{0\leqq u\leqq |x-y|} \Psi^{-1}(Bu^{-2})\,dp(u). \tag{2.21}$$

In [9] and [10] this lemma is stated for the unit interval [0, 1]; but a simple change of variable yields the above statement.

We are now ready for the proof of Theorem 2. For simplicity we assume that $E = R$; the case in which E is an interval requires only notational changes. Note that

$$\begin{aligned}\psi^1(x+u, y+v) &= E^{x+u}e^{-T_{y+v}}\\ &\geqq E^{x+u}e^{-T_x}E^x e^{-T_y}E^y e^{-T_{y+v}}\\ &= \psi^1(x+u, x)\psi^1(x, y)\psi^1(y, y+v) \geqq \{1-p(|u|)\}\{1-p(|v|)\}\psi^1(x, y).\end{aligned}$$

Thus $\psi^1(\cdot, \cdot)$ is lower semicontinuous and $\mathscr{E}\times\mathscr{E}$ measurable. We may therefore begin by assuming that $L_t^x(\omega)$ is a local time satisfying the conditions in the second sentence of Theorem 1. Fix positive integers N and M and define

$$Y(N, x, y) = \sup_{0\leqq t\leqq N} |L_t^x - L_t^y|. \tag{2.22}$$

Then using the estimate (V.3.28) of [3] we obtain for each z and $\delta > 0$

$$P^z[Y(N, x, y) > 2\delta] \leqq 2e^N e^{-\delta/p(|x-y|)}. \tag{2.23}$$

Then for any $\lambda > 0$

$$\begin{aligned}&P^z\left\{\exp\left[\frac{Y(N, x, y)}{4p(|x-y|)}\right] > \lambda\right\}\\ &\qquad = P^z\{Y(N, x, y) > 4p(|x-y|)\log\lambda\} \leqq 2e^N\lambda^{-2},\end{aligned}$$

and so

$$E^z\left\{\exp\left[\frac{Y(N,x,y)}{4p(|x-y|)}\right]\right\} \leqq 4e^N. \tag{2.24}$$

We now are ready to apply lemma 1. Let $\Psi(x) = \exp[|x|/2]$. Then by (2.22) for each $t \leqq N$ we have

$$\begin{aligned}\int_{-M}^{M}\int_{-M}^{M} \Psi\left[\frac{L_t^x - L_t^y}{2p(|x-y|)}\right] dx\,dy \\ \leqq \int_{-M}^{M}\int_{-M}^{M} \exp\left[\frac{Y(N,x,y)}{4p(|x-y|)}\right] dx\,dy \overset{\text{def}}{=} B_{N,M}(\omega).\end{aligned} \tag{2.25}$$

But from (2.24) and Fubini's theorem

$$E^z(B_{N,M}) \leqq (2M)^2 4e^N$$

for all z, and so $B_{N,M}$ is finite almost surely. Applying Lemma 1 to the estimate (2.25) we find that for each t in $[0, N]$

$$|L_t^x(\omega) - L_t^y(\omega)| \leqq 16\int_{0\leqq u\leqq |x-y|} \log\left(\frac{B_{N,M}(\omega)}{u^2}\right) dp(u) \tag{2.26}$$

for almost all (x, y) in $[-M, M]^2$, where the exceptional set may depend on t and ω. Define

$$\phi(u) = \int_{0\leqq t\leqq |u|} [(-\log t)\vee 1]\, dp(t). \tag{2.27}$$

Then (2.17) implies that ϕ is finite and continuous with $\phi(u) \to 0$ as $u \to 0$. Now simple manipulations show that (2.26) may be rewritten as follows: There exists a random variable $c_{N,M}(\omega)$ which is finite almost surely and such that for each $t \in [0, N]$

$$|L_t^x(\omega) - L_t^y(\omega)| \leqq c_{N,M}(\omega)\phi(x-y) \tag{2.28}$$

almost everywhere (Lebesgue) on $[-M, M]^2$ and the exceptional set may depend on both t and ω. However, it is important that $c_{N,M}(\omega)$ does not depend on t.

We now define

$$\bar{L}_t^x(\omega) = \limsup_{n\to\infty} \frac{n}{2}\int_{-1/n}^{1/n} L_t^{x+y}(\omega)\, dy \tag{2.29}$$

for all x, t, and ω. Clearly for each t the map $(s, x, \omega) \to \bar{L}_s^x(\omega)$ from $[0, t]\times E\times\Omega$ to $[0, \infty]$ is $\mathscr{B}_t\times\mathscr{E}\times\mathscr{F}_t$ measurable. It is straightforward to check that for each $t \in [0, N]$ and ω with $c_{N,M}(\omega) < \infty$ one has

$$\bar{L}_t^x(\omega) = \lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} L_t^{x+y}(\omega)\, dy \tag{2.30}$$

for all x in $[-M, M]$,

$$|\bar{L}_t^x(\omega) - \bar{L}_t^y(\omega)| \leqq c_{N,M}(\omega)\phi(x-y) \tag{2.31}$$

for all x, y in $[-M, M]$, and

$$\bar{L}_t^x(\omega) = L_t^x(\omega) \tag{2.32}$$

almost everywhere in x on $[-M, M]$ where the exceptional set may depend on t and ω. It is now a routine, but slightly tedious, exercise in the use of Fubini's theorem to show that almost surely $(t, x) \to \bar{L}_t^x(\omega)$ is continuous and that for each x, $\bar{L}^x$ is a continuous additive functional equivalent L^x. This establishes Theorem 2.

REMARK 1. Note that (2.31) is an explicit Hölder condition for the local time.

3. A necessary condition for the existence of a continuous local time

Again X is a standard process and ψ^α as in (2.3). This time the state space $(E, \mathscr{E})$ does not have to be part of the real line but we assume that it is a locally compact metric space with distance function $d(\cdot, \cdot)$ such that

(3.1) For each $d > 0$, $x \in E$ there exists a $y \in E$ with

$$\frac{d}{2} < d(x, y) < d.$$

We now show that under simple conditions no continuous local time exists, not even when the spatial argument is restricted to a countable dense set. When interpreting Theorem 3 and Corollary 1 one should take into account that (by right continuity of X) there exist a.s. a $t = t(\omega) > 0$ and a compact set $K = K(\omega)$ such that $X_s \in K$ for $s \leqq t$ and consequently $L_s^x = 0$ for all $x \notin K$, $x \in F$ and $s \leqq t$. Thus (3.6) and (3.36) really deal with the continuity of $L_s^{\cdot}$ on K, not just the uniform continuity on E.

THEOREM 3. *Let X be a standard process with metric state space $(E, \mathscr{E})$ satisfying* (3.1), *and such that all points $x \in E$ are regular. Let L_t^x be a local time at x with*

$$v^\alpha(x, y) = E^x \int_0^\infty e^{-\alpha t} d_t L_t^y.$$

(*Note that we do not insist on* (2.2) *here.*) *Moreover, let $F = \{x_1, x_2, \cdots\}$ be a countable dense subset of E and $F_N \subset F$ such that*

(3.2) $$F_N \uparrow F,\ F_N \text{ a closed subset of } F.$$

Let $T_N = \inf\{t > 0 \colon X_t \in F_N\}$. *If*

(3.3) $$E^x e^{-T_N} \to 1\ (N \to \infty) \text{ uniformly in } x \in E,$$

(3.4) $$\psi^\alpha(x, y) \to 0\ (\alpha \to \infty), \text{ for each } d > 0 \text{ uniformly in } x, y \in F,\ |x-y| \geqq \frac{d}{2} > 0,$$

and if there exists a $\delta > 0$ *such that for all* $\gamma_0 > 0$ *there exists a* $\gamma_1 < \infty$ *with*

(3.5) $$\sup_{\gamma_0 \leqq \alpha \leqq \gamma_1} (\log \alpha) v^\alpha(x, x) \geqq \delta \text{ for all } x \in F,$$

then for all t, $d > 0$ *and* $x_0 \in E$

(3.6) $$P^{x_0}\left\{ \sup_{\substack{x, y \in F \\ d(x,y) \leqq d}} \sup_{s \leqq t} |L_s^x - L_s^y| \geqq \frac{3\delta}{32} \right\} = 1.$$

Proof. Fix $x_0 \in E$, $d > 0$ and choose $0 < \varepsilon < 1$. Also, for each $x_i \in F$ pick a $y_i \in F$ with $d/2 \leqq d(x_i, y_i) \leqq d$ (see 3.1). By (3.4) there exists a β such that

(3.7) $$\psi^\alpha(x_i, y_i) \leqq \tfrac{1}{8},\ \psi^\alpha(y_i, x_i) \leqq \tfrac{1}{8} \text{ for } \alpha \geqq \beta.$$

By (3.5) and (3.3) there now exist $\beta \leqq \gamma_0 < \gamma_1 < \infty$, α_i and N such that

(3.8) $$\gamma_0^{-1} + \gamma_0^{-\frac{1}{2}} \varepsilon^{-1} \leqq \varepsilon^2,$$

(3.9) $$\gamma_0 \leqq \alpha_i \leqq \gamma_1,$$

(3.10) $$(\log \alpha_i) v^{\alpha_i}(x_i, x_i) \geqq \frac{\delta}{2},\ i = 1, 2, \cdots,$$

and for all $x \in E$

(3.11) $$(\gamma_1^{\frac{1}{2}} \varepsilon^{-1} + 1)\{1 - \exp -\varepsilon^2(\gamma_1^{\frac{1}{2}} + \varepsilon)^{-1}\}^{-1} E^x(1 - e^{-T_N}) \leqq \varepsilon.$$

Finally, let $\{R_j^i : i \geqq 1, j \geqq 1\}$ be a family of independent exponentially distributed random variables, which is independent of the X process and satisfies

(3.12) $$ER_j^i = \alpha_i^{-1}$$

(it may be necessary to enlarge the probability space in order to construct such random variables, but we do not dwell on this standard construction, see [14] sect. 3 and [15] sect. XIV.39). We take

$$R_j(z) = R_j^i \text{ for } z = x_i \in F$$

and, similarly

$$\alpha(z) = \alpha_i \text{ and } y(z) = y_i \text{ for } z = x_i \in F.$$

A sequence of stopping times is now defined as follows:

$$\begin{aligned}
U_1 &= T_N,\, z_1 = X_{U_1}(\in F_N \text{ on } \{T_N < \infty\} \text{ since } F_N \text{ is closed}),\\
V_1 &= U_1 + R_1(z_1) \quad (R_1(z_1) \equiv 0 \text{ on } \{U_1 = \infty\}),\\
U_i &= V_{i-1} + T_N \circ \theta_{V_{i-1}}, \quad z_i = X_{U_i}(\in F_N \text{ on } \{U_i < \infty\}),\\
V_i &= U_i + R_i(z_i) \quad (R_i(z_i) \equiv 0 \text{ on } \{U_i = \infty\}).
\end{aligned}$$

Notice, that

$$\begin{aligned}
Y_i &\equiv L^{z_i}_{V_i}(\omega) - L^{y(z_i)}_{V_i}(\omega) - (L^{z_i}_{U_i} - L^{y(z_i)}_{U_i}(\omega)) \\
&= L^{z_i}_{R_i(z_i)}(\theta_{U_i}\omega) - L^{y(z_i)}_{R_i(z_i)}(\theta_{U_i}\omega).
\end{aligned} \tag{3.13}$$

Therefore, if

$$\mathscr{I}_{i-1} = \sigma\{Y_j, j \leqq i-1,\, U_k, z_k, k \leqq i\}$$

then, by the strong Markov property and the independence properties of the R^i_j,

$$\begin{aligned}
&P^{x_0}\{|Y_i| \geqq u \,|\, \mathscr{I}_{i-1}\} \\
&\qquad = P^{z_i}\{|L^{z_i}_{S_i} - L^{y(z_i)}_{S_i}| \geqq u\} \text{ on } \{U_i < \infty\},
\end{aligned} \tag{3.14}$$

where S_i is an exponential variable independent of the X process and with mean $\alpha^{-1}(z_i)$. The distribution of

$$B = L^x_{S(\alpha)} - L^y_{S(\alpha)}$$

for an exponential variable $S(\alpha)$ with mean α^{-1} and independent of the X process was explicitly computed by Meyer, [14] theorem 2 (see also [3], V.3.26), under the hypothesis $v^\alpha(x, x) = v^\alpha(y, y) = 1$. But a trivial change in their computations gives in general

$$E^x e^{i\lambda B} = \frac{a\nu_+}{(\nu_+ - i\lambda)} + \frac{b\nu_-}{(\nu_- + i\lambda)}, \tag{3.15}$$

where

$$\begin{aligned}
\nu_\pm &= \tfrac{1}{2}\gamma^{-2}[\pm(v^\alpha(y, y) - v^\alpha(x, x)) \\
&\qquad + \{(v^\alpha(x, x) + v^\alpha(y, y))^2 - 4v^\alpha(x, y)v^\alpha(y, x)\}^{\frac{1}{2}}],\\
\gamma^2 &= \{v^\alpha(x, x)v^\alpha(y, y) - v^\alpha(x, y)v^\alpha(y, x)\},\\
a &= (\nu_+ + \nu_-)^{-1}\{\nu_- + \gamma^{-2}(v^\alpha(y, y) - v^\alpha(x, y))\},\\
b &= (\nu_+ + \nu_-)^{-1}\{\nu_+ - \gamma^{-2}(v^\alpha(y, y) - v^\alpha(x, y))\}.
\end{aligned}$$

But, if $x = x_j \in F$, $y = y(x) = y_j \in F$ and $\alpha \geqq \beta$, one has by [3], V.3.16 and (3.7)

$$v^\alpha(x, y) = \psi^\alpha(x, y)\, v^\alpha(y, y) \leqq \tfrac{1}{8}v^\alpha(y, y), \tag{3.16}$$

$$v^\alpha(y, x) \leqq \tfrac{1}{8}v^\alpha(x, x), \tag{3.17}$$

and simple computations show that (3.5), (3.16) and (3.17) imply

$$\text{(3.18)}\quad 0 < \nu_+ \leqq \gamma^{-2}\, v^\alpha(y, y) \leqq v^\alpha(y, y)\{v^\alpha(x, x) v^\alpha(y, y)(1-\tfrac{1}{4})\}^{-1} = \tfrac{4}{3}\{v^\alpha(x, x)\}^{-1},$$

$$\text{(3.19)}\quad 0 < \nu_- \leqq \tfrac{4}{3}\{v^\alpha(y, y)\}^{-1},$$

$$\text{(3.20)}\quad a \geqq v^\alpha(x, x)\{v^\alpha(x, x)+v^\alpha(y, y)\}^{-1} \geqq 0,\ b \geqq 0,\ a+b = 1.$$

Explicit inversion of the characteristic function (3.15) finally gives

$$P^x\{B \geqq u\} = ae^{-u\nu_+},$$
$$P^x\{B \leqq -u\} = be^{-u\nu_-},\ u \geqq 0.$$

We apply these results to $x = x_j \in F$, $y = y(x) = y_j \in F$, $\alpha = \alpha_j$. Then, by (3.10) and (3.18),

$$0 \leqq \nu_+ \leqq \tfrac{4}{3}\{v^{\alpha_j}(x_j, x_j)\}^{-1} \leqq \frac{8}{3\delta} \log \alpha_j .$$

If

$$\text{(3.21)}\qquad v^{\alpha_j}(y_j, y_j) \geqq v^{\alpha_j}(x_j, x_j)$$

then also

$$0 \leqq \nu_- \leqq \frac{8}{3\delta} \log \alpha_j$$

and

$$\text{(3.22)}\qquad P^{x_j}\left\{|B| \geqq \frac{3\delta}{16}\right\} \geqq (a+b)e^{-\frac{1}{2}\log \alpha_j} = \alpha_j^{-\frac{1}{2}}.$$

If, however, (3.21) fails, then, by (3.20), $a \geqq \frac{1}{2}$ and

$$\text{(3.23)}\qquad P^{x_j}\left\{|B| \geqq \frac{3\delta}{16}\right\} \geqq ae^{-\frac{1}{2}\log \alpha_j} \geqq \tfrac{1}{2}\alpha_j^{-\frac{1}{2}}.$$

We now define

$$\text{(3.24)}\qquad Z_i = \begin{cases} 1 \text{ if } U_i = \infty \text{ or } U_i < \infty \text{ and } |Y_i| \geqq \dfrac{3\delta}{16}, \\ 0 \text{ otherwise,} \end{cases}$$

$$\delta_i = \begin{cases} \gamma_0^{-\frac{1}{2}} \text{ if } U_i = \infty, \\ \{\alpha(z_i)\}^{-\frac{1}{2}} = \alpha_j^{-\frac{1}{2}} \text{ if } U_i < \infty,\ z_i = x_j, \end{cases}$$

$$\text{(3.25)}\qquad M = \inf\{m : \sum_{i=1}^{m} \delta_i \geqq \varepsilon^{-1}\}.$$

Notice that δ_i is $\mathscr{J}_{i-1}$ measurable and that

$$\text{(3.26)}\qquad M \leqq m_0 = [\gamma_1^{\frac{1}{2}}\varepsilon^{-1}+1]$$

since $\delta_i \geqq \gamma_1^{-\frac{1}{2}}$ by (3.9). Therefore, by virtue of (3.14), (3.22) and (3.23)

$$\mu_i \equiv P^{x_0}\{Z_i = 1 | \mathscr{F}_{i-1}\} \geqq \tfrac{1}{2}\delta_i,$$

and by lemma (6) in [8] with $a = \frac{1}{2}$, $b = (4\varepsilon)^{-1}$,

$$\begin{aligned} P^{x_0}&\{Z_i = 0, 1 \leqq i \leqq M\} \\ &\leqq P^{x_0}\{\sum_{i=1}^{M}(Z_i - \mu_i) \leqq -\tfrac{1}{2}\sum_{i=1}^{M}\mu_i(1-\mu_i) - \tfrac{1}{4}\sum_{i=1}^{M}\delta_i\} \\ &\leqq P^{x_0}\{\sum_{i=1}^{M}(\mu_i - Z_i) \geqq \tfrac{1}{2}\sum_{i=1}^{M}\mu_i(1-\mu_i) + (4\varepsilon)^{-1}\} \\ &\leqq \left(1 + \frac{1}{8\varepsilon}\right)^{-1} \leqq 8\varepsilon. \end{aligned}$$

Equivalently

$$P^{x_0}\left\{|Y_i| \geqq \frac{3\delta}{16} \text{ or } U_i = \infty \text{ for some } i \leqq M\right\} \geqq 1 - 8\varepsilon.$$

However, the R_j^i are a.s. finite and clearly, if

$$|Y_i| \geqq \frac{3\delta}{16} \text{ and } U_i < \infty\ ,$$

then

$$|L_{U_i}^{z_i} - L_{U_i}^{y(z_i)}| \geqq \frac{3\delta}{32} \text{ or } |L_{V_i}^{z_i} - L_{V_i}^{y(z_i)}| \geqq \frac{3\delta}{32}.$$

It follows that

$$\begin{aligned} (3.27)\qquad P^{x_0}&\left\{V_M = \infty \text{ or } \sup_{x_i \in F_N}\sup_{s \leqq V_M}|L_s^{x_i} - L_s^{y_i}| \geqq \frac{3\delta}{32}\right\} \\ &\geqq P\left\{|Y_i| \geqq \frac{3\delta}{16} \text{ or } U_i = \infty \text{ for some } i \leqq M\right\} \geqq 1 - 8\varepsilon. \end{aligned}$$

It remains to show that V_M is small with high probability. But

$$V_M = \sum_{i=1}^{M} R_i(z_i) + \sum_{i=1}^{M} T_N \cdot \theta_{V_{i-1}},$$

and (see (3.12))

$$E^{x_0}\{R_i(z_i); M > i-1\} \leqq E^{x_0}\{\delta_i^2; M > i-1\}.$$

Therefore (see (3.8), (3.9), (3.24) and (3.25))

$$\begin{aligned} E^{x_0}\sum_{i=1}^{M} R_i(z_i) &\leqq E^{x_0}\sum_{i=1}^{M}\delta_i^2 \leqq \gamma_0^{-1} + \gamma_0^{-\frac{1}{2}} E^{x_0}\sum_{i=1}^{M-1}\delta_i \\ &\leqq \gamma_0^{-1} + \gamma_0^{-\frac{1}{2}}\varepsilon^{-1} \leqq \varepsilon^2, \end{aligned}$$

whence

$$(3.28)\qquad P^{x_0}\{\sum_{i=1}^{M} R_i(z_i) \geqq \varepsilon\} \leqq \varepsilon.$$

Similarly, taking into account (3.26) and (3.11)

$$(3.29)\quad P^{x_0}\{\sum_{i=1}^{M} T_N \cdot \theta_{V_{i-1}} \geqq \varepsilon\} \leqq \sum_{i=1}^{m_0} P^{x_0}\{V_{i-1} < \infty,\, T_N \circ \theta_{V_{i-1}} \geqq \varepsilon m_0^{-1}\}$$
$$\leqq \sum_{i=1}^{m_0} (1-\exp -\varepsilon m_0^{-1})^{-1} E^{x_0}\{E^{X_{V_{i-1}}}(1-e^{-T_N});\, V_{i-1} < \infty\}$$
$$\leqq (\gamma_1^{\frac{1}{2}}\varepsilon^{-1}+1)\{1-\exp -\varepsilon^2(\gamma_1^{\frac{1}{2}}+\varepsilon)^{-1}\}^{-1} \sup_{x\in E} E^x(1-e^{-T_N}) \leqq \varepsilon.$$

It follows from (3.27)–(3.29) that

$$(3.30)\qquad P^{x_0}\left\{\sup_{x_i\in F_N}\sup_{s\leqq 2\varepsilon} |L_s^{x_i}-L_s^{y_i}| \geqq \frac{3\delta}{32}\right\}$$
$$\geqq (1-8\varepsilon)-P\{V_M > 2\varepsilon\} \geqq (1-10\varepsilon).$$

Since $0 < \varepsilon < 1$ was arbitrary and $y_i \in F$ with $d(x_i, y_i) \leqq d$ the theorem is proved.

Remark 2. If

$$(3.31)\qquad \liminf_{y\to x} \psi^1(x, y) < 1,$$

then there exists a $\delta_1 > 0$ and a sequence $\{y_k\} \to x$ with $P^x\{T_{y_k} \geqq \delta_1\} > \delta_1$. Hence if

$$R_n = \sup_{k\geqq n} T_{y_k},\ R = \lim_{n\to\infty} R_n,$$

then

$$P^x\{R \geqq \delta_1\} \geqq P^x\{R_n \geqq \delta_1 \text{ infinitely often}\} \geqq \delta_1,$$

and by Blumenthal's zero one law

$$(3.32)\qquad P^x\{R = 0\} = 0.$$

Then, for any fixed $s > 0$,

$$(3.33)\quad P^x\{\limsup_{k\to\infty} |L_s^x-L_s^{y_k}| \geqq \delta\} \geqq P^x\{L_s^x \geqq \delta,\, T_{y_k} \geqq s \text{ infinitely often}\}$$
$$\geqq P^x\{L_s^x \geqq \delta\}-P^x\{R \leqq s\}.$$

(Recall that a.s. $L_s^y = 0$ for $s \leqq T_y$, see [3], V.3.5.)

Taking first s small, then δ small, we can make the right hand side of (3.33) as close to 1 as desired, (see (3.32) and Prop. V.3.5 in [3]). Thus under (3.31), we have for all $t > 0$

$$P^x\{\limsup_{k\to\infty}\sup_{s\leqq t} |L_s^x-L_s^{y_k}| > 0\} = 1,$$

i.e., the local time is not continuous at x a.e. $[P^x]$. It is not unreasonable therefore to assume in addition

$$(3.34) \qquad \psi^1(x, y) = E^x e^{-T_y} \to 1$$

uniformly as $d(x, y) \to 0$, x in a compact subset of E.

Under condition (3.34) it is proved in [14], sect. 3, or [3], V.3.29, that for each $t > 0$, [5]

$$\sup_{s \leqq t} |\{v^1(y, y)\}^{-1} L_s^y - \{v^1(x, x)\}^{-1} L_s^x| \to 0 \text{ in } P^x \text{ probability as } y \to x.$$

Consequently, if for some sequence $\{y_k\} \to x$

$$\lim_{k\to\infty} v^1(y_k, y_k) \neq v^1(x, x),$$

then for each $t > 0$

$$P^x\{\limsup_{k\to\infty} |L_t^{y_k} - L_t^x| > 0\} = 1.$$

Thus it is also reasonable to assume

$$(3.35) \qquad v^1(x, x) \text{ is continuous in } x.$$

With (3.34) and (3.35) added one can sharpen (3.6) as in the following

COROLLARY 1. *If* (3.34) *and* (3.35) *hold, in addition to the assumptions of theorem* 3, *and if* F_N *is discrete, then for each* $t > 0$, $x_0 \in E$

$$(3.36) \qquad P^{x_0}\left\{\lim_{f \downarrow 0} \sup_{\substack{d(x,y) \leqq f \\ x,\, y \in F}} |L_t^x - L_t^y| \geqq \frac{\delta}{32}\right\} = 1.$$

PROOF. Fix $\varepsilon > 0$ and $f > 0$. Let $0 < t_0 \leqq t$ and let K be a compact set so large that

$$(3.37) \qquad P^{x_0}\{X_s \notin K \text{ for some } s \leqq t_0\} \leqq \varepsilon.$$

Let $v(K) = \max_{x \in K} v^1(x, x)$. There exists a $0 < d < f$ such that for all $x \in K$, $d(x, y) \leqq d$,

$$(3.38) \qquad \delta(x, y) \equiv 1 - \psi^1(x, y)\psi^1(y, x) \leqq \delta^2 \left\{128 v(K) \log \frac{1}{\varepsilon}\right\}^{-2},$$

as well as

$$(3.39) \qquad |v^1(x, x) - v^1(y, y)| \leqq \min\left(\frac{\varepsilon\delta}{128}, \left(\frac{\varepsilon\delta}{128}\right)^3 \frac{1}{v^2(K)}, v(K)\right).$$

[5] Some modifications have to be made in the argument below in case $v^1(x, x) = 0$, i.e. when $L^x = 0$.

Lastly, let y_i be chosen as in the proof of theorem 3 (with the d as just constructed) and

$$R_N = \inf \left\{ s : \sup_{x_i \in F_N} |L_s^{x_i} - L_s^{y_i}| > \frac{\delta}{16} \right\}.$$

By (3.30), for suitable N

$$P^{x_0}\{R_N < t_0\} \geqq 1-\varepsilon. \tag{3.40}$$

But $F_N = \overline{F_N}$ is discrete, hence $F_N \cap K$ finite and thus a.e. $[P^{x_0}]$ on

$$\{R_N < t_0\} \cap \{X_s \in K \text{ for all } s \leqq t_0\}$$

there exists (by [3], V.3.8) a $z_N \in F_N \cap K$ such that $X_{R_N} = z_N$ or $X_{R_N} = y(z_N)$ and

$$|L_{R_N}^{z_N} - L_{R_N}^{y(z_N)}| \geqq \frac{\delta}{16}. \tag{3.41}$$

Now, on $\{R_N < t_0\}$ one has for any $x_i \in F_N \cap K$ with $v^1(x_i, x_i) \geqq \varepsilon\delta/64$

$$\begin{aligned} &P^{x_0}\left\{|(L_t^{x_i} - L_t^{y_i}) - (L_{R_N}^{x_i} - L_{R_N}^{y_i})| \geqq \frac{\delta}{32} \,|\mathscr{F}_{R_N}\right\} \\ &\leqq P^{X_{R_N}}\left\{\sup_{s \leqq t} |L_s^{x_i} - L_s^{y_i}| \geqq \frac{\delta}{32}\right\} \\ &\leqq P^{X_{R_N}}\{\sup_{s \leqq t} |\{v^1(x_i, x_i)\}^{-1} L_s^{x_i} - \{v^1(y_i, y_i)\}^{-1} L_s^{y_i}| \geqq \delta(64v(K))^{-1}\} \\ &\quad + P^{X_{R_N}}\left\{\sup_{s \leqq t} \left|\frac{1}{v^1(x_i, x_i)} - \frac{1}{v^1(y_i, y_i)}\right| L_s^{y_i} \geqq \delta(64v(K))^{-1}\right\}. \end{aligned} \tag{3.42}$$

By [3], V.3.28, and (3.38) the first term in the last member of (3.42) does not exceed

$$2e^t \exp -\delta(128v(K)\delta^{\frac{1}{2}}(x_i, y_i))^{-1} \leqq 2e^t\varepsilon \text{ (since } x_i \in K),$$

whereas, by (3.39), Chebychev's inequality and [3], V.3.16, the second term in the last member of (3.42) is for $x_i \in K$ with $v^1(x_i, x_i) \geqq \varepsilon\,\delta/64$ bounded by

$$\begin{aligned} &P^{X_{R_N}}\left\{\left(\frac{\varepsilon\delta}{128}\right)^3 \frac{1}{v^2(K)} L_t^{y_i} \geqq \frac{\delta}{64v(K)} v^1(x_i, x_i)v^1(y_i, y_i) \geqq \frac{\varepsilon^2\delta^3}{(128)^3 v(K)}\right\} \\ &\qquad \leqq P^{X_{R_N}}\{e^{-t}L_t^{y_i} \geqq v(K)e^{-t}\varepsilon^{-1}\} \leqq \frac{e^t\varepsilon}{v(K)} E^{X_{R_N}} \int_0^\infty e^{-s} dL_s^{y_i} \\ &\qquad \leqq e^t\varepsilon \frac{v^1(y_i, y_i)}{v(K)} \leqq 2e^t\varepsilon. \end{aligned}$$

Thus

$$P^{x_0}\left\{|(L_t^{x_i}-L_t^{y_i})-(L_{R_N}^{x_i}-L_{R_N}^{y_i})| \geqq \frac{\delta}{32} \,|\mathscr{F}_{R_N}\right\} \leqq 4e^t\varepsilon. \tag{3.43}$$

(3.43) is also valid when

$$v^1(x_i, x_i) \leqq \frac{\varepsilon\delta}{64}. \tag{3.44}$$

For then

$$P^{x_0}\left\{|L_t^{x_i}-L_{R_N}^{x_i}| \geqq \frac{\delta}{64} \,|\mathscr{F}_{R_N}\right\} \leqq P^{X_{R_N}}\left\{L_t^{x_i} \geqq \frac{\delta}{64}\right\} \tag{3.45}$$

$$\leqq 64\delta^{-1}e^tE^{X_{R_N}}\int_0^\infty e^{-s}dL_s^{x_i} \leqq 64\delta^{-1}e^tv^1(x_i, x_i) \leqq e^t\varepsilon.$$

Similarly, when (3.44) holds $v^1(y_i, y_i) \leqq 2\varepsilon\delta/64$ and

$$P^{x_0}\left\{|L_t^{y_i}-L_{R_N}^{y_i}| \geqq \frac{\delta}{64} \,|\mathscr{F}_{R_N}\right\} \leqq 2e^t\varepsilon. \tag{3.46}$$

(3.45) and (3.46) give (3.43) in case (3.44) holds. Now, by (3.37), (3.40) and (3.43)

$$P^{x_0}\left\{\sup_{d(x,y)\leqq d} |L_t^x-L_t^y| \geqq \frac{\delta}{32}\right\}$$

$$\geqq P^{x_0}\left\{R_N < t_0,\, X_s \in K \text{ for all } s \leqq t_0,\, |L_{R_N}^{z_N}-L_{R_N}^{y(z_N)}| \geqq \frac{\delta}{16},\right.$$

$$\text{but} \quad \left.|(L_t^{z_N}-L_t^{y(z_N)})-(L_{R_N}^{z_N}-L_{R_N}^{y(z_N)})| < \frac{\delta}{32}\right\}$$

$$\geqq 1-\varepsilon(2+4e^t).$$

Since $0 < d < f$, and $f > 0$, $\varepsilon > 0$ were arbitrary (3.36) follows.

4. Processes with stationary independent increments

In this section we take $X = \{X_t\}_{t\geqq 0}$ to be a one dimensional process with stationary independent increments and characteristic function

$$E^0e^{i\lambda X_t} = E^xe^{i\lambda(X_{t+s}-X_s)} = e^{-t\psi(\lambda)} \tag{4.1}$$

with

$$\psi(\lambda) = -ia\lambda+\frac{\sigma^2\lambda^2}{2}-\int\left[e^{i\lambda y}-1-\frac{i\lambda y}{1+y^2}\right]\nu(dy), \tag{4.2}$$

ν a Borel measure on $\boldsymbol{R}-\{0\}$ such that

$$\int \min(1, y^2)\nu(dy) < \infty. \tag{4.3}$$

We may restrict ourselves to one dimensional processes, since 'honestly higher dimensional' processes do not hit points and have no regular points. ([13], theorem 3.) Returning to the one dimensional case, one easily sees that if $\sigma^2 = 0$ and $\nu(\boldsymbol{R}-\{0\}) < \infty$ then

$$\psi(\lambda) = -ia'\lambda - \int \{e^{i\lambda y}-1\}\nu(dy),\ a' = a - \int \frac{y}{1+y^2}\nu(dy),$$

and a.s. there exists a $t > 0$ such that

$$X_s = X_0 + a's \text{ for } 0 \leqq s < t$$

(see [13], sect. 2, (iv)). Thus X has regular points only if $a' = 0$. In this case every point is a holding point and for each $t > 0$, the range $\{X_s : 0 \leqq s \leqq t\}$ is a.s. finite. X_0 is not an accumulation point of this range and consequently

$$P^x\{\liminf_{y\to x} L_t^y = 0 < L_t^x\} = 1,\ t > 0.$$

Thus in this case the local time is a.s. not continuous and we shall not consider this case any further. Instead we turn to the

Proof of Theorem 4. (This theorem was stated in the introduction.) The existence of a bounded continuous density u^α satisfying (1.7) is shown in [6], or [17], sect. 6,7. Indeed, if $\sigma^2 > 0$ or $\nu(\boldsymbol{R}-\{0\}) = \infty$ and 0 is regular for 0, then all points are regular, and the density of the potential kernel u^α is continuous and satisfies

$$u^\alpha(x) = E^0 e^{-\alpha T_x} u^\alpha(0) = \psi^\alpha(0, x)u^\alpha(0). \tag{4.4}$$

Thus

$$\psi^\alpha(x, y) = \psi^\alpha(0, y-x) = \frac{u^\alpha(y-x)}{u^\alpha(0)}$$

is continuous and the Lebesgue measure $\xi(dy) = dy$ is a reference measure. Thus, by theorem 1, there exists a local time $L_t^x(\omega)$ satisfying (2.2) and such that the map $(s, x, \omega) \to L_s^x(\omega)$ from $[0, t]\times\boldsymbol{R}\times\Omega \to [0, \infty)$ is $\mathscr{B}_t\times\mathscr{E}\times\mathscr{F}_t$ measurable [3]. There also exists a Borel function g with the properties described in theorem 1. Since both $g(y)v^\alpha(x, y)$ and $u^\alpha(y-x)$ are densities for the potential u^α, they are for each x equal for almost all y. Hence

$$g(y) = \frac{u^\alpha(y-x)}{v^\alpha(x, y)} = \frac{u^\alpha(y-x)}{\psi^\alpha(0, y-x)v^\alpha(y, y)} \text{ a.e.}$$

In particular (see (4.4)), for $\alpha = 1$,

$$g(y) = \frac{u^1(y-x)}{\psi^1(0, y-x)} = u^1(0) \text{ a.e.}$$

Without effect on (2.8) we may take

$$g(y) = u^1(0) \text{ and } l_t^x = u^1(0)L_t^x \text{ for all } y. \tag{4.5}$$

For this choice, the $u^\alpha(x, y) = g(y)\,v^\alpha(x, y)$ of theorem 1 becomes identical with the present $u^\alpha(y-x)$. Indeed,

$$u^1(y, y) = g(y) = u^1(0),$$

and as observed $u^\alpha(x, y) = u^\alpha(y-x)$ for almost all y for all x. Thus, by (2.10), for each x for almost all y

$$\begin{aligned} \frac{u^\alpha(y-x)}{u^1(0)} &= \psi^1(x, y) - (\alpha-1)U^\alpha\psi_y^1(x) \\ &= \psi^1(y-x) - (\alpha-1)\int u^\alpha(z-x)\psi^1(0, y-z)\,dz. \end{aligned} \tag{4.6}$$

But both sides of (4.6) are continuous in y, so that (4.6) holds everywhere. Thus again by (2.10) $u^\alpha(x, y) \equiv u^\alpha(y-x)$ and $l_t^x = u^1(0)L_t^x$ satisfies (2.8) and (2.11), or equivalently (1.9) and (1.8), as required.

The continuity properties of l follow from theorems 2 and 3. Indeed, if 0 is regular, then X hits points and by [13], theorem 2, or [6],

$$\int_{-\infty}^{+\infty} \text{Re}\,\frac{1}{\alpha+\psi(\lambda)}\,d\lambda < \infty, \alpha > 0. \tag{4.7}$$

The inversion formula for characteristic functions (theorem 6.2.1 in [7]) together with (4.7) therefore gives

$$\begin{aligned} U^\alpha I_{[-y, +y]} &= \int_{-y}^{+y} u^\alpha(z)\,dz = E^0 \int_0^\infty e^{-\alpha t} I_{[-y, +y]}(X_t)\,dt \\ &= \int_0^\infty e^{-\alpha t} P\{|X_t| \leqq y\}\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin \lambda y}{\lambda}\,\text{Re}\,(\alpha+\psi(\lambda))^{-1}\,d\lambda \\ &= \frac{1}{\pi}\int_0^y dz \int_{-\infty}^{+\infty} \cos \lambda z\,\text{Re}\,(\alpha+\psi(\lambda))^{-1}\,d\lambda. \end{aligned} \tag{4.8}$$

For the density u^α of U^α we find

$$\begin{aligned} u^\alpha(0)[\psi^\alpha(0, y)+\psi^\alpha(0, -y)] &= u^\alpha(y)+u^\alpha(-y) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \cos \lambda y\,\text{Re}\,(\alpha+\psi(\lambda))^{-1}\,d\lambda. \end{aligned} \tag{4.9}$$

Thus

$$u^\alpha(0)[1-\psi^\alpha(x, y)\psi^\alpha(y, x)] \leqq u^\alpha(0)[1-\psi^\alpha(x, y)+1-\psi^\alpha(y, x)]$$
$$= 2u^\alpha(0)-u^\alpha(y-x)-u^\alpha(x-y)$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}(1-\cos\lambda(x-y))\operatorname{Re}(\alpha+\psi(\lambda))^{-1}\,d\lambda.$$

Consequently, if (1.11) holds, then so thus (2.18), and by theorem 2 there is a version of L_t^x as well as of $l_t^x = u^1(0)L_t^x$ which is continuous in (x, t).

On the other hand,

$$E^y\int_0^\infty \mathrm{e}^{-\alpha t}\,dl_t^y = u^\alpha(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\operatorname{Re}(\alpha+\psi(\lambda))^{-1}\,d\lambda$$

by (2.11) and (4.9). Thus, if (1.12) holds

$$\limsup_{\alpha\to\infty}(\log\alpha)E^y\int_0^\infty e^{-\alpha t}dl_t^y \geqq \delta > 0$$

uniformly in y, for some suitable $\delta > 0$. Also with

(4.10) $\qquad F =$ dyadic rationals, $F_N = \{k2^{-N} : k \text{ integer}\}$,

(3.1), (3.2) and (3.4) become obvious. To obtain (3.3) and (3.34) note that

$$\psi^1(x, y) = \psi^1(0, y-x) \to \psi^1(0, 0) = 1 \text{ as } y-x \to 0$$

by the continuity of ψ and the regularity of 0. Finally (3.35) is immediate again from (2.2). Thus, if (1.12) holds, the conclusions of theorem 3 and corollary 1 hold, and for each $t > 0$, $x_0 \in \boldsymbol{R}$.

(4.11) $\qquad P^{x_0}\{l_t^\cdot$ is continuous on the dyadic rationals$\} = 0$.

Lastly we show that $l_t^\cdot$ has a continuous version with probability zero or one. Let F, F_N be as in (4.10) and

$$T(m, k, N) = \inf\left\{t > 0 : |l_t^x - l_t^y| > \frac{1}{k} \text{ for some } x, y \in F_N, |x-y| \leqq \frac{1}{m}\right\}.$$

Since a.s. $l_\cdot^x$ is a continuous function of t for all $x \in F$

$$T(m, k, N) = \inf\Big\{r > 0\text{: } r \text{ rational, } |l_r^x - l_r^y| > 1/k \text{ for some}$$
$$x, y \in F_N, |x-y| \leqq \frac{1}{m}\Big\}.$$

Thus the $T(m, k, N)$ are stopping times, increasing in m and decreasing in N, k. Hence also

$$T = \lim_{k\to\infty}\lim_{m\to\infty}\lim_{N\to\infty} T(m, k, N)$$

is a stopping time, and clearly $l_t^{\cdot}$ when restricted to F is uniformly continuous for each $t < T$. Moreover, $P\{T = 0\} = 0$ or 1 by Blumenthal's zero-one law. First consider a process with

$$P^0\{T = 0\} = 1, \tag{4.12}$$

and fix $t > 0$, $\varepsilon > 0$. There then exists a $k = k(\varepsilon, t)$ and for each m an $N = N(m, k) = N(m, k, \varepsilon, t)$ such that

$$P^0\left\{T(m, k, N(m, k)) \leqq \frac{t}{2} \text{ for all } m\right\} \geqq 1-\varepsilon. \tag{4.13}$$

From here on one proves essentially as in corollary 1 that $l_t^{\cdot}$ will not be continuous on F. At time $T(m, k, N)$ there exist a.s. $x, y \in F_N$ with

$$|x-y| \leqq \frac{1}{m}, \quad |l^x_{T(m,k,N)} - l^y_{T(m,k,N)}| \geqq \frac{1}{k},$$

and for sufficiently large m this entails $|l_t^x - l_t^y| \geqq (2k)^{-1}$ with probability at least $(1-\varepsilon)$ (compare (3.42) and the estimates following it). One concludes that for each $\varepsilon > 0$ there exists a k such that for all large m,

$$P^0\left\{T(m, k, N(m, k) \leqq \frac{t}{2} \text{ and } |l_t^x - l_t^y| \geqq \frac{1}{2k}\right.$$
$$\left.\text{for some } (x, y) \in F_{N(m,k)},\ |x-y| \leqq \frac{1}{m}\right\} \geqq 1-2\varepsilon,$$

and hence

$$P^0\left\{\text{for arbitrary large } m \text{ there exist } x, y \in F \text{ with}\right.$$
$$\left.|x-y| \leqq \frac{1}{m} \text{ and } |l_t^x - l_t^y| \geqq \frac{1}{2k}\right\} \geqq 1-2\varepsilon.$$

This holds for each $\varepsilon > 0$ and hence (4.12) implies (4.11). Assume now that

$$P^0\{T = 0\} = 0. \tag{4.14}$$

Then, for some $\delta > 0$ and all k

$$P^0\{\lim_{m\to\infty} \lim_{N\to\infty} T(m, k, N) \geqq 2\delta\} \geqq P\{T \geqq 2\delta\} \geqq 2\delta.$$

Hence for each k there exists an $m(k)$ such that

$$P^0\{\lim_{N\to\infty} T(m(k), k, N) \geqq \delta\} \geqq \delta > 0. \tag{4.15}$$

Actually we may write P^x instead of P^0 in (4.15), for any $x \in \boldsymbol{R}$. Firstly, since $F_N - F_N = F_N$ we may take the starting point x in F_N. Secondly, if

$S_N = \inf\{t > 0 : X_t \in F_N\}$ then on $\{S_N < \infty\}$ $X_{S_N} \in F_N$ a.s. and

$$T(m, k, N, \omega) = S_N + T(m, k, N, \theta_{S_N}\omega)$$

so that for any x

$$\begin{aligned} P^x\{T(m, k, N) \geqq \delta\} \geqq E^x\{P^{X_{S_N}}\{T(m, k, N) \geqq \delta\}; S_N < \infty\} \\ +P^x\{S_N = \infty\} = E^x\{P^0\{T(m, k, N) \geqq \delta\}; S_N < \infty\} \\ +P^x\{S_N = \infty\} \geqq P^0\{T(m, k, N) \geqq \delta\}. \end{aligned}$$

This together with (4.15) and the fact that $T(m, k, N)$ is decreasing in N immediately implies

$$(4.16)\quad P^x\{\lim_{N\to\infty} T(m(k), k, N) \geqq \delta\} \geqq \lim_{N\to\infty} P^x\{T(m(k), k, N) \geqq \delta\} \geqq \delta$$

(and by Blumenthal's zero-one law $\lim_{N\to\infty} T(m(k), k, N) > 0$ a.e. $[P^x]$). Put

$$S = S(k) = \lim_{N\to\infty} T(m(k), k, N).$$

By the definition of $T(m, k, N)$, and the continuity of l^x for $x \in F$ one has a.s.

$$(4.17)\qquad \sup_{s\leqq S} |l_s^x - l_s^y| \leqq \frac{1}{k} \text{ for } x, y \in F, |x-y| \leqq (m(k))^{-1}$$

(the important point is that (4.17) also holds for $s = S$). If we define the iterates of $S(k)$ for some fixed k by

$$S_1 = S(k), \cdots, S_j = S_{j-1} + S_1 \cdot \theta_{S_{j-1}}$$

then also

$$\begin{aligned} (4.18)\quad &\sup_{S_{j-1}\leqq s\leqq S_j} |(l_s^x(\omega) - l_{S_{j-1}}^x(\omega)) - (l_s^y(\omega) - l_{S_{j-1}}^y(\omega))| \\ &= \sup_{s\leqq S_1 \cdot \theta_{S_{j-1}}(\omega)} |l_s^x(\theta_{S_{j-1}}\omega) - l_s^y(\theta_{S_{j-1}}\omega)| \leqq \frac{1}{k} \end{aligned}$$

a.s. for all $x, y \in F$, $|x-y| \leqq (m(k))^{-1}$. For any ω which satisfies (4.18) for all j also

$$\sup_{\substack{x, y\in F \\ |x-y|\leqq (m(k))^{-1}}} \sup_{s\leqq S_j} |l_s^x(\omega) - l_s^y(\omega)| \leqq j/k.$$

But by (4.16),

$$P^0\{S_j - S_{j-1} \geqq \delta | \mathscr{F}_{S_{j-1}}\} \geqq \delta \text{ on } \{S_{j-1} < \infty\},$$

and hence

$$(4.19)\quad P^0\{\sup_{s\le t}|l_s^x-l_s^y| \geqq k^{-\frac{1}{4}} \text{ for some } x, y\in F, |x-y| \leqq (m(k))^{-1}\}$$
$$\leqq P^0\{S_{[k^{\frac{1}{2}}]} \leqq t\} \leqq P^0\{S_j-S_{j-1} \geqq \delta \text{ for at most } (t\delta^{-1}+1) \text{ values of } j \leqq k^{\frac{1}{2}} \text{ and } S_{[k^{1/2}]} < \infty\}$$
$$\leqq \sum_{i\leqq t\delta^{-1}+1} \binom{[k^{\frac{1}{2}}]}{i}\delta^i(1-\delta)^{[k^{1/2}]-i} \to 0\ (k\to\infty).$$

Just as in [3], V.3.30, let $\phi_\omega^N: F \to C[0, N]$ be the map which takes $x\in F$ into the continuous function $l_\cdot^x(\omega)$ on $[0, N]$. Then for each $N > 0$, ϕ_ω^N is a.s. uniformly continuous by (4.19) (using the sup norm on $C[0, N]$). As in [3], V.3.31, this shows that under (4.14) l a.s. has a version such that the map $(x, t) \to l_t^x(\omega)$ is continuous for all ω.

The following corollary is immediate from Remark 1.

COROLLARY 2. *If $\{X_t\}_{t\geqq 0}$ is as in theorem 4 and (1.11) holds, then a continuous version of $l_t^\cdot$ satisfies the Hölder condition*

$$(4.20)\quad \sup_{|x-y|\leqq u}\ \sup_{t\leqq N}|l_t^x-l_t^y| \leqq C_N(\omega)\int_{0\leqq v\leqq u}[(-\log v)\vee 1]\,d\delta^{\frac{1}{2}}(v)$$

for some $C_N(\omega)$ which is a.s. finite. (We do not need the restriction $|x|$, $|y| \leqq M$ here since the closed range $\bar{R}_t(\omega) = \{X_s(\omega): 0 \leqq s \leqq t\}$ is a.s. contained in some bounded $[-M(\omega), +M(\omega)]$ and $l_s^x = 0$ for $s \leqq t$, $x\notin \bar{R}_t(\omega)$.)

EXAMPLES.

a) If $\sigma^2 > 0$, X_t has a bounded continuous density p_t which is the convolution of the normal density $(2\pi\sigma^2 t)^{-\frac{1}{2}}\exp\{-x^2(2t\sigma^2)^{-\frac{1}{2}}\}$ and some probability distribution ([13], proof of theorem 1a, p. 17). Hence

$$|p_t(x)-p_t(y)| \leqq \sup_z \frac{1}{(2\pi\sigma^2 t)^{\frac{1}{2}}}\left|\exp\left(-\frac{(x-z)^2}{2t\sigma^2}\right)-\exp\left(-\frac{(y-z)^2}{2t\sigma^2}\right)\right|$$
$$\leqq \sup_z\frac{1}{(2\pi\sigma^2 t)^{\frac{1}{2}}}\min\left(2, \left|\frac{1}{t\sigma^2}\int_{x-z}^{y-z} u\exp-\frac{u^2}{2t\sigma^2}\,du\right|\right)$$
$$\leqq \frac{1}{(2\pi\sigma^2 t)^{\frac{1}{2}}}\min\left(2, \frac{|y-x|}{(t\sigma^2)^{\frac{1}{2}}}\max_v v\exp-\frac{v^2}{2}\right)$$

and

$$|u^1(0)\{1-\psi^1(0,x)\}| = |u^1(0)-u^1(x)|$$
$$\leqq \int_0^{|x|^2}+\int_{|x|^2}^\infty e^{-t}|p_t(0)-p_t(x)|\,dt = 0\left(|x|\log\frac{1}{|x|}\right),\quad x\to 0.$$

In this case all points are regular (since $E^0e^{-T_x}\to 1$, $x\to 0$; see also [6] or [17] sect. 6,7) and a continuous local time exists, even by Boylan's theorem, [4].

b) An asymmetric Cauchy processs has characteristic function (4.1) with

$$\psi(\lambda) = -ia\lambda + C_1|\lambda| + iC_2\lambda \log|\lambda|$$

for some a, $C_1 > 0$, $0 < |C_2| \leqq 2C_1/\pi$. This satisfies (4.7) and all points are regular ([16], [6]). Thus l_t^x exists as a density of the occupation time measure μ_t (i.e. satisfying (1.9)). But (1.12) also holds, since

$$\int_{-\infty}^{+\infty} \operatorname{Re} \frac{1}{\alpha+\psi(x)}\, d\lambda \geqq \int_{|\lambda|\geqq\alpha} \frac{\alpha+C_1|\lambda|}{(\alpha+C_1|\lambda|)^2+(C_2\lambda\log|\lambda|-a\lambda)^2}\, d\lambda$$
$$\sim 2C_1(C_2^2\log\alpha)^{-1}(\alpha\to\infty).$$

By theorem 4 no continuous version of $l_t^{\cdot}$ exists.

c) Let $a = 0$, $\sigma^2 = 0$ and

$$v(dx) = x^{-2}L(|x|)dx$$

for some slowly varying function L as $x \to 0$ as well as $x \to \infty$. Then X_t has a symmetric distribution with characteristic function (4.1) and

$$\text{(4.21)} \qquad \psi(\lambda) = \int_{-\infty}^{+\infty} (1-\cos\lambda x)\frac{L(|x|)}{x^2}\, dx \sim \pi|\lambda|L(|\lambda|^{-1}),\ \lambda\to\infty.$$

If L is such that (4.7) holds then all points are regular ([13], corollary 3.2, or [6]), but if in addition

$$\text{(4.22)} \qquad L(|x|) \leqq C\left(\log\frac{1}{|x|}\right)^2 \text{ for small } x,$$

then, by (4.21),

$$\liminf_{\alpha\to\infty}(\log\alpha)\int_{-\infty}^{+\infty}\operatorname{Re}\frac{1}{\alpha+\psi(\lambda)}\, d\lambda$$
$$\geqq \liminf_{\alpha\to\infty}(2\log\alpha)\int_{\alpha}^{\infty}\{\pi C\lambda(\log\lambda)^2\}^{-1}\, d\lambda > 0.$$

Thus (4.22) implies (1.12) and by theorem 4 no continuous version of $l_t^{\cdot}$ exists in this case. On the other hand, if

$$\text{(4.23)} \quad L(|x|) \geqq C^*\left(\log\frac{1}{|x|}\right)^{3+\varepsilon} \text{ for some } \varepsilon, C^* > 0 \text{ and small } |x|,$$

then, by (4.21), (4.7) holds and

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}(1-\cos x\lambda)\operatorname{Re}\frac{1}{1+\psi(\lambda)}\, d\lambda$$
$$\leqq \frac{x^2}{\pi}\int_{|\lambda|\leqq|x|^{-1}}\frac{\lambda^2}{1+\psi(\lambda)}\, d\lambda + \frac{2}{\pi}\int_{|\lambda|>|x|^{-1}}\frac{d\lambda}{1+\psi(\lambda)}$$
$$= 0\left(\log\frac{1}{|x|}\right)^{-2-\varepsilon};\ x\to 0.$$

Hence, if (4.23) holds, (1.11) is satisfied and a continuous local time exists by Theorem 4. By Corollary 2 a continuous version $l_t^{\cdot}$ even satisfies a.s.

$$\sup_{|x-y|\leqq u} |l_t^x - l_t^y| \leqq C_t^*(\omega)(\log |4|^{-1})^{-\varepsilon/2}$$

for $|u| \leqq e^{-1}$ and some $C^* < \infty$, or

$$|l_t^x - l_t^y| = 0\left(\log \frac{1}{|x-y|}\right)^{-\varepsilon/2} \text{ as } |x-y| \to 0.$$

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