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An analytic construction of degenerating abelian varieties over complete rings

AN ANALYTIC CONSTRUCTION OF DEGENERATING ABELIAN VARIETIES OVER COMPLETE RINGS

by

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This paper is a sequel to the earlier paper on the construction of degenerating curves. The basic ideas of the 2 papers are very similar and we refer the reader to the introduction of that paper for discussion and motivation. We limit ourselves here to an outline of the contents of the present paper.

Throughout this paper, A will stand for a fixed excellent integrally closed noetherian ring with quotient field K, $I \subset A$ will be an ideal such that $I = \sqrt{I}$ and A is complete with respect to the I-adic topology. We will be interested in group schemes over the base scheme:

$$S = \operatorname{Spec}(A)$$

which have particular properties over

(1) the generic point $\eta \in S$

(2) the closed subscheme $S_0 = \operatorname{Spec}(A/I)$.

If X is a scheme over S, X_η and X_0 will stand for the induced schemes over $\{\eta\}$ and S_0. A smooth commutative group scheme G of finite type over any base Z will be called *semi-abelian* if all its fibres G_z are connected algebraic groups without unipotent radical, i.e., each G_z is an extension of an abelian variety by a torus ($=$ a form of G_m^r over $k(z)$). The *rank function* will be the map

$$z \mapsto r(z)$$

associating to each point z the dimension of the torus part of G_z; r is easily checked to be upper $\frac{1}{2}$-continuous (e.g., by looking at the cardinali-

ty of the fibres of the subscheme $\mathrm{Ker}(n_G) \subset G$, étale over Z where n is prime to the residue characteristics of Z). The semi-abelian group schemes G of constant rank are *globally over Z* extensions of an abelian scheme over Z by a torus over Z. We are interested in constructing semi-abelian group schemes G/S such that

(1) G_η is an abelian variety,

(2) G_0 has constant rank.

According to an idea of John Tate, if $r = \mathrm{rank}\,(G_0)$, then G should be canonically represented as a 'quotient' of a semi-abelian group scheme $\tilde{G}$ of constant rank r *over the whole of S* by a discrete subgroup Y of L-valued points of $\tilde{G}$ with $Y \cong \mathbf{Z}^r$. (L a finite algebraic extension of K). To simplify matters, we will consider only the case where G_0 is a split torus, i.e., $G_0 \cong \mathbf{G}_m^r \times S_0$, in which case $\tilde{G} \cong \mathbf{G}_m^r \times S$ and the points of Y are K-valued. Our plan is the following:

i) start with a set of periods $Y \subset \tilde{G}(K)$ where $\tilde{G} \cong \mathbf{G}_m^r \times S$, satisfying suitable conditions;

ii) construct a kind of compactification:
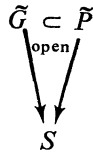
such that the action of Y by translation extends to $\tilde{P}$, and Y acts freely and discontinuously (in the *Zariski* topology) on $\tilde{P}_0$. Unlike the case of curves, $\tilde{P}$ is neither unique nor canonical!

iii) Take the I-adic completion $\tilde{\mathfrak{P}}$ of $\tilde{P}$, construct $\mathfrak{P} = \tilde{\mathfrak{P}}/Y$, algebrize $\mathfrak{P}$ to a scheme P projective over S, and take a suitable open subset $G \subset P$:
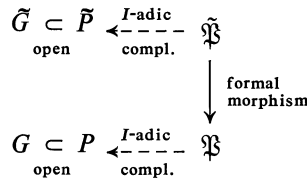
iv) prove that G is a semi-abelian group scheme over S independent of the choice of $\tilde{P}$, G_η is abelian, and $G_0 \cong \tilde{G}_0 \cong \mathbf{G}_m^r \times S_0$.

We will also show that this uniformization $\tilde{G} \to G$ is uniquely determined by G, i.e. if $\tilde{G}/Y_1 \cong \tilde{G}/Y_2$, then $Y_1 = Y_2$. However we will not discuss at all whether all semi-abelian G as in (iv) admit such a uniformization. This seems to be a fairly difficult question. In case dim $A = 1$, and A is

local, Raynaud and I have proven (independently) that all G's as in (iv) do admit uniformizations. Raynaud's method is highly analytic and mine is an application of my theory of 2-adic theta functions*. We both have reason to believe that our methods will extend to the general case, but this has not yet been done. Instead I conclude the paper with many examples. For me, one of the most enjoyable features of this research was the beauty of the examples which one works out without a great deal of extra effort. In fact, the non-uniqueness of $\tilde{P}$ gives one freedom to seek for the most elegant solutions in any particular case.

1. Periods

To begin our program, let $\tilde{G}$ be a given split torus over S. Then if X is the character group of $\tilde{G}$, for all $\alpha \in X$, the character is a canonical element

$$\mathscr{X}^\alpha \in \Gamma(\tilde{G}, \mathcal{O}_{\tilde{G}}).$$

Moreover $\tilde{G}$ is affine and can be described explicitly as:

$$\tilde{G} = \operatorname{Spec} A[\cdots, \mathscr{X}^\alpha, \cdots]_{\alpha \in X} \Big/ \left(\begin{matrix} \mathscr{X}^\alpha \cdot \mathscr{X}^\beta = \mathscr{X}^{\alpha+\beta} \\ \mathscr{X}^0 = 1 \end{matrix} \right)$$

Let:

$$\tilde{G}(K) = \text{group of } K\text{-valued points of } \tilde{G} \cong (K^*)^r.$$

Note that if $y \in G(K)$ and $\alpha \in X$, then the character $\mathscr{X}^\alpha$ takes a value on y which is an element $\mathscr{X}^\alpha(y) \in K^*$.

(1.1) DEFINITION: A set of *periods* is a subgroup $Y \subset \tilde{G}(K)$ isomorphic to $\mathbf{Z}^r$.

The only assumption that we will make about Y is that they admit a polarization, in the following sense:

(1.2) DEFINITION: A *polarization* for the periods Y is a homomorphism

$$\phi \colon Y \to X$$

such that:

i) $\mathscr{X}^{\phi(y)}(z) = \mathscr{X}^{\phi(z)}(y)$, all $y, z \in Y$,

ii) $\mathscr{X}^{\phi(y)}(y) \in I$ for all $y \in Y, y \neq 0$.

Note that by (ii) ϕ must be injective, hence also $[X : \phi Y] < +\infty$. Before going further, we stop to prove a basic lemma of a technical nature concerning periods and polarizations which will be very useful:

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[1] On the equations defining abelian varieties, Inv. Math., Vol 1 and 3.

(1.3) BASIC LEMMA: *Suppose that for every* $y \in Y$, $y \neq 0$, *a positive integer* n_y *is given. Then there exists a finite set of elements* $y_1, \cdots, y_k \in Y$, $y_i \neq 0$ *and a finite subset* $S \subset Y$ *such that for all* $z \in Y - S$,

$$\mathcal{X}^{\phi(z)}(y_i) \in [\mathcal{X}^{\phi(y_i)}(y_i)]^{n_{y_i}} \cdot A \text{ for some } i.$$

PROOF: Since Y is finitely generated, there is only a finite set of minimal prime ideals $\mathfrak{p} \subset A$ such that

$$\text{ord}_{\mathfrak{p}} \, \mathcal{X}^{\phi(y)}(z) \not\equiv 0, \text{ all } y, z \in Y.$$

Let these prime ideals be $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$, and let v_i be the valuation $\text{ord}_{\mathfrak{p}_i}$. The axioms for a polarization tell us that

$$Q_i(y, z) = v_i[\mathcal{X}^{\phi(y)}(z)]$$

is a positive semi-definite quadratic form on Y. We can extend Q_i uniquely to a R-valued quadratic form on $Y \otimes R$. Since A is integrally closed, the assertion to be proved is equivalent to:

$$Q_j(z, y_i) \geqq n_{y_i} Q_j(y_i, y_i), \text{ for all } j.$$

For all $y \in Y$, $y \neq 0$, let

$$C_y = \left\{ z \in Y \otimes R \left| \begin{array}{l} Q_j(z, y) > n_y Q_j(y, y) \text{ for} \\ \text{all } j \text{ such that } y \notin \text{Null-space } (Q_j) \end{array} \right. \right\}$$

Note that C_y is a convex open subset of $Y \otimes R$ such that $\lambda C_y \subset C_y$ if $\lambda \in R$, $\lambda \geqq 1$. To prove the lemma, it suffices to check that:

(*) $$C_{y_1} \cup \cdots \cup C_{y_k} \supset Y \otimes R - (\text{a compact set})$$

for some $y_1, \cdots, y_k$. But first I claim that

(**) $$\bigcup_{N=1}^{\infty} \bigcup_{\substack{y \in Y \\ y \neq 0}} \frac{1}{N} C_y = Y \otimes R - (0).$$

In fact, suppose $z \in Y \otimes R$ and $z \neq 0$. If we approximate z by an element of $Y \otimes Q$ which lies in the same Null-spaces of the Q_j that z does, and multiply this element by a large positive integer, we find a $y \in Y$ such that

$$z \in \text{Null-space } (Q_j) \Rightarrow y \in \text{Null-space } (Q_j)$$

$$z \notin \text{Null-space } (Q_j) \Rightarrow Q_j(y, z) > 0.$$

Therefore $N \cdot z \in C_y$ if $N \gg 0$, hence $z \in 1/N \, C_y$. This proves (**). By the compactness of the unit sphere in $Y \otimes R$, it follows that there are $y_1, \cdots, y_k \in Y$, $y_i \neq 0$, and positive integers N_i such that

$$\bigcup_{i=1}^{k} \frac{1}{N_i} C_{y_i} \supset (\text{unit sphere}).$$

Then $\bigcup_{i=1}^{k} C_{y_i}$ contains all spheres of radius $\geqq$ max (N_i) so $(*)$ is proven.
$$QED$$

Another basic fact which is very useful is:

(1.4) LEMMA: *For any* $\alpha \in X$, *there exists an* $n \geqq 1$ *such that* $\mathscr{X}^{n\phi(y)+\alpha}(y) \in A$ *for all* $y \in Y$.

PROOF. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ and $Q_1, \cdots, Q_n$ be as before. Note that using $\phi: Y \hookrightarrow X$, we can identify X with a subgroup of $Y \otimes R$ so that

$$Q_i(\alpha, y) = \mathrm{ord}_{\mathfrak{p}_i} \mathscr{X}^{\alpha}(y), \qquad \alpha \in X, \, y \in Y.$$

We must show that for $n \gg 0$,

$$Q_i(ny+\alpha, y) \geqq 0, \text{ all } i, \text{ all } y.$$

But

$$nQ_i(ny+\alpha, y) = Q_i\left(ny+\frac{\alpha}{2}, ny+\frac{\alpha}{2}\right) - \tfrac{1}{4}Q_i(\alpha, \alpha).$$

Since Y projects into a lattice in $Y \otimes R/$Null-space Q_i, it follows that if n is large enough, then for all y, either

a) $y \in$ Null-space (Q_i)

or

b) $ny+\alpha/2$ is arbitrarily 'big' in $Y \otimes R/$Null-space Q_i. In either case, $Q_i(ny+\alpha, y) \geqq 0.$ QED

2. Relatively complete models

We now return to our basic problem: given a set of periods Y for which a polarization exists, construct *canonically* a 'quotient' of $\tilde{G}$ by Y. The main tool will be the following:

(2.1) DEFINITION. A *relatively complete model* of $\tilde{G}$ with respect to periods Y and polarization ϕ will be a collection of 5 pieces of data:

a) an integral scheme $\tilde{P}$ locally of finite type over A,

b) an open immersion $i: \tilde{G} \hookrightarrow \tilde{P}$ (we will henceforth identify $\tilde{G}$ with its image in $\tilde{P}$),

c) an invertible sheaf $\tilde{L}$ on $\tilde{P}$,

d) an action of the torus $\tilde{G}$ on $\tilde{P}$ and $\tilde{L}$ (we denote the action of an S'-valued point a of $\tilde{G}$ by

$$T_a : \tilde{P} \to \tilde{P}$$

and

$$T_a^* : \tilde{L} \to \tilde{L}),$$

e) and action of Y on $\tilde{P}$ and $\tilde{L}$ (we denote the action of $y \in Y$ by

$$S_y : \tilde{P} \to \tilde{P}$$

and

$$S_y^* : \tilde{L} \to \tilde{L}),$$

such that

(i) there exists an open $\tilde{G}$-invariant subset $U \subset \tilde{P}$ of finite type over S such that

$$\tilde{P} = \bigcup_{y \in Y} S_y(U),$$

(ii) for all valuations v on $R(\tilde{G})$, the field of rational functions on $\tilde{G}$, for which $v \geqq 0$ on A, v has a center on $\tilde{P}$ if and only if

$$* \; [\forall \; \alpha \in X, \exists y \in Y \text{ such that } v(\mathscr{X}^\alpha(y) \cdot \mathscr{X}^\alpha) \geqq 0].$$

(These valuations are allowed to have rank > 1!)

(iii) The action of $\tilde{G}$ and Y on $\tilde{P}$ extends the 'translation" action of $\tilde{G}$ and Y on $\tilde{G}$ given by the group law on $\tilde{G}$.

(iv) The actions of Y and $\tilde{G}$ on $\tilde{L}$ satisfy the identity:

$$T_a^* S_y^* = \mathscr{X}^{\phi(y)}(a) \cdot S_y^* T_a^*$$

all $y \in Y$ all S'-valued points a of $\tilde{G}$.

(v) $\tilde{L}$ is ample on $\tilde{P}$.[2]

We turn first to the construction of relatively complete models. We need one more definition:

(2.2) DEFINITION. A *star* Σ is a finite subset of X such that $0 \in \Sigma$, $-\Sigma = \Sigma$ and Σ contains a basis of X.

Let θ be an indeterminate and consider the big graded ring:

$$\mathscr{R} = \sum_{k=0}^{\infty} \left(K[\cdots, \mathscr{X}^\alpha, \cdots] \middle/ \left(\begin{matrix} \mathscr{X}^\alpha \cdot \mathscr{X}^\beta - \mathscr{X}^{\alpha+\beta} \\ \mathscr{X}^0 - 1 \end{matrix} \right) \right) \cdot \theta^k.$$

Let Y act on $\mathscr{R}$ via operators S_y^*:

$$S_y^*(c) = c, \text{ all } c \in K,$$
$$S_y^*(\mathscr{X}^\alpha) = \mathscr{X}^\alpha(y) \cdot \mathscr{X}^\alpha, \text{ all } \alpha \in X,$$
$$S_y^*(\theta) = \mathscr{X}^{\phi(y)}(y) \cdot \mathscr{X}^{2\phi(y)} \cdot \theta.$$

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[2] By definition, we take this to mean that the sections of L^n, $n \geqq 1$, form a basis of the topology of $\tilde{P}$. In EGA, II. 4.5., Grothendieck only defines ample on quasi-compact schemes, but this seems to be the best property among his equivalent defining properties for our purposes.

Let

(2.3) DEFINITION. $R_{\phi,\Sigma}$ = subring of $\mathscr{R}$ generated over A by the elements $S_y(\mathscr{X}^\alpha \theta)$, $y \in Y$, $\alpha \in \Sigma$, i.e.,

$$R_{\phi,\Sigma} = A[\cdots, \mathscr{X}^{\phi(y)+\alpha}(y) \cdot \mathscr{X}^{2\phi(y)+\alpha} \cdot \theta, \cdots].$$

This ring is only what we want in case $\mathscr{X}^{\phi(y)+\alpha}(y) \in A$ for all $y \in Y$, $\alpha \in \Sigma$, i.e., in case

$$R_{\phi,\Sigma} \subset A[\cdots, \mathscr{X}^\alpha \cdot \theta, \cdots].$$

However, we can always make sure that this extra condition is satisfied if we replace the polarization ϕ by $n\phi$ for a large enough n. This follows from lemma (1.4). From now on, we assume that ϕ has been so chosen that $\mathscr{X}^{\phi(y)+\alpha}(y) \in A$, all $y \in Y$, $\alpha \in \Sigma$.

Now consider $\mathrm{Proj}(R_{\phi,\Sigma})$. I claim it is a relatively complete model of $\tilde{G}$! Since it is Proj of a graded ring generated by elements of degree 1, it carries a canonical ample invertible sheaf $\mathcal{O}(1)$. Moreover, the automorphisms S_y^* of $R_{\phi,\Sigma}$ induce automorphisms S_y of $\mathrm{Proj}(R_{\phi,\Sigma})$ and a compatible automorphism S_y^* of $\mathcal{O}(1)$, i.e., an action of Y on $\mathrm{Proj}(R_{\phi,\Sigma})$ and on $\mathcal{O}(1)$. To get an action of $\tilde{G}$ on $\mathrm{Proj}(R_{\phi,\Sigma})$ and on $\mathcal{O}(1)$, it suffices to define, for every A-algebra B and every B-valued point a of $\tilde{G}$, an automorphism T_a^* of $B \otimes_A R_{\phi,\Sigma}$, in a way which is functorial and compatible with compositions. In fact, let

$$T_a^*(c) = c, \text{ all } c \in A,$$
$$T_a^*(\mathscr{X}^\alpha) = \mathscr{X}^\alpha(a) \cdot \mathscr{X}^\alpha, \text{ all } \alpha \in X,$$
$$T_a^*(\theta) = \theta.$$

This clearly has all these properties and gives us our action. Since:

$$T_a^* \cdot S_y^* = R \cdot S_y^* \cdot T_a^*$$

where

$$R(c) = c, \text{ all } c \in A$$
$$R(\mathscr{X}^\alpha) = \mathscr{X}^\alpha, \text{ all } \alpha \in X$$
$$R(\theta) = \mathscr{X}^{2\phi(y)}(a) \cdot \theta,$$

it follws that the T_a's and S_y's skew-commute as required for the polarization 2ϕ.

Next $\mathrm{Proj}(R_{\phi,\Sigma})$ is covered by the affine open sets:

$$U_{\alpha,y} = \mathrm{Spec}\, A \left[\cdots, \frac{\mathscr{X}^{\phi(z)+\beta}(z)}{\mathscr{X}^{\phi(y)+\alpha}(y)} \cdot \mathscr{X}^{2\phi(z-y)+\beta-\alpha}, \cdots \right]_{\substack{\beta \in \Sigma \\ z \in Y}}.$$

Although we have here an infinite number of open sets, the action S_{y_0} of $y_0 \in Y$ carries $U_{\alpha,y}$ to $U_{\alpha,y+y_0}$, so there are only a finite number of orbits mod Y in this collection of open sets. All the affine rings are integral domains contained in $K(\cdots, \mathscr{X}^\alpha, \cdots)$. Moreover,

$$U_{0,0} = \operatorname{Spec} A[\cdots, \mathscr{X}^{\phi(z)+\beta}(z) \cdot \mathscr{X}^{2\phi(z)+\beta}, \cdots]_{\substack{z \in Y \\ \beta \in \Sigma}}$$

$$= \operatorname{Spec} A[\cdots, \mathscr{X}^\beta, \cdots]_{\beta \in \Sigma}$$

$$= \operatorname{Spec} A[\cdots. \mathscr{X}^\beta, \cdots]_{\beta \in X} = \tilde{G}.$$

Thus $\operatorname{Proj}(R_{\phi,\Sigma})$ is an integral scheme over A containing $\tilde{G}$ as a dense open subset. Next we prove:

(2.4) PROPOSITION. $U_{\alpha,y}$ is of finite type over A.

PROOF. Since $U_{\alpha,y}$ is isomorphic to $U_{\alpha,z}$ for any $y, z \in Y$, is suffices to check this for $U_{\alpha,0}$. The affine ring of $U_{\alpha,0}$ is generated by the infinite set of monomials:

$$M_{\beta,y} = \mathscr{X}^{\phi(y)+\beta}(y) \cdot \mathscr{X}^{2\phi(y)+\beta-\alpha}.$$

It is easy to check that:

$$(*) \quad M_{\beta,y} = [\mathscr{X}^{2\phi(y-z)-\alpha+\beta}(z)] \cdot M_{\alpha,z} \cdot M_{\beta,y-z}.$$

Now for all z, choose a positive integer n_z such that

$$\mathscr{X}^{\phi(z)}(z)^{2n_z-3} \cdot \mathscr{X}^{-\alpha+\beta}(z) \in A, \text{ all } \beta \in \Sigma.$$

Then by the basic lemma (1.3), there is a finite set $z_1, \cdots, z_k$ such that for all $y \in Y$ except for a finite set $S \subset Y$,

$$\mathscr{X}^{\phi(y)}(z_i) \in [\mathscr{X}^{\phi(z_i)}(z_i)]^{n_{z_i}} \cdot A, \text{ some } i.$$

Therefore

$$\mathscr{X}^{2\phi(y-z_i)-\alpha+\beta}(z_i) = [\text{elt. of } A \cdot \mathscr{X}^{\phi(z_i)}(z_i)^{n_{z_i}}]^2 \cdot \mathscr{X}^{-2\phi(z_i)}(z_i) \cdot \mathscr{X}^{-\alpha+\beta}(z_i)$$

$$= (\text{elt. of } A) \cdot \mathscr{X}^{\phi(z_i)}(z_i) \in I,$$

so that

$$M_{\beta,y} = (\text{elt. of } I) \cdot M_{\alpha,z_i} \cdot M_{\beta,y-z_i}.$$

Thus $M_{\alpha_1 z}, \cdots, M_{\alpha,z_k}$ plus the $M_{\beta,y}$ with $y \in S$ generate the whole ring.

$$QED$$

It follows that if $U = \bigcup_{\alpha \in \Sigma} U_{\alpha,0}$, then U is an open subset of $\operatorname{Proj}(R_{\phi,\Sigma})$ of finite type over A such that

$$\bigcup_{y \in Y} S_y(U) = \operatorname{Proj}(R_{\phi,\Sigma}).$$

It remains to check the following completeness property for $\text{Proj}(R_{\phi,\Sigma})$: if v is a valuation of $R(\tilde{G})$, $v \geqq 0$ on A, then

$$\begin{bmatrix} v \text{ has a center} \\ \text{on Proj } (R_{\phi,\Sigma}) \end{bmatrix} \Leftrightarrow \begin{bmatrix} \forall \alpha \in x, \exists y \in Y \text{ such that} \\ v(\mathscr{X}^\alpha(y) \cdot \mathscr{X}^\alpha) \geqq 0 \end{bmatrix}.$$

Clearly v has a center on $\text{Proj}(R_{\phi,\Sigma})$ if and only if

$$\min_{\substack{\alpha \in \Sigma \\ y \in Y}} v(\mathscr{X}^{\phi(y)+\alpha}(y) \cdot \mathscr{X}^{2\phi(y)+\alpha}) \text{ exists.}$$

As for the other side, note that it is equivalent to any of the statements:

$$(*)'[\forall z \in Y, \exists y \in Y \text{ such that } v(\mathscr{X}^{\phi(z)}(y) \cdot \mathscr{X}^{\phi(z)}) \geqq 0]$$

$$(*)''[\forall z \in Y, \exists \alpha \in X \text{ such that } v(\mathscr{X}^\alpha(z) \cdot \mathscr{X}^{\phi(z)}) \geqq 0]$$

$$(*)'''[\forall z \in Y, \exists n \geqq 1 \text{ such that } v(\mathscr{X}^{\phi(z)}(z)^n \cdot \mathscr{X}^{\phi(z)}) \geqq 0].$$

Now to check the implication '$\Rightarrow$', suppose that:

$$\min_{\substack{\alpha \in \Sigma \\ y \in Y}} v(\mathscr{X}^{\phi(y)+\alpha}(y) \cdot \mathscr{X}^{2\phi(y)+\alpha}) = v(\mathscr{X}^{\phi(y_0)+\alpha_0}(y_0) \cdot \mathscr{X}^{2\phi(y_0)+\alpha_0}).$$

If $(*)''$ is false, then take the $z \in Y$ for which $v(\mathscr{X}^\alpha(z) \cdot \mathscr{X}^{\phi(z)}) < 0$, all $\alpha \in X$, and simply note that

$$v(\mathscr{X}^{\phi(y_0+z)+\alpha_0}(y_0+z) \cdot \mathscr{X}^{2\phi(y_0+z)+\alpha_0})$$
$$= v(\mathscr{X}^{\phi(y_0)+\alpha_0}(y_0) \cdot \mathscr{X}^{2\phi(y_0)+\alpha_0}) + v(\mathscr{X}^{\phi(z)}) + v(\mathscr{X}^{2\phi(y_0)+\phi(z)+\alpha_0}(z) \cdot \mathscr{X}^{\phi(z)})$$
$$< v(\mathscr{X}^{\phi(y_0)+\alpha_0}(y_0) \cdot \mathscr{X}^{2\phi(y_0)+\alpha_0}).$$

Conversely, assume that $(*)'''$ holds. For all $y \in Y$, choose n_y large enough so that

$$v(\mathscr{X}^{\phi(y)}(y)^{n_y} \cdot \mathscr{X}^{2\phi(y)}) \geqq 0$$

and

$$\mathscr{X}^{\phi(y)}(y)^{n_y-1} \cdot \mathscr{X}^\alpha(y) \in A, \text{ all } \alpha \in \Sigma.$$

By the basic lemma (1.3), there exist $y_1, \cdots, y_k \in Y$ such that for all $z \notin S$, a finite set, $\mathscr{X}^{\phi(z)}(y_i) \in [\mathscr{X}^{\phi(y_i)}(y_i)]^{n_{y_i}} \cdot A$ for some i. But then

$$v(\mathscr{X}^{\phi(z)+\alpha}(z) \cdot \mathscr{X}^{2\phi(z)+\alpha})$$
$$= v(\mathscr{X}^{\phi(z-y_i)+\alpha}(z-y_i) \cdot \mathscr{X}^{2\phi(z-y_i)+\alpha})$$
$$\quad + v(\mathscr{X}^{2\phi(z)}(y_i) \cdot \mathscr{X}^{-\phi(y_i)+\alpha}(y_i) \cdot \mathscr{X}^{2\phi(y_i)})$$
$$\geqq v(\mathscr{X}^{\phi(z-y_i)+\alpha}(z-y_i) \cdot \mathscr{X}^{2\phi(z-y_i)+\alpha})$$
$$\quad + v(\mathscr{X}^{\phi(y_i)}(y_i)^{n_{y_i}} \cdot \mathscr{X}^{2\phi(y_i)})$$
$$\quad + v(\mathscr{X}^{\phi(y_i)}(y_i)^{n_{y_i}-1} \cdot \mathscr{X}^\alpha(y_i))$$
$$\geqq v(\mathscr{X}^{\phi(z-y_i)+\alpha}(z-y_i) \cdot \mathscr{X}^{2\phi(z-y_i)+\alpha})$$

so that the minimum in question exists for some $z \in S$. This completes the proof of:

(2.5) THEOREM. *Let $\tilde{G}$ be a split torus over S, let $Y \subset \tilde{G}(K)$ be a set of periods and $\phi: Y \to X$ a polarization. Then if ϕ is replaced by $n\phi$ for $n \in \mathbf{Z}$ sufficiently large, $\mathrm{Proj}(R_{\phi, \Sigma})$ is a relatively complete model of $\tilde{G}$ over S relative to Y and 2ϕ.*

3. The Construction of the quotient

We can now forget about $\mathrm{Proj}(R_{\phi, \Sigma})$ and deal with an arbitrary relatively complete model $\tilde{P}$. The first thing I want to prove is that $\tilde{P}$ is not 'too much' bigger than $\tilde{G}$:

(3.1) PROPOSITION. *Let $y \in Y$ and let $f = \mathscr{X}^{\phi(y)}(y)$. Then in the open set $\tilde{P}_f \subset \tilde{P}$, $\mathscr{X}^{\phi(y)}$ is a unit, i.e., $\in \Gamma(\tilde{P}_f, \mathcal{O}_{\tilde{P}}^*)$.*

PROOF. By the axioms for a polarization, it follows easily that for all $\alpha \in X$,

$$\mathscr{X}^{n\phi(y)+\alpha}(y) \in A \text{ if } n \gg 0.$$

Therefore $\mathscr{X}^{\alpha}(y) \in A_f$. In other words, the section y of $\tilde{G}$ over $\{\eta\}$ extends to a section over S_f too. For clarity, call this y'. Then we have automorphisms S_y and $T_{y'}$ of $\tilde{P}_f$ and S_y^* and $T_{y'}^*$, of $\tilde{L}_f = \tilde{L}|_{\tilde{P}_f}$. Since $S_y = T_{y'}$, on G_f, S_y equals $T_{y'}$, everywhere, so

$$S_y^* = \begin{pmatrix} \text{mult. by a unit } \lambda \\ \text{in } \tilde{P}_f \end{pmatrix} \cdot T_{y'}^*.$$

The law of skew-commutativity of S_y^* with the operators T_a^* shows that

$$\lambda(a+b) = \mathscr{X}^{\phi(y)}(b) \cdot \lambda(a)$$

for all points a, b of $\tilde{G}_\eta$ in some K-algebra B. Therefore, on $\tilde{G}_\eta$, the function $\lambda/\mathscr{X}^{\phi(y)}$ is constant along the fibres, i.e.,

$$\lambda = \zeta \cdot \mathscr{X}^{\phi(y)}, \zeta \in K.$$

But λ and $\mathscr{X}^{\phi(y)}$ are units on the bigger open set $\tilde{G}_f$, hence ζ must be a unit in A_f. Then λ and ζ^{-1} are units in $\tilde{P}_f$ so $\mathscr{X}^{\phi(y)}$ is also.   QED

(3.2) COROLLARY. $\tilde{G}_\eta = \tilde{P}_\eta$.

Next, let's look at the closed subscheme $\tilde{P}_0$ of $\tilde{P}$.

(3.3) PROPOSITION. *Every irreducible component of $\tilde{P}_0$ is proper over A/I.*

PROOF. First apply the completeness condition (ii) to prove that if Z

is any component of $\tilde{P}_0$ and if v is any valuation of its quotient field $R(Z)$, with $v \geqq 0$ on A/I, then v has a center on Z. In fact, let v_1 be a valuation of $R(\tilde{G})$, $v \geqq 0$ on A, whose center is Z, and let v_2 be the composite of the valuations v and v_1. Since for all $z \in Y$, if $n \gg 0$, $\mathscr{X}^{\phi(z)}(z)^n \cdot \mathscr{X}^{\phi(z)}$ is regular and zero at the generic point of Z by (3.1), it follows that $v_1(\mathscr{X}^{\phi(z)}(z)^n \cdot \mathscr{X}^{\phi(z)}) > 0$, hence $v_2(\mathscr{X}^{\phi(z)}(z)^n \cdot \mathscr{X}^{\phi(z)}) > 0$. So by the completeness condition, v_2 has a center on $\tilde{P}$, hence v has a center on Z.

The Proposition now follows from the rather droll:

(3.4) LEMMA. *Let $f: X \to Y$ be a morphism locally of finite type, with X an irreducible scheme but Y arbitrary. If f satisfies the valuative criterion for properness for all valuations, then f is proper.*

PROOF. The usual valuative criterion (cf. EGA. II-7-3) would hold if we know that f was of finite type. It suffices to prove that f is quasi-compact. To prove this, we may as well replace X by X_{red}, Y by Y_{red}; then looking locally on the base, we can assume Y is affine, say $\text{Spec}(A)$; and finally we can assume f is dominating. Then A is a subring of the function field $R(X)$ of X: let $\mathscr{X}$ be Zariski's Riemann Surface of $R(X)/A$ (cf. Zariski-Samuel, vol. II, p. 110). Set-theoretically, $\mathscr{X}$ is the set of valuations v of $R(X)$, $v \geqq 0$ on A. By the valuation criterion, every v has a center on X, so there is a natural map $\pi: \mathscr{X} \to X$ taking v to its center. Now $\mathscr{X}$ is a quasi-compact topological space and π is continuous and surjective. Therefore X is quasi-compact.   QED

(3.5) COROLLARY. *The closure $\bar{U}_0$ of U_0 in the scheme $\tilde{P}_0$ is proper over S_0.*

PROOF. Since U_0 is of finite type over S_0, $\bar{U}_0$ has only a finite number of irreducible components, and by the Proposition, each is of finite type over S_0.   QED

The next thing I want to prove is that Y acts freely and discontinuously on $\tilde{P}_0$ in the Zariski-topology:

(3.6) PROPOSITION. *Let $U \subset \tilde{P}$ be the open set given by the definition of a relatively complete model. Let $\bar{U}_0$ be the closure of U_0 in $\tilde{P}_0$. There is a finite subset $S \subset Y$ such that*

$$S_y(\bar{U}_0) \cap S_z(\bar{U}_0) = \phi$$

if $y - z \notin S$.

PROOF. Let $F \subset \tilde{P}$ be the closed subset which is the locus of geometric points left fixed by the action of $\tilde{G}$. The action of $\tilde{G}$ on the invertible sheaf $\tilde{L}|_F$ is a 1-dimensional representation of $\tilde{G}$ over the base scheme

F. Therefore for every *connected* subset $F' \subset F$, there is a character $\alpha \in X$ such that $\tilde{G}$ acts on $\tilde{L}|_{F'}$ via the character α. Moreover, if $y \in Y$, then $S_y(F')$ will be another connected subset of F, and by the skew-commutativity of the actions of Y and $\tilde{G}$, $\tilde{G}$ will act on $\tilde{L}|_{S_y(F')}$ through the character $\alpha + \phi(y)$. Now $F \cap \overline{U}_0$ has only a finite number of connected components: let $\alpha_1, \cdots, \alpha_n$ be the characters $\tilde{G}$ associated to the action of $\tilde{G}$ on $\tilde{L}$ on these sets. Then $\tilde{G}$ acts on $\tilde{L}$ at the points of $F \cap S_y(\overline{U}_0)$ through the characters $\alpha_1 + y, \cdots, \alpha_n + y$. Now suppose $S_y(\overline{U}_0) \cap S_z(\overline{U}_0) \neq \phi$. Since this intersection is proper over $\mathrm{Spec}(A/I)$, by the Borel fixed point theorem (cf. A. Borel-Linear algebraic groups; Benjamin, 1959; pag. 242, th. 10.4):

$$F \cap S_y(\overline{U}_0) \cap S_z(\overline{U}_0) \neq \phi.$$

Looking at the action of $\tilde{G}$ on $\tilde{L}$ here, it follows that one of the characters $\alpha_i + y$ must equal one of the characters $\alpha_j + z$. Let $S = \{\cdots, \alpha_i - \alpha_j, \cdots\}$. Thus $y - z \in S$.

<div align="right">QED</div>

(3.7) COROLLARY. *Y acts freely on* $\tilde{P}_0$.

PROOF. If some $y \in Y$, $y \neq 0$, had a fixed point x in $\tilde{P}_0$, then since y has infinite order, x would be left fixed by an infinite subgroup of Y and this would contradict the Proposition.   QED

(3.8) THEOREM. $\tilde{P}_0$ *is connected.*

PROOF. Note that since A is complete in the I-adic topology, and A has no idempotents, A/I has no idempotents either, i.e., S_0 is connected. Therefore $\tilde{G}_0$ is a connected open subset of $\tilde{P}_0$ and it determines a canonical connected component of $\tilde{P}_0$. Now suppose there is a 2nd connected component. Choose a point $x \in \tilde{P}_0$ in this 2nd component and let v be a discrete rank 1 valuation of $R(\tilde{G})$, $v \geqq 0$ on A, with center x. Let $A' = \{x \in K | v(x) \geqq 0\}$. Let $S' = \mathrm{Spec}(A')$ and $\tilde{P}' = \tilde{P} \times_s S'$. Now A' is a discrete, rank 1, valuation ring with quotient field K, so S and S' have the 'same' generic point, and $\tilde{P}, \tilde{P}'$ have the 'same' generic fibre. Let $\tilde{P}''$ be the closure in $\tilde{P}'$ of its generic fibre. $\tilde{P}''$ is an integral scheme with the same quotient field as $\tilde{P}$, namely $R(\tilde{G})$, and it is locally of finite type over the valuation ring A'. Now S' has only 2 points – its generic point and its closed point. Let $\tilde{P}_0''$ be the fibre of $\tilde{P}''$ over the closed point. There is a natural morphism

$$\tilde{P}_0'' \to \tilde{P}_0.$$

I claim that (*a*) $x \in$ Image, and (*b*) $\tilde{G}_0 \subset$ Image, hence the image meets $\geqq 2$ connected components, hence $\tilde{P}_0''$ is disconnected too. The reason

for (a) is that if R_v = valuation ring of v, then we are given morphisms:

$$
\begin{array}{ccc}
\mathrm{Spec}\ (R_v) & \longrightarrow & \tilde{P} \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
$$

hence we get a morphism $\mathrm{Spec}(R_v) \to \tilde{P}'$, which takes the generic point of $\mathrm{Spec}(R_v)$ to the generic fibre of $\tilde{P}'$. Therefore this morphism factors through $\tilde{P}''$. The image $x' \in \tilde{P}_0''$ of the closed point lies over x. The reason for (b) is that $\tilde{G} \subset \tilde{P}$, and $\tilde{G}$ is smooth of S; so $\tilde{G} \times_s S' \subset \tilde{P}'$, and it is smooth over S'; so $\tilde{G} \times_s S' \subset \tilde{P}''$.

Now because of the completeness property (ii) of $\tilde{P}$ we know exactly which valuations of $R(\tilde{G})$ have centers in $\tilde{P}''$ too. We have reduced the Theorem to:

(3.9) LEMMA. *Let A be a discrete rank 1 valuation ring with maximal ideal (π), let G be a split torus over $S = Spec(A)$, let P be an integral scheme, locally of finite type over S containing G as a dense open subset. Assume:*

1) *the generic fibres G_η, P_η are equal,*

2) *for all valuations v of $R(G)$, $v \geqq 0$ on A and $v(\pi) > 0$, v has a center on P if and only if for all $\alpha \in X$, (the character group of G), $-nv(\pi) \leqq v(\mathscr{X}^\alpha) \leqq nv(\pi)$, if $n \gg 0$.*

Then the closed fibre P_0 of P is connected.

PROOF. Introduce a basis in X, and let $\mathscr{X}_1, \cdots, \mathscr{X}_r$ be the corresponding characters. For all n, let

$$
\begin{aligned}
P^{(n)} &= \mathrm{Spec}\ A[\pi^n \mathscr{X}_1 \cdot \pi^n \mathscr{X}_1^{-1}, \cdots, \pi^n \mathscr{X}_r, \pi^n \cdot \mathscr{X}_r^{-1}] \\
&\cong \mathrm{Spec}\ A[U_1, V_1, \cdots, U_r, V_r]/(U_1 V_1 - \pi^{2n}, \cdots, U_r V_r - \pi^{2n}).
\end{aligned}
$$

This scheme is a relative complete intersection in A_s^{2r} over S and is smooth over S, hence regular, outside a subset of codimension 2. Therefore $P^{(n)}$ is a normal scheme. $P^{(n)}$ and P have the same function field, so let $Z^{(n)} \subset P^{(n)} \times_A P$ be the join of this birational correspondence. By (1) and (2), all valuations v of $R(G)$ with a center on $P^{(n)}$ also have a center on P, so $Z^{(n)} \to P^{(n)}$ satisfies the valuative criterion for properness. Since $Z^{(n)}$ is at least locally of finite type over $P^{(n)}$, by lemma (3.4), $Z^{(n)}$ is proper over $P^{(n)}$. Therefore by Zariski's connectedness theorem, all fibres of $Z^{(n)}$ over $P^{(n)}$ are connected. Now the closed fibre of $P^{(n)}$ is isomorphic to:

$$
\mathrm{Spec}\ (k[U, V]/(U \cdot V)) \underset{\mathrm{Spec}\ k}{\times} \cdots \underset{\mathrm{Spec}\ (k)}{\times} \mathrm{Spec}\ (k[U, V]/(U \cdot V))
$$

where $k = A/(\pi)$, which is certainly connected. Therefore the closed fibre of $Z^{(n)}$ is connected. Therefore if

$$W_n = \overline{p_2(Z_0^{(n)})} \subset P,$$

W_n is connected too. But I claim that $P_0 = \bigcup_n W_n$. In fact, every valuation v with a center x on P_0 has a center on *some* $P^{(n)}$ because of (2). Therefore x lifts to a point of $Z^{(n)}$ for some n, hence $x \in W_n$. This shows that P_0 is connected.   QED

We are now ready to begin the construction of G. The first step is:

(3.10) THEOREM. *For every $n \geq 1$, there exists a scheme P_n, projective over A/I^n, an ample sheaf $\mathcal{O}(1)$ on P_n, and an étale surjective morphism*:

$$\pi: \tilde{P} \times_A A/I^n \to P_n$$

such that set-theoretically, $\pi(x) = \pi(y)$ if and only if x and y are in the same Y-orbit, and such that $\mathcal{O}(1)$ on $\tilde{P} \times_A A/I^n$ is the pull-back of $\mathcal{O}(1)$ on P_n.

PROOF. First, let $k \geq 1$ be an integer such that under the action of the subgroup $kY \subset Y$, no 2 points of any open set

$$S_y(U) \times_A A/I^n$$

are identified. Then we can form a quotient:

$$\pi' : \tilde{P} \times_A A/I^n \to P_n'$$

by the subgroup kY by the simple device of gluing these basic open sets together on bigger overlaps. Observe that since Y acts on $\mathcal{O}(1)$ we get a 'descended' form of $\mathcal{O}(1)$ on P_n'.

Choose coset representatives $y_1, \cdots, y_l \in Y$ for the cosets of kY in Y. Now notice that the restriction of π':

$$\text{Res}(\pi') : \bigcup_{y \in \{y_1, \cdots, y_l\}} S_y(U) \times_A A/I \to P_n'$$

is surjective, hence so is the restriction:

$$\text{Res}(\pi') : \bigcup_{y \in \{y_1, \cdots, y_l\}} S_y(\overline{U_0}) \to P_n'.$$

But the scheme on the left is a finite union of schemes proper over A/I. Therefore P_n' is proper over A/I^n. Moreover $\mathcal{O}(1)$ on P_n' pulls back to $\mathcal{O}(1)$ on the left, and $\mathcal{O}(1)$ here is ample. Therefore $\mathcal{O}(1)$ on P_n' is ample too (for instance, by Nakai's criterion, cf. S. Kleiman – A note on the Nakai-Moisezon test for ampleness of a divisor; Amer. J. Math. *87* (1965), 221–226).

Finally, the finite group Y/kY acts freely on the projective scheme P'_n and on the ample sheaf $\mathcal{O}(1)$, so a quotient $P_n = P'_n/(Y/kY)$ exists. Moreover by descent, P_n carries an ample $\mathcal{O}(1)$ too, so P_n has all the required properties.   QED

These schemes P_n obviously fit together to form a formal scheme $\mathfrak{P}$ proper over A. Moreover the sheaves $\mathcal{O}(1)$ fit together into an ample sheaf $\mathcal{O}(1)$ on $\mathfrak{P}$. We now apply the fundamental *formal existence* theorem (EGA Ch. 3, (5.4.5)) of Grothendieck: this shows that $\mathfrak{P}$ is the formal completion of a unique scheme P, proper over A, and that $\mathcal{O}(1)$ on $\mathfrak{P}$ comes by completion from an $\mathcal{O}(1)$ on P relatively ample over A.

Now inside all of our schemes, we want to pick out a big open set:

(I.) $\displaystyle\bigcup_{y\in Y} S_y(\tilde{G}) \subset \tilde{P}$

(II.) $G_n =^{def} \displaystyle\bigcup_{y\in Y} (S_y(\tilde{G}) \times_A A/I^n)/Y \subset P_n$

(III.) $\displaystyle\varprojlim_n G_n = \mathfrak{G} \subset \mathfrak{P}.$

Note that $G_n \cong \tilde{G} \times_A A/I^n$, so that $\mathfrak{G} \cong I$-adic completion of $\tilde{G}$. To pick out an open subset of P whose completion is $\mathfrak{G}$, proceed as follows:

(I.)   Let $\tilde{B} = \tilde{P} - \bigcup_{y\in Y} S_y(\tilde{G})$ and make $\tilde{B}$ into a reduced closed subscheme of $\tilde{P}$.

(II.)   Let $B_n = (\tilde{B} \times_A A/I^n)/Y \subset P_n$.

(III.)   Let $\displaystyle\varprojlim_n B_n = \mathfrak{B} \subset \mathfrak{P}.$

Then $\mathfrak{B}$ is the formal completion of a reduced closed subscheme $B \subset P$. Finally let $G = P - B$. Then, by construction the I-adic completion of G is $\mathfrak{G}$, i.e., the I-adic completions of G and of $\tilde{G}$ are canonically isomorphic.

This G is our final goal. We will eventually prove that G is a semi-abelian group scheme.

4. G is semi-abelian

We begin by proving that G is smooth over S. This is best proved in a more general context:

(4.1) PROPOSITION. Given

(1) 2 *schemes, locally of finite type over S*:

$$\begin{array}{ccc} X_1 & & X_2 \\ {\scriptstyle f_1}\searrow & & \swarrow{\scriptstyle f_2} \\ & S & \end{array}$$
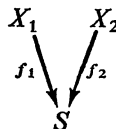
with X_2 proper over S.

(2) *an étale surjective morphism of their I-adic completions*:

$$\pi \colon \mathscr{X}_1 \to \mathscr{X}_2$$

(3) *closed reduced subschemes* $B_1 \subset X_1$, $B_2 \subset X_2$, *assume that the following holds*:

(4) $X_1 - B_1$ *is smooth over* S, *of relative dimension* r,

(5) *if* $\mathfrak{B}_i = I$-*adic completion of* B_i, *then we have the inclusion of formal subschemes (not just subsets)*

$$\mathfrak{B}_1 \subset \pi^{-1}(\mathfrak{B}_2),$$

then we conclude that $X_2 - B_2$ *is smooth over* S, *of relative dimension* r.

PROOF. First, let's check that $X_2 - B_2$ is flat over S. Let $\mathscr{M} \subset \mathscr{N}$ be 2 A-modules. Consider the 2 kernels:

$$0 \to \mathscr{K}_1 \to \mathscr{M} \otimes \mathcal{O}_{X_1} \to \mathscr{N} \otimes \mathcal{O}_{X_1}$$
$$0 \to \mathscr{K}_2 \to \mathscr{M} \otimes \mathcal{O}_{X_2} \to \mathscr{N} \otimes \mathcal{O}_{X_2}.$$

Since $X_1 - B_1$ is flat over S, $\mathrm{Supp}(\mathscr{K}_1) \subset B_1$, hence for all $x \in X_1$, $(\mathscr{K}_1)_x \cdot (I_{B_1})_x^n = (0)$ for some n. Taking I-adic completions, we get exact sequences:

$$0 \to \hat{\mathscr{K}}_1 \to \mathscr{M} \otimes \mathcal{O}_{\mathscr{X}_1} \to \mathscr{N} \otimes \mathcal{O}_{\mathscr{X}_1}$$
$$0 \to \hat{\mathscr{K}}_2 \to \mathscr{M} \otimes \mathcal{O}_{\mathscr{X}_2} \to \mathscr{N} \otimes \mathcal{O}_{\mathscr{X}_2}.$$

and since π is flat, it follows that $\hat{\mathscr{K}}_1 \cong \pi^* \hat{\mathscr{K}}_2$. Since for all $x \in \mathscr{X}_1$, $(\hat{\mathscr{K}}_1)_x \cdot (I_{\mathfrak{B}_1})_x^n = (0)$ for some n, and since by (5), $I_{\mathfrak{B}_1} \supset \pi^*(I_{\mathfrak{B}_2})$, it follows that $(\hat{\mathscr{K}}_2)_{\pi(x)} \cdot (I_{\mathfrak{B}_2})_{\pi(x)}^n = (0)$. This means that in an open neighborhood of $f_2^{-1}(S_0)$, $\mathscr{K}_2$ is killed by I_{B_2}, hence $\mathrm{Supp}(\mathscr{K}_2) \subset B_2$. Since X_2 is proper over S, all closed points of X_2 lie over S_0 so that $\mathrm{Supp}(\mathscr{K}_2) \subset B_2$ everywhere.

To show that $X_2 - B_2$ is actually smooth over S, it suffices to show that in addition to being flat, it is differentiably smooth: i.e., $\Omega^1_{X_2/S}$ is locally free of rank r outside B_2, and $S^n(\Omega^1_{X_2/S}) \to \mathfrak{P}^n_{X_2/S}$ is an isomorphism outside B_2 (cf. EGA, IV^4. 16). Since we know these are true for X_1/S, we deduce, in particular, that at all points x of $f_1^{-1}(S_0)$,

a) for all $g \in I(B_1)_x$, $(\Omega^1_{X_1/S})_x \otimes_{\mathcal{O}_{x,X_1}} \mathcal{O}_{x,X_1}[1/g]$ is locally free of rank r over $\mathcal{O}[1/g]$,

b) Ker and Coker of $S^n(\Omega^1_{X_1/S})_x \to \mathfrak{P}^n_{X_1/S,x}$ are killed by powers of $I(B_1)_x$.

These 2 facts imply the corresponding facts for the formal scheme $\mathscr{X}_1$. Since π is étale, $\pi^*(\Omega^1_{\mathscr{X}_2/S}) \cong \Omega^1_{\mathscr{X}_1/S}$ and $\pi^*(\mathfrak{P}^n_{\mathscr{X}_2/S}) \cong \mathfrak{P}^n_{\mathscr{X}_1/S}$, so by

assumption (5), we get the corresponding facts for $\mathscr{X}_2$. Finally, these imply that (a) and (b) hold for X_2/S at points $x \in f_2^{-1}(S_0)$. Since X_2 is proper over S, they hold everywhere on X_2. Thus $X_2 - B_2$ is differentiably smooth over S. *QED*

(4.2) COROLLARY. *G is smooth over S.*

PROOF. Take $X_1 = \tilde{P}$, $X_2 = P$, $B_1 = \tilde{B}$, $B_2 = B$, and use the fact that $\tilde{G}$ is smooth over S. *QED*

Our next step is to show that P and hence G is irreducible:

(4.3) PROPOSITION. *P is irreducible.*

PROOF. To prove this, we may assume that P is normal. In fact, we can replace $\tilde{P}$ by its normalization and then, since A is excellent, $\tilde{\mathfrak{P}}$ is normal, hence $\mathfrak{P}$ and P are normal. Then to show that P is irreducible, it suffices to show that P is connected. But we know that $\tilde{P}_0$ is connected. Therefore $P_0 = \tilde{P}_0/Y$ is connected. But P is proper over S and so therefore P is connected too. *QED*

Next, we take up the key problem of proving that G is independent of the choice of the polarization ϕ and the model $\tilde{P}$.

(4,4) DEFINITION. A subtorus $\tilde{H} \subset \tilde{G}$ is *integrable* if

$$\operatorname{rank}(Y \cap \tilde{H}(K)) = \dim \tilde{H}.$$

The key step in our proof of independence is that an integrable subtorus $\tilde{H} \subset \tilde{G}$ defines a closed subscheme $H \subset G$ in the following way:

a) let W_1 be the closure of $\tilde{H}$ in $\tilde{P}$ (considered as a reduced closed subscheme of $\tilde{P}$). If $Y^* = Y \cap \tilde{H}(K)$, then W_1 is Y^*-invariant.

b) let $\mathfrak{W}_1$ be the I-adic completion of W_1. It is also Y^*-invariant. Then

$$\mathfrak{W}_2 = \bigcup_{y \in Y/Y^*} S_y(\mathfrak{W}_1)$$

turns out to be a locally finite union, so this defines $\mathfrak{W}_2$ as a reduced closed subscheme of $\tilde{\mathfrak{P}}$.

c) Let $\mathfrak{W}_3 = \mathfrak{W}_2/Y \subset \mathfrak{P}$.

d) Let $W_3 \subset P$ be the reduced closed subscheme whose I-adic completion is $\mathfrak{W}_3$.

e) $H = W_3 \cap G$.

The finiteness assertion in (b) is the only non-trivial step where integrability has to be used. It results from:

(4.5) PROPOSITION. *Let $\tilde{H} \subset \tilde{G}$ be an integrable subtorus and let $Y^* = Y \cap \tilde{H}(K)$. Let W_1 be the closure of $\tilde{H}$ in a relatively complete model $\tilde{P}$ of $\tilde{G}$. Then there is a finite set $S \subset Y$ such that*

$$W_1 \cap S_y(U_0) = \phi \text{ if } y \notin S + Y^*.$$

PROOF. Let $X^* \subset X$ be the group of characters that are identically 1 on $\tilde{H}$. Then $\mathscr{X}^\alpha(y) = 1$ if $\alpha \in X^*$, $y \in Y^*$. In particular, $Y^* \cap \phi^{-1}(X^*) = (0)$ since if $y \in Y^* \cap \phi^{-1}(X^*)$, $y \neq 0$, then $\mathscr{X}^{\phi(y)}(y)$ is in I and is 1. If $r = \dim \tilde{G}$, $s = \dim \tilde{H}$, then rank $Y^* = s$ and rank $\phi^{-1}(X^*) = $ rank $X^* = r{-}s$. Therefore $Y^* + \phi^{-1}(X^*)$ has finite index in Y. Let $k \geq 1$ be an integer such that

$$kY \subseteq Y^* + \phi^{-1}(X^*) \subseteq Y.$$

Let $Y^{**} = \{y \in Y | ny \in Y^*, \text{ some } n \geq 1\}$. Then Y^{**}/Y^* is a finite group killed by k and Y/Y^{**} is torsion-free. Consider the quotient torus $\tilde{G}_1 = \tilde{G}/\tilde{H} \cdot Y^{**}$. Its character group is the subgroup $X^{**} \subset X^*$ of characters which are 1 on all of Y^{**}. Note that $kX^* \subset X^{**}$. In $\tilde{G}_1$, consider the group of periods $Y_1 = Y/Y^{**}$. Define a polarization $\psi: Y_1 \to X^{**}$ as follows:

> let $\bar{y} \in Y_1$,
>
> lift $\bar{y}$ to an element $y \in Y$,
>
> write $ky = y^* + w$, where $y^* \in Y^*$, $\phi(w) \in X^*$;
>
> set $\psi(\bar{y}) = k \cdot \phi(w)$.

If $\bar{y}_0$ is a second element of Y_1, and $ky_0 = y_0^* + w_0$ as above, then

$$
\begin{aligned}
\mathscr{X}^{\psi(\bar{y})}(\overline{y_0}) &= \mathscr{X}^{k\phi(w)}(y_0) \\
&= \mathscr{X}^{\phi(w)}(ky_0) \\
&= \mathscr{X}^{\phi(w)}(y_0^* + w_0) \\
&= \mathscr{X}^{\phi(w)}(w_0) \\
&= \mathscr{X}^{\phi(w_0)}(w) = \cdots = \mathscr{X}^{\psi(\overline{y_0})}(\bar{y})
\end{aligned}
$$

so that ψ is a bona fide polarization. We shall apply the basic lemma (1.3) to this torus and this polarization.

To find the appropriate n_y's, recall that for all $y \in Y$, $\mathscr{X}^{\phi(y)}$ is a regular function on $\tilde{P}$ outside of the locus $\mathscr{X}^{\phi(y)}(y) = 0$. In particular, since U is of finite type over A,

$$[\mathscr{X}^{\phi(y)}(y)]^n \cdot \mathscr{X}^{\phi(y)}$$

is a regular function on all of U if n is large. Increasing n by one, we can even make it a function that vanishes on U_0. Now if $\bar{y} \in Y_1$ and $y \in Y$

lies over $\bar{y}$ with $ky = y^* + w$ as above, then

$$\mathscr{X}^{\psi(\bar{y})}(\bar{y}) = \mathscr{X}^{\phi(w)}(w)$$

and

$$\mathscr{X}^{\psi(\bar{y})} = [\mathscr{X}^{\phi(w)}]^k.$$

So choose an integer $n_{\bar{y}}$ such that

(*) $[\mathscr{X}^{\psi(\bar{y})}(\bar{y})]^{n_{\bar{y}}} \cdot \mathscr{X}^{\psi(\bar{y})}$ is regular on U, zero on U_0.

Applying the lemma, we find a finite set $y_1, \cdots, y_k \in Y$ and a finite subset $S \subset Y$ such that

(**) $\begin{cases} \text{for all } z \in Y, z \notin S + Y^{**}, \\ \mathscr{X}^{\psi(\bar{z})}(\bar{y}_i) = (\text{elt. of } A) \cdot \mathscr{X}^{\psi(\bar{y}_i)}(\bar{y}_i)^{n_{y_i}}, \text{ for some } i. \end{cases}$

Now consider the function $\mathscr{X}^{\psi(\bar{y}_i)}$ on $S_z(U)$. Via the isomorphism S_z of $S_z(U)$ and U, it corresponds to the function

$$\mathscr{X}^{\psi(\bar{y}_i)}(z) \cdot \mathscr{X}^{\psi(\bar{y}_i)}$$

on U. Combining (*) and (**), it follows that this function is regular on U and zero on U_0. Therefore $\mathscr{X}^{\psi(\bar{y}_i)}$ is regular on $S_z(U)$ and zero on $S_z(U_0)$. But $\mathscr{X}^{\psi(\bar{y}_i)} \equiv 1$ on W_1 so this shows that $W_1 \cap S_z(U_0) = \phi$. Since z was an arbitrary element of Y outside $S + Y^{**}$, this proves the Proposition. QED

We are now ready to prove:

(4.6) THEOREM. *Let $(\tilde{G}_i, Y_i, \phi_i, \tilde{P}_i)$, $i = 1, 2$ be 2 tori plus periods, polarizations, and relatively complete models. Let G_i, $i = 1, 2$ be the 2 schemes constructed as above. Then for all S-homomorphisms $\tilde{\alpha}: \tilde{G}_1 \to \tilde{G}_2$ such that $\tilde{\alpha}(Y_1) \subset Y_2$, there is a unique S-morphism $\alpha: G_1 \to G_2$ such that under the canonical isomorphisms of the I-adic completions of G_i and $\tilde{G}_i$, α and $\tilde{\alpha}$ are formally identical.*

PROOF OF THEOREM. Consider the torus $\tilde{G}_1 \times_s \tilde{G}_2$; $Y_1 \times Y_2$ is a set of periods for this torus, $X_1 \times X_2$ is its character group and $\phi_1 \times \phi_2$: $Y_1 \times Y_2 \to X_1 \times X_2$ is a polarization. Moreover $P_1 \times_S P_2$ is easily seen to be a relatively complete model for $\tilde{G}_1 \times \tilde{G}_2$ relative to these periods and this polarization. Now suppose $\tilde{\alpha}: \tilde{G}_1 \to \tilde{G}_2$ is an S-homomorphism such that $\tilde{\alpha}(Y_1) \subset Y_2$. Look at the graph

$$\tilde{H} = \text{Image of } (1, \tilde{\alpha}) : \tilde{G}_1 \to G_1 \times_s \tilde{G}_2.$$

It is a subtorus of $G_1 \times_s G_2$, and because $\tilde{\alpha}(Y_1) \subset Y_2$, it is integrable. As in the last Proposition, it induces a closed subscheme of $G_1 \times_s G_2$ as

follows:

$$W_1 = \text{closure of } \tilde{H} \text{ in } \tilde{P}_1 \times_S \tilde{P}_2$$

$$\mathfrak{W}_2 = \bigcup_{y \in Y_1 \times Y_2} S_y(\mathfrak{W}_1)$$

$$\mathfrak{W}_3 = \mathfrak{W}_2/Y_1 \times Y_2 \subset \mathfrak{P}_1 \times_S \mathfrak{P}_2$$

$$H = W_3 \cap (G_1 \times_S G_2).$$

I claim that H is the graph of a morphism from G_1 to G_2. First of all, we prove that the projection

$$p_1 : W_3 \to P_1$$

is smooth of relative dimension 0 outside $B_1 = P_1 - G_1$. This follows by essentially the same argument used in the proof of Proposition (4.1). In fact, $p_1 : W_1 \to \tilde{\mathfrak{P}}_1$ is smooth of rel. dim. 0 outside $\tilde{B} = \tilde{\mathfrak{P}}_1 - \tilde{G}_1$. Now locally at every point $\mathfrak{W}_2$ is the formal completion of a finite union $S'_{y_1}(W_1) \cup \cdots \cup S'_{y_k}(W_1)$, $y_i \in Y_1 \times Y_2$, and since this is also smooth of rel. dim. 0 outside $\tilde{B}_1$, so is $p_1 : \mathfrak{W}_2 \to \tilde{\mathfrak{P}}_1$. Here by 'smooth outside $\tilde{B}_1$', we do *not* mean merely smooth at points of $\tilde{\mathfrak{P}}_1 - \tilde{B}_1$: instead we mean smoothness in the sense of properties (a) and (b) in the proof of the proposition, viz. smoothness after localizing by the ideal $I(\tilde{B}_1)$. This property descends to smoothness for $p_1 : \mathfrak{W}_3 \to \tilde{\mathfrak{P}}_1$, hence for $p_1 : W_3 \to P_1$. Secondly, we prove that $W_3 \cap (P_1 \times_S B_2) \subset B_1 \times_S B_2$. This follows by the same method, by descending a stronger ideal-theoretic property on the $\tilde{\ }$-schemes. In fact, since for every finite set of y_i's:

$$[S_{y_1}(W_1) \cup \cdots \cup S_{y_k}(W_1)] \cap (\tilde{P}_1 \times_S \tilde{B}_2) \subset \tilde{B}_1 \times_S \tilde{B}_2,$$

it follows that on $\tilde{\mathfrak{P}}_1 \times \tilde{\mathfrak{P}}_2$,

$$I(\mathfrak{W}_2) + I(\tilde{\mathfrak{P}}_1 \times_S \tilde{\mathfrak{B}}_2) \supset I(\tilde{\mathfrak{B}}_1 \times_S \tilde{\mathfrak{B}}_2)^N$$

for some N. This property descends, and on algebraizing, proves that $W_3 \cap (P_1 \times_S B_2) \subset B_1 \times_S B_2$. Since $p_1 : W_3 \to P_1$ is a proper morphism, this proves that the restriction $p_1 : H \to G_1$ is also proper. Combining these 2 halves, it follows that $p_1 : H \to G_1$ is finite and étale. But finally the formal I-adic completion of $\mathfrak{H}$ is obviously the graph of the formal morphism from G_1 to G_2 defined by $\tilde{H}$. Therefore $p_1 : H \to G_1$ has degree 1 over S_0, hence because G_1 is irreducible, it has degree 1 everywhere. This proves that there is an S-morphism $\alpha : G_1 \to G_2$ extending the formal morphism defined by $\tilde{\alpha}$. Finally, since G_1 is irreducible, such an α is clearly determined by its restriction to $\mathfrak{G}_1$.   *QED*

(4.7) COROLLARY. *The scheme G depends only on the torus $\tilde{G}$ and the periods Y, and is independent of the polarization ϕ and the relatively complete model $\tilde{P}$.*

PROOF. Apply the theorem to 2 4-tuples $(\tilde{G}, Y, \phi_1, \tilde{P}_1)$ and $(\tilde{G}, Y, \phi_2, \tilde{P}_2)$ and to the identity $1_{\tilde{G}} : \tilde{G} \to \tilde{G}$.

(4.8) COROLLARY. *G is a group scheme over S.*

PROOF. Apply the theorem to $(\tilde{G} \times_S \tilde{G}, \; Y \times Y, \; \phi \times \phi, \; \tilde{P} \times_S \tilde{P})$ and $(\tilde{G}, Y, \phi, \tilde{P})$ and the multiplication map from $\tilde{G} \times_S \tilde{G}$ to $\tilde{G}$. This yields a map $\mu \colon G \times_S G \to G$. Apply the theorem to the inverse from $\tilde{G}$ to $\tilde{G}$. This yields a map $i \colon G \to G$. Since μ and i make the formal completion $\mathfrak{G}$ of G into a formal group scheme and since G is irreducible, μ and i satisfy the same identities on G on as $\mathfrak{G}$ and therefore G is a group scheme.   *QED*

(4.9) COROLLARY. G_η *is an abelian variety.*

PROOF. Since $\tilde{G}_\eta = \tilde{P}_\eta$, the generic fibre $\tilde{B}_\eta$ of $\tilde{B}$ is empty. It follows that the structure sheaf $\mathcal{O}_{\tilde{B}}$ is killed by some non-zero $\tau \in A$. Therefore the structure sheaf $\mathcal{O}_B$ is also killed by τ. Therefore $B_\eta = \phi$, hence $G_\eta = P_\eta$ is proper over K. Since G is irreducible, G_η is also irreducible, hence an abelian variety.   *QED*

Before we can prove that G is, in fact, a semi-abelian group scheme, we have to prove that *every* fibre of G over S is connected. This follows from a description of the points of G of finite order:Start with $\tilde{G}, Y, \phi, \tilde{P}$ again.

Let $G^* = \bigcup_{y \in Y} S_y(\tilde{G}) \subset \tilde{P}$.

Let $\sigma_y \colon S \to G^*$ be the section such that $\sigma_y(\eta) = y$.

Let $Z_y^{(n)}$ be the subscheme of points z such that $nz = y$, i.e., the fibre product:

$$\begin{array}{ccc} Z_y^{(n)} & \hookrightarrow & G^* \\ \downarrow & & \downarrow n \\ S & \hookrightarrow & G^* \\ & \sigma_y & \end{array}$$

Note that S_z induces an isomorphism of $Z_y^{(n)}$ with $Z_{y+nz}^{(n)}$.

(4.10) THEOREM. *The kernel* $G^{(n)} \subset G$ *of multiplication by n is isomorphic over S to the disjoint union of the schemes* $Z_y^{(n)}$ *as y runs over a set of cosets of* Y/nY.

PROOF. First consider the closure $\bar{Z}_y^{(n)}$ of $Z_y^{(n)}$ in $\tilde{P}$. By the valuation property of the relatively complete model $\tilde{P}$, it follows that all valuations of $\bar{Z}_y^{(n)}$ have centers on $\bar{Z}_y^{(n)}$. By Lemma (3.4), it follows that $\bar{Z}_y^{(n)}$ is proper over S. Let $\overline{}_y^{(n)}$ be its I-adic completion. We have seen above that if

$$W_1^{(n)} = \text{closure in } \tilde{P} \times_S \tilde{P} \text{ of the graph of } x \mapsto nx$$

then

$$\mathfrak{W}_2^{(n)} = \bigcup_{y \in Y} S_{(0,\,w)}(\mathfrak{W}_1^{(n)}) \subset \tilde{\mathfrak{P}} \times_S \tilde{\mathfrak{P}}$$

is a locally finite union. Since $\bar{Z}_y^{(n)} \times_S \sigma_y(S) \subset W_1^{(n)}$, hence $\bar{Z}_y^{(n)} \times_S \sigma_0(S) \subset S_{(0,\,-y)}(W_1^{(n)})$, it follows that

$$\bigcup_{y \in Y} \bar{\mathfrak{Z}}_y^{(n)} \subset \tilde{\mathfrak{P}}$$

is a locally finite union, and that

$$\bigcup_{y \in Y} \bar{\mathfrak{Z}}_y^{(n)} = \mathfrak{W}_2^{(n)} \cap (\tilde{\mathfrak{P}} \times_S \sigma_0(S)).$$

Taking quotients by Y and $Y \times Y$, we obtain a formal closed subscheme $\bigcup_{y \in Y} \bar{\mathfrak{Z}}_y^{(n)}/Y \subset \mathfrak{P}$ such that

$$\bar{\mathfrak{Z}}^{(n)} = \bigcup_{y \in Y} \bar{\mathfrak{Z}}_y^{(n)}/Y = \mathfrak{W}_3^{(n)} \cap (\mathfrak{P} \times_S \sigma_0(S))$$

where $\mathfrak{W}_3^{(n)} = \mathfrak{W}_2^{(n)}/Y \times Y$. It follows that $\bar{\mathfrak{Z}}^{(n)}$ algebrizes to a subscheme $\bar{Z}^{(n)} \subset P$ such that

$$\bar{Z}^{(n)} = W_3^{(n)} \cap (P \times_S \sigma_0(S)),$$

hence $Z^{(n)} = \bar{Z}^{(n)} \cap G = H^{(n)} \cap (G \times_S \sigma_0(s))$ where $H^{(n)} \subset G \times_S G$ is the graph of the morphism $x \mapsto nx$. Thus $Z^{(n)}$ is the kernel in G of multiplication by n. Now for every *finite* subset $Y_0 \subset Y$, we have a formal morphism as follows:

$$\alpha : \bigcup_{y \in Y_0} \bar{\mathfrak{Z}}_y^{(n)} \to \bar{\mathfrak{Z}}^{(n)}.$$

Since these formal schemes are the completions of the (algebraic) schemes $\bigcup_{y \in Y_0} \bar{Z}_y^{(n)}$ and $\bar{Z}^{(n)}$, which are *proper* over S, this formal morphism extends an (algebraic) morphism:

$$\alpha : \bigcup_{y \in Y_0} \bar{Z}_y^{(n)} \to \bar{Z}^{(n)}.$$

If $\tilde{B} = \tilde{P} - \tilde{G}$ and $B = P - G$, recall that $\tilde{\mathfrak{P}}$ is the inverse image of $\mathfrak{P}$ in the étale map $\tilde{\mathfrak{P}} \to \mathfrak{P}$. Therefore $\tilde{\mathfrak{P}} \cap \bigcup_y \bar{\mathfrak{Z}}_y^{(n)}$ is the inverse image of $\mathfrak{P} \cap \bar{\mathfrak{Z}}^{(n)}$, hence $\tilde{B} \cap \bigcup_{y \in Y_0} \bar{Z}_y^{(n)}$ is the inverse image of $B \cap \bar{Z}^{(n)}$. Therefore α restricts to a proper morphism:

$$\text{res}\,(\alpha) : \bigcup_{y \in Y_0} Z_y^{(n)} \to Z^{(n)}.$$

Next, note that

$$\alpha : \bigcup_{y \in Y} \bar{\mathfrak{Z}}_y^{(n)} \to \bar{\mathfrak{Z}}^{(n)}$$

is étale and surjective. It follows that for each fixed $y_0 \in Y$, there is a finite set $Y_0 \subset Y$ such that $\alpha : \bigcup_{y \in Y_0} \bar{\mathfrak{Z}}_y^{(n)} \to \bar{\mathfrak{Z}}^{(n)}$ is étale at all points of

$\overline{\mathfrak{Z}}_{y_0}^{(n)}$, and is surjective. Therefore its algebraization has the same proper-
ties. On the other hand, when we intersect with $\tilde{G}$, the union $\bigcup_{y \in Y_0} Z_y^{(n)}$
is disjoint, so it follows that

$$\text{res}(\alpha) : \bigcup_{y \in Y_0} Z_Y^{(n)} \to Z^{(n)}$$

is étale for every Y_0, and surjective for Y_0 big enough.

Now it is clear that the diagram:

commutes, so, in fact, if Y_0 is a set of coset representatives of Y/nY

$$\text{res}(\alpha) : \bigcup_{y \in Y_0} Z_Y^{(n)} \to Z^{(n)}$$

is already surjective. More than that: identifying formally

$$\bigcup_{y \in Y_0} Z_y^{(n)} \text{ with } \bigcup_{y \in Y} Z_Y^{(n)} / Y$$

we see that $\bigcup_{y \in Y_0} Z_y^{(n)}$ has a natural group scheme structure and it is
clear that $\text{res}(\alpha)$ is a homomorphism. Therefore $\text{res}(\alpha)$ has degree 1
everywhere if it has degree 1 over the points $\sigma_0(S) \subset G$, and this will
follow if $\text{res}(\alpha)$ has degree 1 over the points $\sigma_0(S_0) \subset G_0$. But it is easy
to see that *over S_0, $Z_y^{(n)} = \phi$ unless $y \in nY$, $Z_0^{(n)}$ is the kernel of n in the
part of the torus $\tilde{G}$ over S_0*, and $\text{res}(\alpha)$ is just the restriction to the kernel
of n of the canonical isomorphism of $\tilde{G}$ and G over S_0, Thus $\text{res}(\alpha)$ has
degree 1.   *QED*

(4.11) COROLLARY. *Let $s \in S$. Let*

$$Y_1 = \{y \in Y | \mathcal{X}^{\phi(y)}(y) \notin m_{s,s}\}.$$

*Then Y_1 is a subgroup of Y (in fact, it is a direct summand) and the kernel
of n in G_s fits into an exact sequence*:

$$0 \to (\text{ker of } n \text{ in } \tilde{G}_s) \to (\text{ker of } n \text{ in } G_s) \to \frac{1}{n} Y_1/Y_1 \to 0.$$

As n increases, we obtain in the limit an exact sequence

$$0 \to (\text{torsion in } \tilde{G}_s) \to (\text{torsion in } G_s) \to Y_1 \otimes Q/Z \to 0.$$

PROOF. This follows immediately from the Theorem and the remark
that $Z_y^{(n)}$ has a non-empty fibre over y if and only if $y \in Y_1 + nY$.

(4.12) COROLLARY. G_s is connected, hence G is semi-abelian.

PROOF. By Corollary 1, its torsion is p-divisible for every prime p, hence G_s is connected and without unipotent radical.

5. Examples-dim $G/S = 1$

Let's look first at the 1-dimensional case $\tilde{G} = G_m \times S$ where $r = 1$.

Then $X \cong \mathbf{Z}$. Choose such an isomorphism. Then we have a distinguished generator $1 \in X$, and we denote the character $\mathscr{X}^1$ imply by $\mathscr{X}$, so that

$$\tilde{G} = \operatorname{Spec} A[\mathscr{X}, \mathscr{X}^{-1}].$$

All possible ϕ's will be positive multiples of one basic ϕ which is an isomorphism of Y and X. We assume this isomorphism chosen as our ϕ. All periods are multiples of the basic period $y_0 = \phi^{-1}(1)$: let $\tau = \mathscr{X}(y_0)$.

Note that $\tau \in I$. Identifying $\tilde{G}(K)$ with K^*, Y becomes the set of powers $\{\tau^m\}_{m \in \mathbf{Z}}$. The simplest possible Σ consists in the 3 elements $\{1, 0, -1\}$. Then

$$R_{\phi, \Sigma} = A[\cdots, \mathscr{X}^{\phi(ky_0)+\{\varepsilon\}}(ky_0) \cdot \mathscr{X}^{2\phi(ky_0)+\{\varepsilon\}} \cdot \theta, \cdots]_{\varepsilon=0, \pm 1}$$

$$= A[\cdots, \tau^{k^2+k} \cdot \mathscr{X}^{2k+1} \cdot \theta, \tau^{k^2} \cdot \mathscr{X}^{2k} \cdot \theta, \tau^{k^2-k} \cdot \mathscr{X}^{2k-1} \cdot \theta, \cdots]$$

with basic automorphism:

$$S(\mathscr{X}) = \tau \cdot \mathscr{X}$$
$$S(\theta) \ = \tau \cdot \mathscr{X}^2 \cdot \theta$$

One checks easily that

$$U_{1, 0} \ = \operatorname{Spec} A[\tau \mathscr{X}, \mathscr{X}^{-1}]$$
$$U_{0, 0} \ = \operatorname{Spec} A[\mathscr{X}, \mathscr{X}^{-1}]$$
$$U_{-1, 0} = \operatorname{Spec} A[\mathscr{X}, \tau \mathscr{X}^{-1}]$$

Using the automorphism S, it follows that

$$U_{1, ky_0} = \operatorname{Spec} A[\tau^{k+1} \mathscr{X}, \tau^{-k} \mathscr{X}^{-1}]$$
$$U_{0, ky_0} = \operatorname{Spec} A[\tau^k \mathscr{X}, \tau^{-k} \mathscr{X}^{-1}]$$
$$U_{-1, ky_0} = \operatorname{Spec} A[\tau^k \mathscr{X}, \tau^{-k+1} \mathscr{X}^{-1}]$$

In particular $U_{1, ky_0} = U_{-1, (k-1)y_0}$. One checks immediately that the the closed fibre of $\operatorname{Proj}(R_{\phi, \Sigma})$ is an infinite union of non-singular rational curves, connected in a chain and crossing each other transversely.

We may visualize $\operatorname{Proj}(R_{\phi, \Sigma})$ like this (when A is a discrete valuation ring):
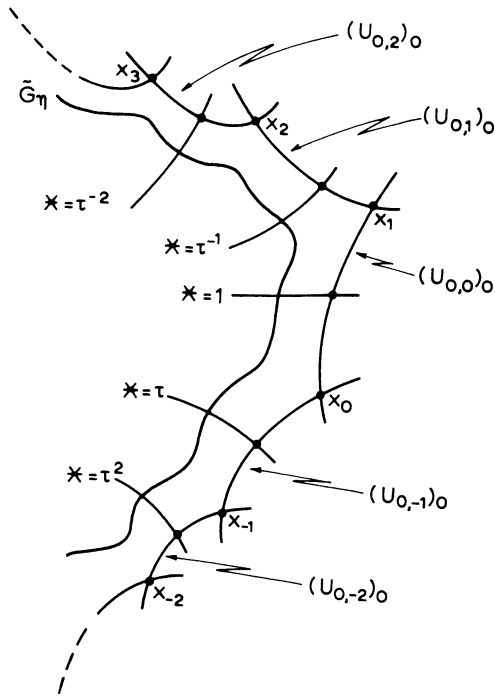
Dividing by Y, we obtain a formal scheme $\mathfrak{P}$ whose closed fibre is an irreducible rational curve with one ordinary double point. P has the same property and looks like this:
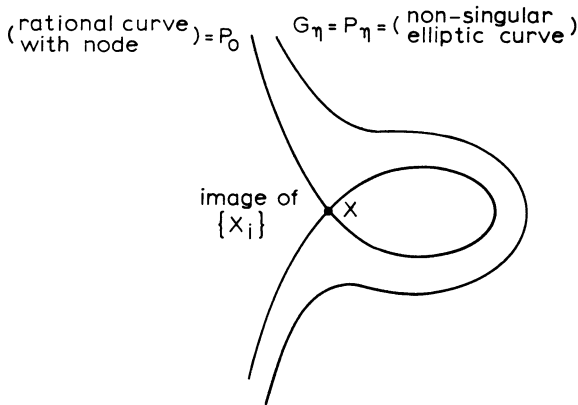
The part S which is to be thrown out will be just the single point x if A is a discrete valuation ring; in general, it will be a section of P through x over the subscheme $(\tau = 0)$ of Spec (A).

6. Examples-dim S = 1

Next, I want to look at higher dimensional tori $\tilde{G}$, but in the case where dim $A = 1$, i.e. A is a complete discrete valuation ring. Let r be the dimension of $\tilde{G}$ over S. In this case, we can describe the models $\tilde{P}$ very intuitively by a polyhedral decomposition of r-dimensional Euclidean space and we can see in this way the degree of choice in $\tilde{P}$ very clearly. We shall limit ourselves to *normal* relative complete models $\tilde{P}$ which admit an affine open covering $\{U_i\}$ such that each U_i is $\tilde{G}$-invariant. (It seems to me likely that every $\tilde{P}$ has such a covering; at least the $\text{Proj}(R_{\phi, \Sigma})$'s do). Our method is this: for every finite algebraic extension $L \supset K$, let A_L be the integral closure of A in L. A_L is again a complete discrete valuation ring. By the completeness property of $\tilde{P}$, every L-valued point ξ of $\tilde{G}$ extends uniquely to an A_L-valued point of $\tilde{P}$. We shall describe various subsets Σ of $\tilde{P}_0$ by describing those ξ whose closure meets Σ.

To begin with, the only thing that will really matter about a ξ is the *rate of growth of the various characters of $\tilde{G}$ on ξ*. This can be conveniently measured as follows:

(a) let $E = Hom(X, \mathbf{R})$, $X = $ char. gp. of $\tilde{G}$ (this is a covariant functor of $\tilde{G}$),

(b) $\forall \alpha \in X$, let $l_\alpha : E \to \mathbf{R}$ be the linear functional given by evaluation at α,

(c) fix, once and for all, an embedding of the value group of A in $\mathbf{R}$. This naturally extends to an embedding of the value groups of each A_L in $\mathbf{R}$,

(d) $\forall \xi \in \tilde{G}(L)$, define $\|\xi\| \in E$ by $\|\xi\|(\alpha) = \text{ord } \mathscr{X}^\alpha(\xi)$.

Note that for each L, $\{\|\xi\| \mid \xi \in \tilde{G}(L)\} = N_L$ is a lattice in E; as L becomes more and more ramified over K, N_L gets denser and denser. Inside N_K, we have the smaller lattice $\{\|\xi\| \mid \xi \in Y\}$ induced by the periods. We shall identify this with Y:

$$Y \subset N_K \subset \cdots \subset N_L \subset \cdots \subset E.$$

Now consider the finitely generated normal affine schemes U/S ($S = \text{Spec } A$), such that $U_\eta = \tilde{G}_\eta$ and such that $\tilde{G}$ acts on U extending the translation action of $\tilde{G}_\eta$ on U_η. It is easy to check that all such U are all of the type:

$$U = \text{Spec } A[\cdots, \pi^{r_i}\mathscr{X}^{\alpha_i}, \cdots]_{1 \leq i \leq N}$$

$$\text{where } (\pi) = \text{maximal ideal of } A$$

$$r_i \in \mathbf{Z}$$

$$\alpha_i \in X.$$

Concerning such U, we have the following:

(6.1) PROPOSITION. *There is a 1–1 correspondence between*:

(I) *normal affine schemes U, with $U_\eta = \tilde{G}_\eta$ and invariant under the action of $\tilde{G}$, and*

(II) *closed, bounded polyhedra $\Delta \subset E$ with vertices in $\boldsymbol{Q} \cdot N_K$.*

This is set up by the relation: $\forall L \supset K$, $\forall \xi \in \tilde{G}(L)$,

$$\left\{ \begin{array}{l} \xi \text{ extends to an} \\ A_L\text{-valued point of } U \end{array} \right\} \Leftrightarrow \{\|\xi\| \in \Delta\}.$$

PROOF. In fact, if $\Gamma(U, \mathcal{O}_U)$ is generated by $\pi^{r_i} \mathcal{X}^{\alpha_i}$, then define Δ by

$$\Delta = \{x \in E | l_{\alpha_i}(x) \geqq -r_i \operatorname{ord}(\pi), 1 \leqq i \leqq N\}.$$

Conversely, given Δ, define

$$U = \operatorname{Spec} A[\cdots, Y_{r,\alpha}, \cdots],$$

with $Y_{r,\alpha} = \pi^r \mathcal{X}^\alpha$ for all (r, α) such that $l_\alpha + r \cdot \operatorname{ord}(\pi)$ is non-negative on Δ. We leave it to the reader to check that these set up inverse maps between the sets (I) and (II) of the Proposition.

(6.2) DEFINITION. Let $U_\Delta = \operatorname{Spec}(R_\Delta)$ be the affine scheme associated to Δ by the Proposition.

By a *closed face* of a polyhedron Δ, we mean Δ itself or a polyhedron $\Delta_1 = H \cap \Delta$, where H is a hyperplane, and Δ lies completely on side of H. By an *open face*, we mean a closed face minus all properly smaller subfaces. It is clear that every polyhedron Δ is a finite disjoint union of its open faces.

(6.3) PROPOSITION. *There is a 1–1 correspondence between*:

(I) *orbits Z of $\tilde{G}_0$ on the closed fibre $(U_\Delta)_0$ of U_Δ, and*

(II) *open faces $\sigma \subset \Delta$.*

This is set up by the relation: $\forall L \supset K$, $\forall \xi \in \tilde{G}(L)$,

(*) $\left\{ \begin{array}{l} \xi \text{ extends to a} \\ A_L\text{-valued point meeting } Z \end{array} \right\} \Leftrightarrow \{\|\xi\| \in \sigma\}.$

Moreover $\dim Z + \dim \sigma = r$.

PROOF. Since the union of the open faces is Δ, and the union of the orbits is $U_\Delta \times \operatorname{Spec}(\bar{k})$ it will suffice to show that for all σ there is an orbit Z satisfying (*). Let $R_\Delta = A[\cdots, \pi^{r_i} \mathcal{X}^{\alpha_i}, \cdots]_{1 \leqq i \leqq N}$. Then

$\Delta = \{x \in E | l_{\alpha_i}(x) + r_i \cdot \text{ord } \pi \geqq 0\}$. Let σ be defined by:

$$l_{\alpha_i}(x) + r_i \cdot \text{ord } \pi = 0, \; i \in I_1$$
$$l_{\alpha_i}(x) + r_i \cdot \text{ord } \pi > 0, \; i \in I_2$$

where $I_1 \cup I_2 = \{1, \cdots, N\}$. Write $\pi^{r_i} \mathcal{X}^{\alpha_i} = Y_i$ for simplicity. Then for all $L \supset K$, and A_L-valued points ξ such that $||\xi|| \in \Delta$, we see that

$$||\xi|| \in \sigma \Leftrightarrow \begin{cases} Y_i (\xi) \text{ a unit for } i \in I_1 \\ Y_i (\xi) \in \text{max. ideal of } A_L, \, i \in I_2 \end{cases}$$
$$\Leftrightarrow \xi \text{ meets } Z$$

where $Z \subset (U_\Delta)_0$ is defined by $Y_i \neq 0$, $i \in I_1$ (resp. $= 0$, $i \in I_2$). Now Z is clearly $\tilde{G}_0$-invariant. Moreover Z is easily checked to be the locus in the affine Y_i-space ($i \in I_1$) defined by:

i) $Y_i \neq 0$, all $i \in I_1$

ii) $\Pi Y_i^{s_i} = 1$ whenever $s_i \in \mathbf{Z}$, $s_i \geqq 0$ satisfy $\sum s_i \alpha_i = 0$ (Since $\sigma \neq \emptyset$, $\sum s_i \alpha_i = 0 \Rightarrow \Sigma s_i r_i = 0$!).

If $\tilde{H} \subset \tilde{G}$ is the subtorus defined by $\mathcal{X}^{\alpha_i} = 1$, $i \in I_1$, then it follows immediately that Z is isomorphic to $\tilde{G}_0/\tilde{H}_0$ if a point of Z with coordinates $Y_i = a_i$ is associated by a point of G_0/H_0 with coordinate $\mathcal{X}^{\alpha_i} = a_i$.

(6.4) COROLLARY. $(U_\Delta)_0$ *is a finite union of orbits, hence each component contains a unique open orbit, and the set of components is in 1–1 correspondence with the vertices of Δ.*

(6.5) COROLLARY. *If Δ_1, Δ_2 are polyhedra, then U_{Δ_1} is an open subset of U_{Δ_2} if and only if Δ_1 is a closed face of Δ_2.*

PROOF. By the Prop., it is certainly necessary that Δ_1 be a closed face of Δ_2. Conversely, if Δ_1 is a closed face of Δ_2, then $\Delta_1 = \Delta_2 \cap H$ where H is a hyperplane defined by $l_\alpha(X) + r \cdot \text{ord } \pi = 0$ and $l_\alpha(x) + r \cdot \text{ord } \pi \geqq 0$ on Δ_2. Then $\pi^r \cdot \mathcal{X}^\alpha \in \Gamma(\mathcal{O}_{\Delta_2})$ and U_{Δ_1} is the affine open subset of U_{Δ_2} defined by $\pi^r \cdot \mathcal{X}^\alpha \neq 0$.

(6.6) COROLLARY. *There is a 1–1 correspondence between:*

(I) *normal schemes $\tilde{P}$ locally of finite type over S such that*

(a) $\tilde{P}_\eta = \tilde{G}_\eta$,

(b) *the translation action of $\tilde{G}$ extends to $\tilde{P}$ and P is covered by $\tilde{G}$-invariant affine open sets, and*

(c) *for all valuations v on $\mathbf{R}(\tilde{G})$ if $v \geqq 0$ on A and if $[\forall \alpha \in X, \exists n$ such that $n \cdot v(\pi) \geqq v(\mathcal{X}^\alpha) \geqq -nv(\pi)]$ hold, then v has a centre on $\tilde{P}$, and*

(II) *Polyhedral decompositions of E, i.e. a set of polyhedron $\Delta_\alpha \subset E$ such that $\bigcup \Delta_\alpha = E$, every closed face of a Δ_α is a Δ_β, $\Delta_\alpha \cap \Delta_\beta \neq \phi \Rightarrow \Delta_\alpha \cap \Delta_\beta = \Delta_\gamma$ (some γ), all of whose vertices are in $\mathbf{Q} \cdot N_K$.*

PROOF. Clear

In the 1–1 correspondence of this last corollary, certain properties of $\tilde{P}$ carry over nicely to properties of $\{\Delta_\alpha\}$:

(A) $\tilde{P} \supset \tilde{G}$ if and only one of the Δ_α's is the origin $\{0\}$.

(B) If $Y \subset \tilde{G}(K)$ is a set of periods, the action of Y on $\tilde{G}_\eta$ extends to $\tilde{P}$ if and only if the polyhedral decomposition is Y-invariant, i.e. $\forall \alpha, y$, $\Delta_\alpha + y = $ some Δ_β.

(C) $\tilde{P}$ is smooth over A at all generic points of $\tilde{P}_0$ if and only if the vertices of all Δ_α are points of N_K.

(D) $\tilde{P}$ is regular everywhere and smooth A at all generic points of $\tilde{P}_0$ if and only if the vertices of all Δ_α are in N_K, each Δ_α of dimension r is an r-simplex with volume $1/r!$ in a coordinate system in E making N_K into the integral points.

Here (A), (B) and (C) are almost immediate, and we omit the proof of (D), which is harder, because we do not need it. So far, there appears to be tremendous choice in constructing a relatively complete model $\tilde{P}$. However, the existence of an ample $\tilde{L}$ corresponding to a polarization $\phi: Y \rightarrow X$ puts very strong restrictions on the polyhedral decomposition $\{\Delta_\alpha\}$. The exact conditions on $\{\Delta_\alpha\}$ for the existence of $\tilde{L}$ are very messy, so we will only state, without proof, one partial result. First we must recall some constructions used in sphere packing problems. For all the following, cf. Rogers, *Packing and Covering*, Camb. Univ. Press, Ch. 7, § 1 and Ch. 8, § 1. Assume given a Euclidean metric on E. Start with a discrete set of points $\Sigma \subset E$ such that for some r, $E =$ union of balls around points of Σ with radius r. We may now construct 2 canonical polyhedral decompositions E:

(A) the *Voronoi decomposition*: $\forall \sigma \in \Sigma$, let

$$\Delta_\sigma = \{y \in E | \, ||y-\sigma|| \leqq ||y-\tau||, \text{ all } \tau \in \Sigma\}.$$

These are the top dimensional polyhedra. We get their faces by looking, for all $\sigma_1, \cdots, \sigma_k \in \Sigma$, at:

$$\Delta_{\sigma_1, \cdots, \sigma_k} = \left\{ y \in E \, \left| \begin{array}{c} ||y-\sigma_1|| = \cdots = ||y-\sigma_k|| \leqq ||y-\tau|| \\ \text{all } \tau \in \Sigma \end{array} \right. \right\}.$$

(B) *the Delaunay decomposition*: For all $\sigma_1, \cdots, \sigma_k$ such that

$\Delta_{\sigma_1,\dots,\sigma_k} \neq \phi$ and such that for y in its interior,

$$||y-\sigma_1|| = \cdots = ||y-\sigma_k|| < ||y-\tau|| \ (\tau \in \Sigma, \tau \neq \sigma_i),$$

let $\Delta^* = $ convex hull of $\sigma_1, \cdots, \sigma_k$. These polyhedra Δ^* form a decomposition of E whose vertices are exactly the points of Σ.

Now returning to our tori, let $Y \subset \tilde{G}(K)$ be a set of periods and let $\phi: Y \to X$ be a polarization. The quadratic form

$$||y||^2 = \text{ord } \mathscr{X}^{\phi(y)}(y), \ y \in Y$$

extends rationally to a Euclidean norm $|| \ ||$ on E. We find:

(6.7) PROPOSITION. *Let $\tilde{G}$, Y and ϕ be given as above. Let $\tilde{G} \subset \tilde{P}$ be given where $\tilde{P}$ is a normal scheme locally of finite type over S such that*

  i) *$\tilde{P}_\eta = \tilde{G}_\eta$,*

  ii) *$\tilde{G}$ and Y act on $\tilde{P}$,*

  iii) *P has the valuative completeness property of Corollary (6.6),*

  iv) *$\tilde{P}$ is covered by $\tilde{G}$-invariant affine open sets.*

Let $\tilde{P}$ correspond to the polyhedral decomposition $\{\Delta_\alpha\}$ and let Σ be the 0-simplices in $\{\Delta_\alpha\}$. If $\{\Delta_\alpha\}$ is the Delaunay decomposition of E associated to Σ, with the Euclidean norm $|| \ ||$ defined by ϕ, then there exists an ample invertible sheaf $\tilde{L}$ on $\tilde{P}$, plus an action of $\tilde{G}$ and Y on $\tilde{L}$, making $\tilde{P}$ into a relatively complete model corresponding to the polarization $n \cdot \phi$ (some $n \geq 1$).

(Proof omitted)

7. A final example

I want to give one final example which

(i) illustrates the situation where both the base and the fibre are more than one dimensional, and

(ii) is literally the keystone in the compactification of the moduli space of 2-dimensional abelian varieties.

This example was proposed by Deligne before we saw the general theory. Take as the ground ring:

$$A = k[[a, b, c]]$$

and take $\tilde{G}$ to be 2-dimensional. To express the symmetry of the situation, it is convenient not to introduce a basis of the character group X of $\tilde{G}$, but rather 3 generators $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ for X, related by one identity:

$$\mathscr{X} \cdot \mathscr{Y} \cdot \mathscr{Z} \equiv 1$$

The group of periods Y will similarly be generated by the 3 periods r, s, t with $r+s+t = 0$, given by

$$\mathcal{X}(r) = bc, \qquad \mathcal{Y}(r) = c^{-1}, \qquad \mathcal{Z}(r) = b^{-1},$$
$$\mathcal{X}(s) = c^{-1}, \qquad \mathcal{Y}(s) = ac, \qquad \mathcal{Z}(s) = a^{-1},$$
$$\mathcal{X}(t) = b^{-1}, \qquad \mathcal{Y}(t) = a^{-1}, \qquad \mathcal{Z}(t) = ab.$$

The polarization ϕ will be defined by

$$\phi(r) = \mathcal{X}$$
$$\phi(s) = \mathcal{Y}$$
$$\phi(t) = \mathcal{Z}$$

and Σ will consist in the seven characters:

$$\Sigma = \{1, \mathcal{X}, \mathcal{X}^{-1}, \mathcal{Y}, \mathcal{Y}^{-1}, \mathcal{Z}, \mathcal{Z}^{-1}\}.$$

The resulting ring $R_{\phi, \Sigma}$ will have some automorphisms, in addition to those given by translation with respect to Y:

a) an automorphism σ such that $\sigma^3 = \mathrm{id}$,

$$\sigma(a) = b, \sigma(b) = c, \sigma(c) = a$$
$$\sigma(\mathcal{X}) = \mathcal{Y}, \sigma(\mathcal{Y}) = \mathcal{Z}, \sigma(\mathcal{Z}) = \mathcal{X}$$
$$\sigma(\theta) = \theta$$

b) an involution $\tau(\tau^2 = \mathrm{id})$,

$$\tau = \mathrm{id} \text{ on } A$$
$$\tau(\mathcal{X}) = \mathcal{X}^{-1}, \tau(\mathcal{Y}) = \mathcal{Y}^{-1}, \tau(\mathcal{Z}) = \mathcal{Z}^{-1}$$
$$\tau(\theta) = \theta.$$

Now $\mathrm{Proj}\,(R_{\phi, \Sigma})$ is covered by the $U_{\alpha,y}$'s with $\alpha \neq 0$ and these are all isomorphic under these various automorphisms. We must calculate the ring of one of them. Look at $U_{\mathcal{X},0}$; its ring is generated by:

$$\mathcal{X}^{\phi(y)+\alpha}(y) \cdot \mathcal{X}^{2\phi(y)+\alpha} \cdot \mathcal{X}^{-1}, \text{ all } y \in Y, \alpha \in \Sigma.$$
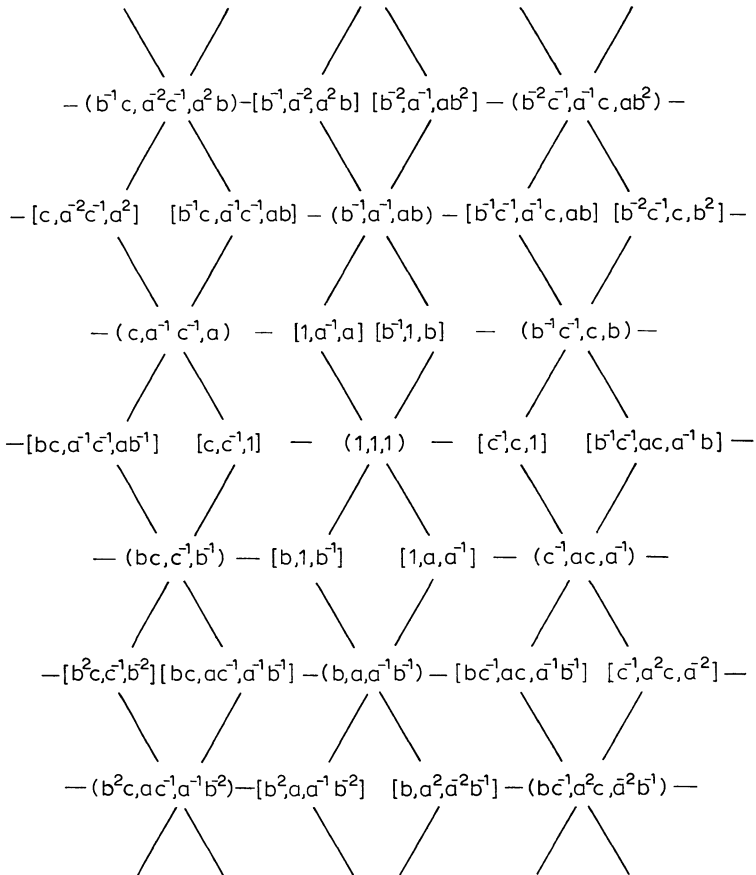
One then checks easily that:

$$U_{\mathcal{X},0} = \mathrm{Spec}\,A[\mathcal{Y}, c\mathcal{Y}^{-1}, \mathcal{Z}, b\mathcal{Z}^{-1}]$$
$$\cong \mathrm{Spec}\,A[X_1, X_2, X_3, X_4]/(X_1 X_2 - c, X_3 X_4 - b).$$

In particular, this is regular, so $\mathrm{Proj}(R_{\phi, \Sigma})$, $\mathfrak{P}$ and P are all regular 5-dimensional schemes. Moreover, $(U_{\mathcal{X}, 0})_0$ has 4 components which are the orbits of $\tilde{G}_0$ containing the 4 sections:

$$(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = (1, 1, 1)$$
$$= (c^{-1}, c, 1)$$

$$= (b^{-1}, 1, b)$$
$$= (b^{-1}c^{-1}, c, b).$$

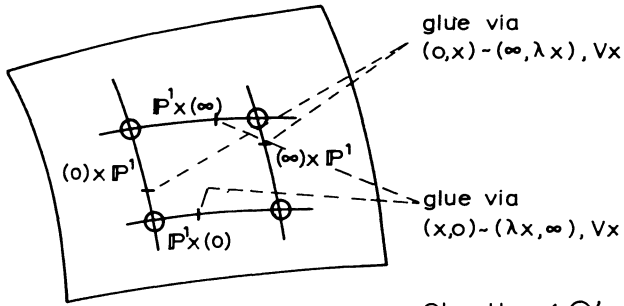Applying the automorphisms σ, τ and translation periods, it is not hard to construct the whole closed fibre of $\text{Proj}(R_{\phi, \Sigma})$. We give the result in the following diagram (strangely isometric to the root diagram of G_2). Each triple stands for the component which is the $\tilde{G}_0$-orbit of the closed point on the section defined by that triple; 2 components meet along a curve if they are joined by a line. For each diamond, there is one point at which the 4 components corresponding to its 4 vertices meet. The components indicated by triples in square brackets are projective planes. each of them meets 3 other components along the three lines of a triangle, The components indicated by triples in round brackets are projective planes with 3 non-collinear points blown up; the 3 exceptional lines, plus the 3 lines joining pairs of blown-up points, form a hexagon along which these components meet 6 neighbouring components.

$$- (b^{-1}c, a^{-2}c^{-1}, a^2 b) - [b^{-1}, a^{-2}, a^2 b] \quad [b^{-2}, a^{-1}, ab^2] - (b^{-2}c^{-1}, a^{-1}c, ab^2) -$$

$$- [c, a^{-2}c^{-1}, a^2] \quad [b^{-1}c, a^{-1}c^{-1}, ab] - (b^{-1}, a^{-1}, ab) - [b^{-1}c^{-1}, a^{-1}c, ab] \quad [b^{-2}c^{-1}, c, b^2] -$$

$$- (c, a^{-1}c^{-1}, a) \quad - \quad [1, a^{-1}, a] \quad [b^{-1}, 1, b] \quad - \quad (b^{-1}c^{-1}, c, b) -$$

$$-[bc, a^{-1}c^{-1}, ab^{-1}] \quad [c, c^{-1}, 1] \quad - \quad (1, 1, 1) \quad - \quad [c^{-1}, c, 1] \quad [b^{-1}c^{-1}, ac, a^{-1}b] -$$

$$- (bc, c^{-1}, b^{-1}) - [b, 1, b^{-1}] \quad [1, a, a^{-1}] \quad - \quad (c^{-1}, ac, a^{-1}) -$$

$$-[b^2c, c^{-1}, b^{-2}][bc, ac^{-1}, a^{-1}b^{-1}] - (b, a, a^{-1}b^{-1}) - [bc^{-1}, ac, a^{-1}b^{-1}] \quad [c^{-1}, a^2c, a^{-2}] -$$

$$- (b^2c, ac^{-1}, a^{-1}b^{-2}) - [b^2, a, a^{-1}b^{-2}] \quad [b, a^2, a^{-2}b^{-1}] - (bc^{-1}, a^2c, a^{-2}b^{-1}) -$$
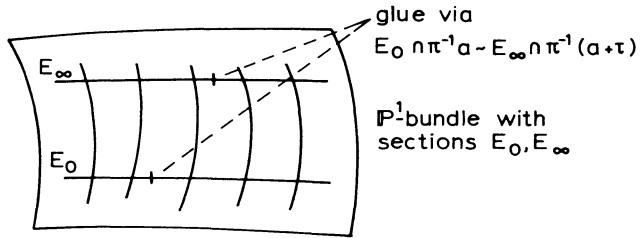
To find the structure of the closed fibre of P, we must divide by the action of Y. All the components of $\text{Proj}(R_{\phi, \Sigma})_0$ collapse to 3 components: one isomorphic to the three times blown-up projective plane, which is the image of $(1, 1, 1)$; two isomorphic to projective planes which are the images, for example, of $[1, a^{-1}, a]$ and $[1, a, a^{-1}]$ respectively. They are glued as indicated below (I).

Glue the 4 ○'s
Glue the 4 △'s
Glue the 4 □'s
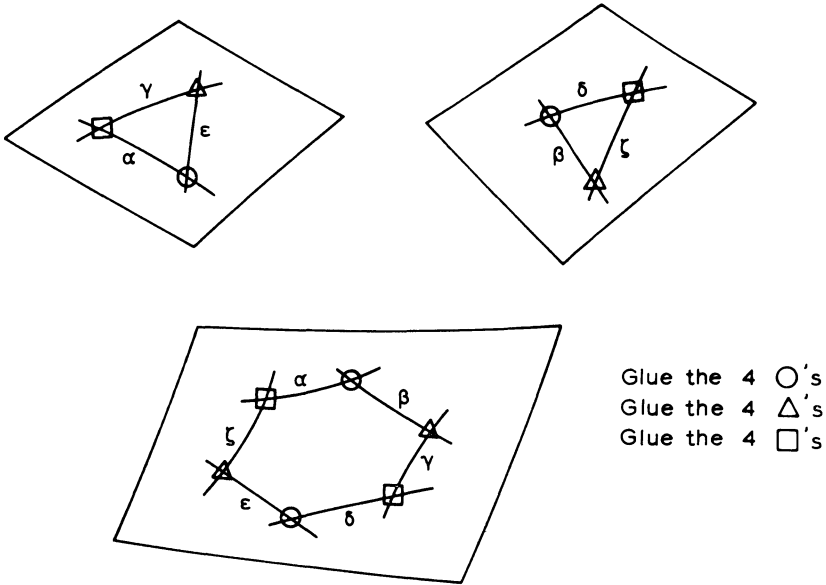
The other fibres of P over Spec (A) have the following structure: over any point x where $a \cdot b \cdot c \neq 0$, the fibre is a 2-dimensional abelian variety. Over a point where *only one* of the 3 coordinates a, b, c is 0, the fibre is a P^1-bundle over an elliptic curve glued to itself along 2 disjoint sections of this fibration but with a shift (cf. illustration III). Over a point where exactly *two* of the coordinates a, b, c are 0, the fibre is $P^1 \times P^1$ glued to itself along 2 pairs of lines (cf. illustration II).

glue via
$(o,x) \sim (\infty, \lambda x)$, $\forall x$

$\mathbb{P}^1 x (\infty)$

$(o) x \mathbb{P}^1$

$(\infty) x \mathbb{P}^1$

glue via
$(x,o) \sim (\lambda x, \infty)$, $\forall x$

$\mathbb{P}^1 x (o)$

Glue the 4 ○'s

E_∞

E_0

glue via
$E_0 \cap \pi^{-1} a \sim E_\infty \cap \pi^{-1}(a+\tau)$

$\mathbb{P}^1$-bundle with
sections E_0, E_∞

π

Elliptic curve

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