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EMBEDDINGS, ISOTOPY AND STABILITY OF BANACH MANIFOLDS

by

K. D. Elworthy

Introduction

The results of [7], [28], [13] combine to show that homotopy equivalent separable metrisable infinite dimensional C^∞ Hilbert manifolds are C^∞ diffeomorphic and suggest that the basic differential topology of infinite dimensional manifolds can be reduced to the study of the homotopy theory of vector bundle maps between Banach space bundles over C.W. complexes: in other words the classification problem for manifolds, embeddings, and diffeomorphisms, depends on little more than the tangential homotopy type of the manifold or smooth map. For C^0 manifolds it is essentially only the homotopy type that is involved and there is ample evidence [1] to support this rather vaguely worded assertion. In this paper we are concerned with separable metrisable differentiable manifolds and will give further supporting evidence for the case of parallelisable manifolds modelled on fairly general Banach spaces which are C^∞-smooth (i.e. admit C^∞ partitions of unity). In fact our conclusions and scope can be summed up in a rather crude and exaggerated way as follows:

For separable metrisable manifolds modelled on a wide class of Banach spaces E:

(i) If E is C^∞-smooth the differential topology of parallelisable C^∞ E-manifolds is trivial in the sense above (the problem at the end of § G indicates how much this remains an exaggerated statement).

(ii) If E is C^∞-smooth it seems likely that our methods can be extended to give (i) for the case of non-parallelisable manifolds and in particular to show that such manifolds are diffeomorphic to the total space of a vector bundle over some open subset of a Banach space. However, we hardly consider this case at all.

(iii) If E is C^r smooth, $r \geqq 1$ and has an infinite dimensional smooth factor space then the C^r diffeomorphism classes of open subsets of E are determined by their homotopy types.

(iv) the cases (a) C^r classification of C^r manifolds with C^∞-smooth model, (b) C^r classification with C^r-smooth model, (c) C^∞ classification with C^r-smooth model, get steadily more intractable and we have little or nothing to say about these cases. For example, I do not know whether every smooth manifold modelled on $C(0, 1)$, or l_1, even admits a differentiable real valued function which is not constant. An exception to this is case (a) when $E = l_{2n}$, $n = 1, 2, \cdots$ or $E = c_0$: the work of Nicole Moulis [29] implies that a C^r-manifold with one of these spaces as model admits a compatiable C^∞-structure and so (i) applies. We have only one, isolated and rather weak, result, Theorem 27 of § H, which comes under the heading (c).

Initially the paper is in two parts. The first part is concerned with closed embeddings; their existence, extension, and ambient isotopy classification. The main results here are in Theorem 12 and 14. The second part which occurs at the beginning of § F, is centered around a proof of the Palais stability of open subsets of certain Banach spaces, Proposition 16. In the Hilbert space case this reduces to Kuiper and Burghelea's theorem [7] but the proof which does not use Morse theory is much simpler than the original one. After this the two parts are combined to discuss the existence and isotopy classification of diffeomorphisms (Theorem 19) and the classification of manifolds with boundary, and manifold pairs, (Theorem 22 and 33). In § G we give another version of the isotopy theorem and then prove some results on open embeddings, including manifolds with boundary and manifold pairs. We attack open subsets of some non-smooth spaces in § H: this leads to (iii) above and in particular to a diffeomorphism classification of open subsets of Lp spaces, $2 \leqq p < \infty$.

Finally, in an appendix, we discuss manifolds modelled on certain Banach spaces which are not isomorphic to their squares, and give an example of a separable metrisable manifold which has no C^1 immersion into any finite direct sum of its model space. Some aspects of this are considered in more detail in [21].

The section on embeddings and isotopy gives a strengthened version of the embedding theorem announced without proof in [15], and uses the technique of layer manifolds and maps. This was also one of the basic tools in the paper by J. Eells and the author: 'Open Embeddings of Certain Banach Manifolds' [13], henceforth denoted by OECBM. In fact, we have to use the layer version of the main theorem of that paper (our Theorem 00 § C). The basic theory of layer structures can be found in [15], [16]. However, it turned out that we needed a more refined version of the existence theorem for layer structures, and this led to a better proof of their existence and classification (Theorem 10) than the ones previously given. This means that this paper taken together with OECBM forms a fairly self-contained exposition. The main exception to this lies in the use of Kuiper and Burghelea's tubular neighbourhood theorem which is stated without proof.

Many of the results given here were announced at the SMS, Montreal 1969, and the proofs are sketched in the lecture notes [30]. The main improvement since then is that for stability results we no longer need assume that our model spaces have a split subspace isomorphic to l_2. In particular, the differential topology of c_0-manifolds can be included. This is a worthwhile gain in view of the fact that it covers the case of many function space manifolds modelled on spaces of functions which satisfy Hölder type conditions [4]. On the other hand the attempt at as much generality as possible has led to an annoying plethora of side conditions on the model spaces. The reader who does not want to get involved in this could restrict himself to considering only the Hilbert manifold case, and would then find that the proofs from § F onwards become extremely simple.

Some of the developments in the theory of differentiable Banach manifolds since Eells' survey [12] are described in the Montreal notes [20], [30] and also in [21].

I would like to thank J. Eells for a lot of help: in particular for suggesting the approach that is used in the basic stability theorem (Proposition 16). I am also very grateful to N. H. Kuiper for many helpful discussions.

Unless the contrary is specifically stated, from now on all manifolds considered will be assumed to be separable, metrisable, and modelled on Banach spaces.

A. Preliminaries

We first recall some of the basic definitions concerning layer manifolds [15], [16].

A map $k : X \to E$ of a topological space into a linear space is called *locally finite dimensional* (l.f.d.) if each point x of X has a neighbourhood N_x with $k(N_x)$ contained in some finite dimensional subspace of E. If T lies in the space $L(E, F)$ of bounded linear operators between the Banach spaces E and F and if U is open in E then $f : U \to F$ is an $L(T)$-*map* if $f - T : U \to F$ is l.f.d. When $E = F$ and T is the identity map, I, of E we obtain a group $GL_F(E)$ consisting of the invertible linear $L(I)$-maps. This is a subgroup of the group of units $GL(E)$ of $L(E, E)$.

A C^p *layer structure*, modelled on E, M_1, on a C^p manifold M, is a maximal C^p atlas $\{(U_i, \varphi_i)\}_i$ for M, $\varphi_i : U_i \to E$, such that when defined

$\varphi_i \circ \varphi_j^{-1}$ is an $L(I)$-map. A *layer manifold* is a manifold together with a preferred layer structure. It then has a *tangent layer bundle*, TM_1, which for our purposes may simply be taken to be TM together with its induced reduction to $GL_F(E)$: although this is not the definition in [16] nor the one needed for OECBM. A map between layer manifolds M and N is an $L(T)$-map if it is represented as an $L(T)$-map by the layer charts of M and N.

We shall need a slight strengthening of this last notion. Recall that a sequence $\{e\}_{i=1}^{\infty}$ in a Banach space is a *basic sequence* if it is a Schauder base for its closed linear span. Suppose we have a direct sum $E_1 \times E_2$ of Banach spaces with natural injection $T: E_1 \to E_1 \times E_2$ and projection $\pi: E_1 \times E_2 \to E_2$, together with an $L(T)$-map $f: M_1 \to E_1 \times E_2$. Then f will be called a 'strong $L(T)$-map' if there is a basic sequence $\{e_1, e_2, \cdots\}$ of E_2 with an open cover $\{U_i\}_{i=1}^{\infty}$ of M such that, for each i, $\pi f(U_i)$ lies in the span of $\{e_1, \cdots, e_{\bar{i}}\}$ for some $\bar{i}$. When E_2 is a Hilbert space this is no extra restriction on f; I do not know if the same is true for more general E_2. We shall often wish to specify $\{e_i\}_{i=1}^{\infty}$ in advance and say that f is strong with respect to a given basic sequence.

Manifolds of class C^p are called *C^p-smooth* if they admit C^p partitions of unity. In the separable metrisable case under consideration this is ensured if the model E is C^p-smooth, and this will follow if E has a C^p norm, i.e. an admissible norm which is of class C^p on $E-\{0\}$, [3]. Since it is unknown whether every C^p-smooth E admits such a norm $(p > 1)$ we will often have recourse to the referee's method in OECBM lemma 6 which gives the existence of smooth functions which behave sufficiently like norms for our purposes. The version of this which we will use is given below as a lemma, in a form that will also be needed in the proof of stability, even for some spaces, like c_0, which are known to have smooth norms. A subset D of E is *radial* if whenever x is in D and $0 \leqq t \leqq 1$ then tx is in D.

LEMMA A. *Let D and D' be closed bounded radial subsets of the C^p-smooth Banach space $E, p \geqq 1$, with*

$$0 \in \operatorname{Int} D \subset kD \subset \operatorname{Int} D'$$

for some $k > 1$. Then there exists a C^p map $\mu: E \to [0, 1]$ and a continuous $\rho: E \to R\ (\geqq 0)$ satisfying:

(i) Supp $\mu \subset$ int D', $\mu | D \equiv 1$, *and for each x in $E-\{0\}$ $\mu(tx)$ decreases strictly in t except when its value is* 0 *or* 1;

(ii) $\rho | E-\{0\}$ *is C^p and for each x in $E-\{0\}$ the map $t \to \rho(tx)$ is a submersion of $R\ (> 0)$ onto itself. Further $\rho(tx) \to \infty$ as $t \to \infty$ uniformly in x, in any subset of E which is bounded away from* 0, *and the family*

$\{\rho^{-1}[(0, 1/n)]\}_{n=1}^{\infty}$ *forms a base for the neighbourhood system of* E *at* 0;

(iii) $D \subset \rho^{-1}[(0, 1)] \subset \rho^{-1}[(0, 2)] - \operatorname{int} D'$.

PROOF. Choose α with $1 < \alpha < \alpha^3 < k$. As in OECBM for any continuous $\psi : E \to R$ with bounded support set

$$s\psi(x) = \int_1^{\infty} \psi(tx)dt \qquad \text{for } x \in E-\{0\}.$$

Take a C^p map $\lambda : E \to [0, 1]$ with $\operatorname{supp} \lambda \subset D'$ and $\lambda|\alpha^3 D \equiv 1$. Then $s\lambda : E-\{0\} \to R$ is C^p and decreases strictly to zero along each ray. Also, if $x \in \alpha^2 D$,

$$s\lambda(x) \geqq \int_1^{\alpha} \lambda(tx)dt = \alpha-1.$$

Let $\varphi : R(>0) \to [0, 1]$ be C^∞ with $\varphi(0) = 0$, $\varphi(t) = 1$ for $t \geqq \alpha-1$, and with φ strictly increasing in $(0, \alpha-1)$. Set $\mu(x) = \varphi \circ s\lambda(x)$ for $x \in E-\{0\}$, $\mu(0) = 1$. Then μ satisfies (i) and even has $\mu|\alpha^2 D \equiv 1$. It follows, as above, that $s\mu(x) \geqq \alpha-1$ for $x \in \alpha D$. Since the interval $(0, 1)$ contains only regular values of the map $t \mapsto s\mu(tx)$ for each x in E the set $\tilde{S} = (s\mu)^{-1}((\alpha-1)/2)$ is a C^p submanifold of E and if $\theta : \tilde{S} \times R(>0) \to E-(0)$ is defined by $\theta(v, t) = tv$ then θ is a C^p diffeomorphism. Now take a C^∞ diffeomorphism $\rho_1 : R(\geqq 0) \to R(\geqq 0)$ with $\rho_1(1/\alpha) = 1$ and $\rho_1(1) = 2$ and define $\rho_0 : \tilde{S} \times R(>0) \to R(>0)$ by $\rho_0(v, t) = \rho_1(t)$. Now set $\rho(x) = \rho_0 \theta^{-1}(x)$ for $x \in E-\{0\}$ and $\rho(0) = 0$. This is seen to satisfy our conditions.//

Any continuous $\rho : E \to R(\geqq 0)$ satisfying (ii) will be called *norm-like*. For $t > 0$, the subset $\rho^{-1}[0, t]$ is then a closed bounded radial domain, containing 0 in its interior, whose boundary is a C^p submanifold of E which is transverse to each ray through 0. Such subsets, and their translates will be referred to as *pseudo discs* and *pseudo spheres*.

REMARKS. (i) Suppose that D, D' are closed subsets of the total space B of a C^p-smooth vector bundle $\pi : B \to X$, having the zero section $S(X) \subset \operatorname{int} D \subset kD \subset \operatorname{int} D'$ for some $k > 1$. Assume also that D' is bounded with respect to some Finsler on B and that the intersections of D and D' with the fibres of B are radial. Then a straightforward modification of the proof of lemma A shows that there is a continuous $\rho : B \to R(\geqq 0)$, C^p on $B-S(X)$, which restricts to a norm-like function on each fibre and has $D \subset \rho^{-1}[0, 1] \subset \rho^{-1}[0, 2] \subset \operatorname{int} D'$. Such a function will be called *Finsler-like*.

(ii) Even though c_0 admits a C^∞ norm there exists no norm-like function on it which is C^∞ on the whole space. This is because such a function would have bounded first and second derivatives in a closed neighbourhood N of 0 but would also have to be bounded away from zero on ∂N; according to J. Wells [36] this is impossible.

Next we collect some fairly well known results which will enable us to avoid drops in differentiability class in our constructions.

LEMMA. *Any vector bundle over a C^p-smooth manifold is equivalent to a vector bundle of class C^p, $0 \leqq p \leqq \infty$.*

PROOF. Let $\pi : B \to M$ be an E-vector bundle over the C^p-smooth manifold M. The Grassmanian of all those split subspaces of a Banach space F which are isomorphic to E and have co-space isomorphic to a space H will be denoted by $G_E(F, H)$. According to Douady, [9], this has the structure of a real analytic manifold modelled on $L(E, H)$, in general non-separable. As in the finite dimensional case, consider the subset $\gamma_E(F, H)$ of $F \times G_E(F, H)$ consisting of those pairs (x, V) with $x \in V$. This is easily seen to have the structure of a C^∞ (even real analytic) E-vector bundle over $G_E(F, H)$.

Now take $F = H = c_0(E)$. Then π can be considered a sub-bundle of the trivial $c_0(E)$-bundle over M (see I(iii) § D below). This induces a continuous map $f : M \to G_E(F, F)$ with $f^*(\gamma_E(F, F))$ equivalent to π. Approximating f by a homotopic C^p map $\bar{f}$ we obtain a C^p bundle $\bar{f}^*(\gamma_E(F, F))$ as required.//

LEMMA. *Let $f : M \to N$ be a closed C^p embedding into a manifold of class C^{p+2}. If both manifolds are C^p-smooth and $p \geqq 1$ there exists a C^p tubular neighbourhood of $f(M)$ in N.*

PROOF. To use Lang's proof [23] we need a C^p spray on N and a C^p transverse bundle to $Tf(TM)$ in TN. There is no problem about the spray but the standard method only gives a C^{p-1} transverse bundle, $\pi : B \to f(M)$ say. However, by the lemma above, the bundle π is C^0 equivalent to a C^p bundle π' over $f(M)$. It is a straightforward exercise in partitions of unity to approximate the induced C^0 bundle injection of π' into $TN|_{f(M)}$ by a C^p injection which gives the required normal bundle.//

A proof of the following is given by Kuiper and Burghelea in [7] for the case of manifolds whose models admit C^∞ norms. The modifications of that proof needed for our statement are immediate using our lemma and Remark (i) above:

HIRSCH'S LEMMA. *Let $f : M \times R \to N \times R$ be a C^p isotopy of closed embeddings, $p \geqq 1$, $f(m, t) = (h(m, t), t) = (h_t(m), t)$, such that $h|M \times [-\varepsilon, 1+\varepsilon]$ is a closed embedding into N for some $\varepsilon > 0$. Assume also that N is of class C^{p+2} and is C^p-smooth. Then there is a C^p isotopy $\Phi : N \times R \to N \times R$ with $\Phi_0 = id$ and $\Phi_1 \circ h_0 = h_1$. The support of Φ can be chosen to lie in any given neighbourhood of $h(M \times [-\varepsilon, 1+\varepsilon])$.//*

B. Layer embeddings and isotopy

PROPOSITION 1. *Let M be a C^p-smooth layer manifold over $E, p \geqq 1$. Suppose that $T : E \to F$ is a continuous linear injection of E onto a closed, infinite co-dimensional, complemented subspace of a Banach space F and that $f : M \to F$ is a $C^pL(T)$-map.*

Then for any continuous $\delta : M \to R(>0)$ there is a C^p map $k : M \to F$ satisfying:

(i) *k is compact and locally finite dimensional*
(ii) *$||k(x)|| < \delta(x)$ for all x in M*
(iii) *$f+k : M \to E$ is a $1-1$ immersion.*

Moreover if X_1 is an open neighbourhood of a closed subset X of M and if f is an embedding on a neighbourhood of $\overline{X}_1$, which is closed on $\overline{X}_1$, such that $\overline{f(M)}$ does not reintersect $f(\overline{X}_1)$, we may choose k to vanish on X. Also if f is a strong $L(T)$-map with respect to some basic sequence in E' for some splitting of F into $T(E)\times E'$ we may arrange k so that $f+k$ remains strong with respect to the same basic sequence.

PROOF. Take open subsets X_0, X_2, X_3 of M so that $f|X_0$ is an embedding and $X \subset X_3, \overline{X}_3 \subset X_2, \overline{X}_2 \subset X_1, \overline{X}_1 \subset X_0$. Identify $T(E)$ with E and write $F = E \times E'$ where E' is infinite dimensional and $T(x) = (x, 0)$.

Choose a star-finite open cover $\{U_i\}_{i=1}^{\infty}$ of M together with positive numbers $m(i)$ and finite dimensional subspaces A_i of F to satisfy:

(a) δ is bounded below by $m(i)$ on U_i, and $m(i) < 1$;

(b) there is a layer chart $\varphi_i : U_i \to E$ with

$$f \circ \varphi_i^{-1} - T : \varphi_i(U_i) \to F$$

having bounded range in A_i;

(c) for these φ_i, each $\varphi_i(U_i)$ is contained in the unit ball of E;

(d) if $U_i \cap \overline{X}_r \neq \emptyset$ then $U_i \subset X_{r-1}, r = 1, 2, 3$.

For each x in M choose an integer $i(x)$ with $x \in U_{i(x)}$. If $U_{i(x)} \subset X_i$ take a neighbourhood V'_x of x and an $r_x > 0$ with $V'_x \subset U_{i(x)}$ and $f^{-1}(B_{r_x}(f(y)) \subset U_{i(x)}$ for all $y \in V'_x$; otherwise take $r_x = 1$ and V'_x to be an arbitrary neighbourhood of x with $\overline{V}'_x \subset U_{i(x)}$. If $U_{i(x)} \cap \overline{X}_2 = \emptyset$ take a neighbourhood V''_x of x and an $s_x > 0$ with $\overline{V}''_x \subset U_{i(x)}$ and $B_{s_x}(f(y)) \cap f(\overline{X}_2) = \emptyset$ for $y \in V''_x$; otherwise choose V''_x so that $\overline{V}''_x \subset U_{i(x)}$ and take $s_x = 1$.

Set $V_x = V'_x \cap V''_x$ and take a star-finite open refinement $\{V_j\}_{j=1}^{\infty}$ of the cover $\{V_x\}$. Make the following further definitions for each j:

$r_j = r_x, s_j = s_x, i(j) = i(x)$, for some x with $V_j \subset V_x$;

$$\rho_j = \min\left\{r_a, s_a, \frac{m(i(a))}{2^a}\right\}$$

taken over all a with $V_a \cap V_j \neq \emptyset$;

$\varepsilon_j = 0$ if $U_{i(j)} \cap \bar{X}_3 \neq \emptyset$ and $\varepsilon_j = 1$ otherwise;

$d_j = \dim A_{i(j)}$;

$\gamma_j = \{i : 0 < i \leqq \max_{n \leqq j} i(n)\} \cup \{i : U_i \cap U_{i(j)} \neq \emptyset\}$.

Since E' is infinite dimensional we can find by induction a sequence $\{B_j\}_{j=1}^{\infty}$ of subspaces of E' with $\dim B_j = d_j$ together with a sequence $\{Z_j\}_{j=1}^{\infty}$ of elements of E' with $||Z_j|| = 1$ such that for each $n = 1, 2, \cdots$ the subspaces B_n, $\boldsymbol{R}Z_n$ are linearly independent of each other, and of the span of the other with

$$(\bigcup_{i \in \gamma_n} A_i) \cup (\bigcup_{j=1}^{n-1} B_j) \cup (\{Z_1, \cdots Z_{n-1}\}).$$

Let $p_j : F \to B_j$ be the composition of a projection of F onto $A_{i(j)}$ with a linear isomorphism of $A_{i(j)}$ onto B_j, having $||p_j|| \leqq 1$.

Take a shrunk refinement $\{V_j'\}$ of $\{V_j\}$: i.e. an open cover with $\bar{V}_j' \subset V_j$ for each j. Set $N_0 = X_0$ and $N_n = X_0 \cup \bigcup_{j=1}^{n} V_j'$. Define $k_0 : M \to F$ by $k_0(M) = 0$, and assume inductively that we have a set $\{k_j\}_{j=0}^{n-1}$ of C^p functions $k_j : M \to F$ satisfying:

(1) $k_j(X_1) = \{0\}$

(2) $k_j(M - \bigcup_{a=1}^{j} V_a) = 0$

(3) $k_j|(M - V_j) = k_{j-1}|(M - V_j)$

(4) $k_j(M)$ lies in the linear span of $\bigcup_{a=1}^{j} B_a$

(5) for x in V_a, $||k_j(x)|| \leqq \frac{1}{2^j}\left(\frac{1}{2} + \cdots + \frac{1}{2^j}\right) \min\left(\varepsilon_a, r_a, s_a, \frac{m(i(a))}{2^a}\right)$

(6) if $h_j = f + k_j : M \to F$ then $h_j|N_j$ is an immersion.

If $U_{i(n)} \cap \bar{X}_1 \neq \emptyset$ set $k_n = k_{n-1}$. If not, take a C^p map $\psi_n : M \to [0, 1]$ with support in V_n and with $\psi_n|V_n' \equiv 1$. Define $k_n : M \to F$ by

$$k_n(x) = k_{n-1}(x) + \frac{1}{2^{n+1}} \rho_n \psi_n(x) p_n T\varphi_{i(n)}(x)$$

for $x \in V_n$ and by $k_n(x) = k_{n-1}(x)$ otherwise.

Certainly k_n satisfies (1)–(4). For condition (5) suppose $x \in V_a$. If $V_a \cap V_n = \emptyset$ there is no problem since $k_n(x) = k_{n-1}(x)$. When $V_a \cap V_n \neq \emptyset$ and $\varepsilon_a = 0$ then $U_{i(a)} \cap \bar{X}_3 \neq \emptyset$ so that $U_{i(a)} \subset X_1$ and again $k_n(x) = k_{n-1}(x)$. Finally when $V_a \cap V_n \neq \emptyset$ and $\varepsilon_a = 1$ we have

$$\begin{aligned} ||k_n(x)|| &\leqq ||k_{n-1}(x)|| + \frac{1}{2^{n+1}} \rho_n \\ &\leqq ||k_{n-1}(x)|| + \frac{1}{2^{n+1}} \min\left(\varepsilon_a, r_a, s_a, \frac{m(i(a))}{2^a}\right) \end{aligned}$$

giving (5).

For condition (6), if $U_{i(n)} \cap \overline{X}_1 \neq \emptyset$ there is no problem. When $U_{i(n)} \cap \overline{X}_1 = \emptyset$ first note that $h_n|N_{n-1}$ is an immersion since $h_{n-1}|N_{n-1}$ was an immersion and $(k_n - k_{n-1})N_{n-1}$ is linearly independent of $h_{n-1}(N_{n-1})$. Suppose therefore that $x \in V'_n$. By (b) and (4)

$$\ker D(h_{n-1}\varphi_{i(n)}^{-1})(\varphi_{i(n)}(x)) \subset A_{i(n)} \cap E.$$

Since

$$D(h_n \varphi)_{i(n)}^{-1})(\varphi_{i(n)}(x)$$

is the sum of

$$D(h_{n-1}\varphi_{i(n)}^{-1})(\varphi_{i(n)}(x))$$

with the map $v \mapsto (\frac{1}{2})^{n+1}\rho_n p_n Tv$ which is injective on $A_{i(n)} \cap E$, we see that it is injective. Also h_n splits since it is an $L(T)$-map, and so (6) is satisfied.

Taking the limit of $\{k_j\}_{j=1}^{\infty}$ we obtain a C^p map $k' : M \to F$ satisfying:

(1)′ $k'(X_1) = 0$

(2)′ $k'(M) \subset \mathrm{Sp}\,(\bigcup_{a=1}^{\infty} B_a)$

(3)′ k' has finite dimensional range on each V_a

(4)′ for $x \in V_a$, $||k'(x)|| \leqq \frac{1}{2}\min\left(\frac{m(i(a))}{2^a}, r_a, s_a, \varepsilon_a\right)$

(5)′ if $h' = f + k'$ then $h' : M \to F$ is an immersion.

In particular, by (4)′, $||k'(x)|| < 1/2^a$ for $x \in V_a$. Hence (3)′ gives:

(6)′ k' is compact.

Since h' is an immersion, there is a star-finite open cover $\{W_b\}_{b=1}^{\infty}$ of M with h' injective on each W_b. We may choose $W_b \subset V_{j(b)}$ for some $j(b)$, and also (taking $W_b = \emptyset$ when necessary) we may assume $j(b) \leqq b$. Set

$$\rho(b) = \frac{1}{2}\min\left\{r_j, s_j, m(i(j)), \frac{1}{2^a}, \varepsilon_{j(b)}\right\}$$

taken over all j with $V_j \cap W_b \neq \emptyset$ and all a with $W_a \cap W_b \neq \emptyset$. Take a C^p partition of unity $\{\mu_b\}$ subordinate to $\{W_b\}$ and define $k'' : M \to F$ by

$$k''(x) = \sum_b \mu_b(x)\rho(b)Z_b$$

Then

(1)″ $k''(\overline{X}_3) = 0$ and if $k''(x) = 0$ and $\mu_b(x) \neq 0$ then $U_{i(j(b))} \cap \overline{X}_3 \neq \emptyset$

(2)″ k'' has finite dimensional range on each W_b

(3)″ if $x \in W_b$ then $2\|k''(x)\| \leqq \min\{\frac{1}{2}b, r_{j(b)}, s_{j(b)}, \delta(x)\}$.

By (2)″ and (3)″ we see that k'' is also compact.

Now set $k = k'+k''$ and $h = f+k$. By linear independence h is an immersion. To show that it is injective suppose $x, y \in M$ with $h(x) = h(y)$. Let a, b be the largest integers with $\rho(a)\mu_a(x) \neq 0$ and $\rho(b)\mu_b(y) \neq 0$, or zero when no such integers exist.

Case (i): $a \neq 0$, $b \neq 0$. Then $x \in W_a$ and $y \in W_b$, so that

$$h(x) \in \mathrm{Sp}\,(E \cup A_{i(j(a))} \cup \bigcup_j B_j \cup \{Z_1, \cdots, Z_a\})$$

and

$$h(y) \in \mathrm{Sp}\,(E \cup A_{i(j(b))} \cup \bigcup_j B_j \cup \{Z_1, \cdots, Z_b\}).$$

By linear independence we see $a = b$. Hence $x, y \in W_a$. Since $f+k'$ was injective on W_a we have $x = y$, again by the linear independence.

Case (ii): $a = 0$, $b \neq 0$. For some α, $\mu_\alpha(x) \neq 0$. Then (1)″ gives $U_{i(j(\alpha))} \cap \overline{X}_3 \neq 0$. Hence $x \in X_2$ and $h(x) = f(x)$. Since $y \in W_b$, $\|k(y)\| < r_j(b), s_j(b)$. It follows that $U_{i(j(b))} \cap \overline{X}_2 \neq \emptyset$: otherwise we would have $B_{s_{j(b)}}(f(y)) \cap f(\overline{X}_2) = \emptyset$. Hence $U_{i(j(b))} \subset X_1$, and so $f^{-1}(B_{r_{j(b)}}(f(y))) \subset U_{i(j(b))}$, giving $x \in U_{i(j(b))}$. But then $h(y) = h(x) = f(x)$ lies in the span of $E \cup A_{i(j(b))}$ and so $k(y) = 0$, showing that this case is impossible.

Case (iii): $a = 0$, $b = 0$. Here both x and y lie in $\overline{X}_2$ and so $f(x) = h(x) = h(y) = f(y)$, giving $x = y$.

Thus h is a $1-1$ immersion.

In the case where f is a strong $L(T)$-map we may choose the splitting $F = E \times E'$ so that f is strong with respect to a basic sequence $\{e_i\}_{i=1}^\infty$ of E' and then choose the Z_n from among this sequence and the subspaces B_n so that each B_n is spanned by finitely many e_i. The resulting $1-1$ immersion h will then be strong with respect to $\{e_i\}$. This completes the proof of proposition 1.//

Next consider an open subset X of a Banach space E together with a commuting family of projections $\{p_n\}_{n=1}^{\infty}$ on E. Set $p_n(E) = H_n$, $H_n \cap X = X_n$, $p^n = 1-p_n$, and $p^n(E) = H^n$. Assume that $H_n \subset H_{n+1}$, each n. If $r_n : X_n \to R(>0)$ the *standard open tube* about X_n in X of radius r_n is

$$Z_n = \{x \in X : \|p^n x\| < r_n(p_n x)\}.$$

LEMMA 2. *Suppose that $r_n = r|X_n$ for some continuous $r : X \to R(>0)$, and that $\{p_n\}$ converges strongly to the identity. Then for any compact set K in X there is an integer N with $K \subset Z_n$ for all $n \geqq N$. In particular $X = \bigcup_n Z_n$.*

PROOF. Set $\delta = \inf\{r(x) : x \in K\}$. Suppose U is a neighbourhood of a point x in X. By the uniform boundedness theorem, since $\{p_n\}$ converges strongly to the identity, there is a neighbourhood V of X with $p_n(V) \subset U$ for all sufficiently large n. Hence, by compactness of K, there is an integer N_1 with $r(p_n x) > \delta/2$ for all $x \in K$ and all $n \geqq N_1$. Similarly, there is also an N_2 with $\|p^n x\| < \delta/2$ for $n \geqq N_2$ and all $x \in K$. Set $N = \max(N_1, N_2)$.//

The next proposition is the basis of our method of obtaining approximations by proper maps and hence by closed embeddings.

PROPOSITION 3. *Let X be an open subset of the direct sum $E = E_1 \times E_2$ of two Banach spaces, and M a C^p layer manifold over E_1, $p \geqq 1$. Suppose that $f : M \to X$ is a strong $L(T)$-map of class C^p, where $T : E_1 \to E_1 \times E_2$ is the natural inclusion, and that $r : X \to R(>0)$ is continuous. Assume that E_1 is C^p-smooth and E_2 is infinite dimensional. Then there is a proper, C^p, $L(T)$-map $\bar{f} : M \to X$ with $\|f(x)-\bar{f}(x)\| < r(f(x))$ for all x in M, which is strong with respect to the same basic sequence as f.*

Moreover, there is an open cover $\{U_j\}_{j=1}^{\infty}$ of M, independent of r, with each $f|\bar{U}_j$ proper as a map into E, such that $\bar{f}$ may be chosen with each $(f-\bar{f})|\bar{U}_j$ having bounded finite dimensional range. In particular $\bar{f}|\bar{U}_j$ is also proper as a map into E.

PROOF. Since f is a strong $L(T)$-map there is a basic sequence $\{e_1, e_2, \cdots\}$ for E_2, a star-finite open cover $\{U_j\}_{j=1}^{\infty}$ of M with layer chart maps $\{\varphi_j\}$ defined over each $\bar{U}_j$, and a strictly increasing sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ such that each $f \circ \varphi_j^{-1}|\varphi_j(\bar{U}_j)$ has the form $x \mapsto T_x + \alpha_j(x)$ where α_j is bounded and has range in $E_1 \times \mathrm{Sp}\{e_1, \cdots, e_{m_j}\}$. Note that this implies that $f|\bar{U}_j$ is proper, even as a map into E.

We may assume that $\{e_1, e_2, \cdots\}$ is a basis for all of E_2. Define the closed subspaces F_n, H_n, and H^n of E by

$$F_n = \mathrm{Sp}\{e_1, \cdots, e_{m_n}\}, H_0 = E_1, \quad H_n = E_1 \times F_n,$$
$$H^0 = E_2, \quad H^n = \overline{\mathrm{Sp}}\{e_{m_n+1}, e_{m_n+2}, \cdots\}$$

and let $p_n : E \to H_n$, $p^n : E \to H^n$ be the corresponding, complementary, projections. We can inductively define a strictly increasing sequence $\{\bar{j}\}_{j=1}^{\infty}$ of integers such that $e_{\bar{j}}$ does not lie in F_n whenever U_n intersects U_j.

As in [8; Chapt. IV, § 3 Theorem 1], if necessary, we may take a new equivalent norm on E to ensure that $||p_n|| = 1$ and hence $||p^n|| \leqq 2$, for each n.

Set $X_n = X \cap H_n$ and let Z_n, Z_n' be the standard open tubes about X_n of radius $r|X_n$, $\frac{1}{2}r|X_n$. We may assume that r is so small that $B_{r(x)}(x) \subset X$ for all x in X, and hence

$$Z_n = \{(x, y) \in X_n \times H^n \text{ s.t. } ||y|| < r(x)\}.$$

Take a shrunk refinement $\{V_j\}_{j=1}^{\infty}$ of $\{U_j\}$ and set $V_0 = U_0 = \emptyset$.

We shall inductively construct a sequence of C^p functions $\varphi_n : M \to R(\geqq 0)$ and define $k_n : M \to E$ by

$$k_n(x) = \sum_{j=0}^{n} \varphi_j(x) e_{\bar{j}}$$

to satisfy:

1) $\operatorname{Supp} \varphi_j \subset U_j$,

2) $||k_n(x)|| < r(f(x))$ all $x \in M$,

3) $(f+k_n)(\bar{V}_m) \subset X - Z_m'$ for $1 \leqq M \leqq n$.

From these follow

4) $k_n|(M-U_n) = k_{n-1}|(M-U_n)$,

5) $k_n(M) \subset \operatorname{Sp}\{e_1, \cdots, e_{\bar{n}}\}$

6) $(f+k_n)(M) \subset X$.

To do this define $\varphi_0 \equiv 0$ and assume that $\varphi_0, \cdots, \varphi_{n-1}$ have been constructed. Note that $f(U_n) \subset X_n$, that

$$(f+k_{n-1})(U_n) \subset H_n + \operatorname{Sp}\{e_1, \cdots, e_{\overline{n-1}}\} = G,$$

say, and that $k_{n-1}(U_n) \subset H^n$ by construction of the $\bar{j}$.

Set $A = \{z \in G \text{ s.t. } ||p^n z|| < \frac{3}{4} r(p_n z)\}$ and for each $z \in A$ define

$$K(z) = \{t \in R(\geqq 0) \text{ s.t. } \tfrac{2}{3} r(p_n z) \leqq ||t e_{\bar{n}} + p^n z|| \leqq \tfrac{3}{4} r(p_n z)\}.$$

Suppose $0 \leqq t_1 < t_2$ and $z \in A$. Then

$$||p^n z|| = ||p_{\overline{n-1}}(t_2 e_{\bar{n}} + p^n z)|| \leqq ||t_2 e_{\bar{n}} + p^n z||$$

and so

$$||t_1 e_n + p^n z|| \leqq \frac{t_1}{t_2} ||t_2 e_{\bar{n}} + p^n z|| + \left(\frac{1-t_1}{t_2}\right) ||p^n z|| \leqq ||t_2 e_{\bar{n}} + p^n z||.$$

It follows that each $K(z)$ is a non-empty closed interval, and so by a straightforward exercise in partitions of unity (or alternatively by appeal to Michael's theory of continuous selections [24]) we may construct a C^p function $\xi : G \to R(\geqq 0)$ with $\xi(z) \in K(z)$ for z in A and $\xi(z) = 0$ elsewhere.

Now take a C^p function $\lambda : M \to [0, 1]$ with $\operatorname{supp} \lambda \subset U_n$ and $\lambda | V_n \equiv 1$, and define $\varphi_n : M \to R$ by

$$\varphi_n(x) = \lambda(x)\xi(f(x)+k_{n-1}(x)).$$

Then

a) $\operatorname{supp} \varphi_n \subset U_n$

b) $(f+k_{n-1})(x)+\varphi_n(x)e_{\bar{n}} \in Z_n$ if $\varphi_n(x) \neq 0$

c) $(f+k_{n-1})(x)+\varphi_n(x)e_{\bar{n}} \notin Z_n'$ if $x \in \overline{V}_n$.

Certainly 1) is satisfied, as is 2) by b). Also c) says that $(f+k_n)(\overline{V}_n) \subset X-Z_n'$, and by adding $\varphi_n(x)e_{\bar{n}}$ to $f(x)+k_{n-1}(x)$ we have if anything increased the distances of points in the image of $\overline{V}_m$, $m < n$, from H_m, by the same argument as that used in the discussion of the $K(z)$. Hence 3) is true.

Define $k : M \to E$ by $k(x) = \sum_{j=0}^{\infty} \varphi_j(x)e_{\bar{j}}$. For each x, $k(x) = k_m(x)$ for any m greater than or equal to the largest n with $x \in U_n$. Consequently, if $\bar{f} = f+k, \bar{f}(\overline{V}_n) \subset X-Z_n'$ for all $n, \bar{f}$ is a strong $L(T)$-map as required, $||\bar{f}(x)-f(x)|| < r(f(x))$ and $(f-\bar{f})|\overline{U}_n$ has bounded finite dimensional range.

It remains to show that $\bar{f} : M \to X$ is proper. Suppose therefore that K is a compact subset of X. By lemma 2 there is an integer N with $K \subset Z_n'$ for all $n > N$. Then $\bar{f}^{-1}(K) \subset \bigcup_{j=1}^{N} \overline{V}_j$. Since $\bar{f}|\overline{V}_j$ is proper for each j it follows that $\bar{f}^{-1}(K)$ is compact.//

In the above we used the fact that the sum of a proper map into E with a compact map is still proper, as a map into E. More care seems needed in the case of maps which are proper into open subsets of E. Hence the next lemma.

LEMMA 4. *Given a countable locally finite open cover $\{U_i\}_{i=1}$ of a metric space M and an open subset X of Banach space E, there is a continuous $\delta : M \to R(> 0)$ such that for all continuous proper maps $f : M \to X$ with $f|\overline{U}_i$ proper as a map into E for each i, and for all compact $k : M \to E$ with $||k(x)|| < \delta(x)$, all $x \in M$, if $(f+k)(M) \subset X$ then $f+k : M \to X$ is proper.*

PROOF. Choose a continuous $\delta : M \to R(> 0)$ with $\delta(x) < 1/2^i$ for all $x \in U_i$. Suppose we have f and k as stated, with a sequence $\{x_i\} \subset M$ such that $f(x_i)+k(x_i) = y_i \to y$ in X. We may assume $\{k(x_i)\}$ converges to a point z in E. If the x_i belong to an infinite number of distinct U_j then $z = 0$ and $f(x_i) \to y$. Since f is proper this implies that $\{x_i\}$ has a convergent subsequence. Otherwise there is an n with $\{x_i\}_{i=1}^{\infty} \subset \bigcup_{j=1}^{n} \bar{U}_j$ and since $(f+k)| \bigcup_{j=1}^{n} \bar{U}_j$ is proper as a map into E we can again conclude that $\{x_i\}$ is subconvergent.//

LEMMA 5. *Suppose E, T, M are as in proposition* 3 *and that E_2 has a basis $\{e_1, e_2, \cdots\}$. Then given continuous maps $\delta : M \to R(> 0)$ and $h : M \to E$ there is a C^p, $L(T)$-map $h : M \to E$ which is a δ-approximation to h and is strong with respect to $\{e_i\}$.*

Moreover, if A is a closed subset of M, V is a neighbourhood of A in M, and $h_0 : V \to E$ a $C^pL(T)$-map, strong with respect to $\{e_i\}$, and a δ-approximation to $h|V$, we may choose $\bar{h}$ equal to h_0 on A.

PROOF. Let F be the linear subspace of E_2 consisting of the finite linear combinations of the e_i. Since F is dense in E_2 there is a continuous approximation to h with image in $E_1 \times F$, equal to h_0 near A. The lemma now follows from the standard $L(T)$ approximation result [15], [16] by considering the range of our maps to be $E_1 \times F$.//

We now come to the main result of this section. As for proposition 1 the main complications in the proof arise in obtaining an extension rather than an approximation. In fact, essentially, we first obtain an approximation, making it sufficiently close so that it may then be modified to give an extension.

THEOREM 6. *Suppose E, T, M, and X are as in proposition* 3, *and that E_2 has a basis $\{e_1, e_2, \cdots\}$. Given continuous maps $r : X \to R(> 0)$ and $h : M \to X$ there is a closed $C^p, L(T)$-embedding $\bar{h} : M \to X$ with $\|\bar{h}(x) - h(x)\| < r(h(x))$, for all x in M, which is strong with respect to $\{e_i\}$.*

Furthermore if V_1 is an open neighbourhood of a closed subset A of M and h is a C^p $L(T)$-embedding on a neighbourhood of $\bar{V}_1$, strong with respect to $\{e_i\}$ and closed on $\bar{V}_1$, with $\overline{h(M)}$ not reintersecting $h(\bar{V}_1)$, we may choose $\bar{h}$ equal to h on A.

PROOF. There is a continuous $\rho : X \to R(> 0)$ such that, for each $y \in X$, $B_{\rho(y)}(y) \subset X$ and $\rho(x) < \frac{1}{2}r(y)$ for all $x \in B_{\rho(y)}(y)$. Take a family $V_2, V_3, \cdots$ of open neighbourhoods of A in M with $\bar{V}_{i+1} \subset V_i$, $i \geqq 1$. By lemma 5, choosing a suitable δ, we obtain a C^p $L(T)$-map $f : M \to X$, strong with respect to $\{e_i\}$, with $f|\bar{V}_2 = h|\bar{V}_2$ and with $\overline{f(M)}$ not reintersecting $f(\bar{V}_2)$, such that $\|h(x) - f(x)\| < \rho(h(x))$.

Take a locally finite open cover $\{U_j\}_{j=1}^{\infty}$ of M as in the statement of proposition 3. For this cover take a $\delta : M \to R(> 0)$ as given by lemma 4, with $\delta(x) < 1/2^i$ for x in U_i. The restriction $f|\bar{V}_2$ is a homeomorphism of $\overline{V}_2$ onto the closed subset $f(\overline{V}_2)$ of X. Using the inverse function we obtain a continuous $r_1 : f(\overline{V}_2) \to R(>0)$ with $r_1 \circ (f|\overline{V}_2) = \delta|\overline{V}_2$; this extends to a continuous map $r_2 : X \to R(>0)$.

Since $\overline{f(M-V_4)} \cap f(\overline{V}_5) = \emptyset$ there is an $r_3 : X \to R(>0)$ satisfying $B_{r_3}(x) \cap f(\overline{V}_5) = \emptyset$ for $x \in \overline{f(M-V_4)}$.

Define $\bar{r} : X \to R(>0)$ by $\bar{r}(x) = \frac{1}{3}\min\{\rho(x), r_2(x), r_3(x)\}$. By proposition 3 there is a proper C^p $L(T)$-map $\bar{f} : M \to X$, strong with respect to $\{e_i\}$, with $||f(x)-\bar{f}(x)|| < \bar{r}(f(x))$ and with each $(f-\bar{f})|\overline{U}_j$ having bounded finite dimensional range.

Set $k_1 = (f-\bar{f})|\overline{V}_2$. By the construction of r_2 we see that k_1 is compact. Take a C^p function $\lambda : M \to [0, 1]$ with $\operatorname{supp}\lambda \subset V_2$ and $\lambda|\overline{V}_3 \equiv 1$. Define $h_1 : M \to X$ by $h_1(x) = \bar{f}(x) - \lambda(x)k_1(x)$. By lemma 4, h_1 is proper. Since $h_1|\overline{V}_3 = f|\overline{V}_3 = h|\overline{V}_3$ this restriction is an embedding, and so, by construction of r_3, $\overline{h_1(M)}$ does not reintersect $h_1(\overline{V}_5)$.

Finally apply proposition 1 to h_1 to obtain a compact, C^p map, $k : M \to E$ with $||k(x)|| < \min\{\bar{r}(f(x)), \delta(x)\}$, having $k|\overline{V}_6 \equiv 0$, and such that if $\bar{h} = h_1 + k$ then $\bar{h}$ is a $1-1$ immersion, strong with respect to $\{e_i\}$. Then $\bar{h}(M) \subset X$ and using lemma 4 again we see that $\bar{h} : M \to X$ is proper and hence a closed embedding. Also, for all x in M,

$$||\bar{h}(x) - h_1(x)|| < \tfrac{1}{3}\rho(f(x))$$

and

$$||h_1(x) - f(x)|| < \tfrac{2}{3}\rho(f(x)).$$

Since $||h(x)-f(x)|| < \rho(h(x))$, by construction of ρ, $\rho(f(x)) < \frac{1}{2}r(h(x))$, and we see that $\bar{h}$ satisfies the required conditions.//

COROLLARY 6.1. *Any proper continuous map $h : M \to X$ may be approximated in $C^0_{\text{fine}}(M, X)$ by a closed embedding $\bar{h} : M \to X$ which is a C^p $L(T)$-map.*

PROOF. It suffices to show that for any such h and any continuous $\delta : M \to R(>0)$ there is a continuous $r : X \to R(>0)$ with $r(h(x)) \leqq \delta(x)$ for each x in M.

For each y in the closed subset $h(M)$ of X set $R(y) = \inf\{\delta(x) : x \in h^{-1}(y)\}$. We need only show that $R(y)$ is locally bounded away from zero on $h(M)$ to obtain an $r_1 : h(M) \to R(>0)$ with $r_1(y) < R(y)$ which will then extend to a suitable map r.

Suppose $R(y)$ is not bounded away from zero on any neighbourhood of the point z of $h(M)$. Then there is a sequence $\{z_i\}$ in $h(M)$ with $z_i \to z$ and $R(z_i) \to 0$. This means there is a sequence $\{x_i\}$ in M with $h(x_i) = z_i$ and $\delta(x_i) \to 0$. Since h was proper $\{x_i\}$ is subconvergent, contradicting the fact that $\delta(x_i) \to 0$.//

REMARK. We cannot deduce from 6.1 that h is properly homotopic to a closed embedding, unless M is finite dimensional: in contrast with the finite dimensional situation for infinite dimensional M and X the proper maps do not form an open subset of $C^0_{\text{fine}}(M, X)$. However, the theorem of [1a] shows that such a homotopy exists.

COROLLARY 6.2. *Suppose E, T, M and X are as in proposition* 3 *(E_2 is not assumed to have a basis), then there is a closed, C^p, strong $L(T)$-embedding $f: M \to E$ with $f(M) \subset X$.*

PROOF. Every infinite dimensional Banach space contains a closed infinite dimensional subspace with a basis [8a]. Take such a subspace E_2' of E_2. Then by the theorem we obtain a closed embedding $f: M \to E_1 \times E_2' \subset E$ with $f(M) \subset X \cap (E_1 \times E_2) \subset X$ as required.//

COROLLARY 6.3. *Suppose that E_2 has a basis $\{e_i\}_{i=1}^{\infty}$, that E is C^p smooth, and that T, M and X are as above. Then given homotopic C^p, $L(T)$, closed embeddings $h_0, h_1 : M \to X$, both strong with respect to $\{e_i\}$, there is a C^p isotopy $\Phi : R \times X \to R \times X$ sending h_0 to h_1: i.e. a level preserving diffeomorphism Φ with $\Phi_0 = \mathrm{id}_X$ and $\Phi_1 \circ h_0 = h_1$. If there is a closed subset Y of X containing $h_1(M)$ and $h_0(M)$ with h_0 homotopic to h_1 as maps into Y the isotopy may be chosen to have support in an arbitrary neighbourhood of Y.*

PROOF. Let U_1, U_2 be open neighbourhoods of Y with $\bar{U}_1 \subset U_2$. Since $h_0(M) \cup h_1(M)$ is homotopy negligible in U_1 (i.e. the inclusion $M-(h_0(M) \cup h_1(M)) \to M$ is a homotopy equivalence [14]), the map h_0 is homotopic in U_1 to an $\tilde{h}$ with $\tilde{h}(M)$ contained in U_1 but disjoint from both $h_0(M)$ and $h_1(M)$. By the theorem this map can be taken to be a closed C^p $L(T)$-embedding, strong with respect to $\{e_i\}$. Thus we may assume that $h_0(M)$ and $h_1(M)$ are disjoint.

By lemma 7, below, there is a continuous $h: R \times M \to U_1$ with $h|\{0\} \times M = h_0$ and $h|\{1\} \times M = h_1$ which satisfies the conditions of theorem 6 with $A = \{0, 1\} \times M$. The theorem then gives a closed $L(T)$-embedding $\bar{h} . R \times M \to X$ with $\bar{h}(R \times M) \subset U_2$. We can now apply Hirsch's lemma as stated at the end of § A, and obtain Φ as required.//

REMARK. By taking a layer tubular neighbourhood of $\bar{h}(R \times M)$ (see [16]) in 6.3 we could arrange for each Φ_t to also be an $L(T)$-map.

LEMMA 7. *Suppose $f: \{0\} \times M \to U$ is closed C^p $L(T)$-embedding, $p \geqq 1$, into an open set U of $E = E_1 \times E_2$, which is strong with respect to a basic sequence $\{e_1, e_2, \cdots\}$ of E_2. Then if E_1 is C^p smooth, f extends to a C^p $L(T)$-map $\bar{f}: M \times R \to E$ strong with respect to $\{e_i\}$, which is a closed embedding into U on some neighbourhood of $\{0\} \times M$ in $R \times M$.*

Proof. There is a star-finite open cover $\{U_i\}_{i=1}^{\infty}$ of M, a C^p partition of unity $\{\mu_i\}$ on M subordinate to it, and a sequence $\{\bar{\imath}\}_{i=1}^{\infty}$ of integers such that if

$$k(x) = \Sigma\mu(x)e_i$$

then the subsets $k(U_i)$ and $f(U_i)$ are linearly independent.

Define $\bar{f}(t, x) = f(x)+tk(x)$. By linear independence $\bar{f}$ is an immersion. Since it is a closed embedding on $\{0\}\times M$ and is locally proper standard results show that it behaves as required.//

C. Reduction to the layer case

We will recall two known results. The first concerns the relationship between Fredholm maps and layer structures.

Theorem 0 [15] [16]. *For a C^p manifold M, $p \geqq 1$, and a Banach space E, a C^p Φ_0-map $f: M \to E$ induces a unique C^p layer structure $\{M,f\}$ on M, modelled on E, with respect to which f becomes an $L(I)$-map into E.*

If $\tau : M\times E \to TM$ is a trivialization, the layer tangent bundle of $\{M,f\}$ is trivial iff the map $t(f) : M \to \Phi_0(E)$, defined by $Tf\circ\tau$, is homotopic to a map into $GL(E)$.//

A C^∞ manifold X over the infinite dimensional Banach space E will be said to satisfy *the conditions of OECBM* if X is parallelizable and E is C^∞-smooth and has a Schauder base. The following apparent strengthening of the basic result of OECBM is in fact evident from the construction in lemma 5A of that paper and from the 'strong layer' construction of the tubular neighbourhoods there:

Theorem 00. *Suppose that X and E satisfy the conditions of OECBM and that X_1 is a C^∞ layer structure on X, with TX_1 trivial. Assume that there exists a proper bounded $C^\infty L(I)$-map $f: X_1 \to E$. Then for any basis $\{e_1, e_2, \cdots\}$ for E there is a $C^\infty L(I)$-map $\bar{f}: M \to E$ such that*

(i) *$\bar{f}$ is an open embedding,*

(ii) *$f-\bar{f}: M \to E$ has range in the subspace consisting of all finite linear combinations of the e_i.//*

Next we have an extension lemma for Fredholm maps and proper Fredholm maps (c.f. lemma 4.2 of [16]).

Lemma 8. *Suppose that X is an open subset of a C^p-smooth Banach space E, $p \geqq 1$, and that G, G_1, G_0 are open subsets of X with $\bar{G} \subset G_1$, $\bar{G}_1 \subset G_0$. Let $f: G_0 \to E$ be a C^p Φ_0-map such that $Df|\bar{G}_1 \to \Phi_0(E)$ extends to a map $h: X \to \Phi_0(E)$. Then $f|\bar{G}$ extends to a C^p Φ_0-map $\bar{f}: X \to E$ having $D\bar{f}: X \to \Phi_0(E)$ homotopic to h.*

If $f|\bar{G}_1$ was proper but not surjective then $\bar{f}$ may be chosen to be proper, and if $f|G_1$ was bounded we may choose $\bar{f}$ to be bounded as well.

PROOF. Take an open set G_2 of X with $\bar{G} \subset G_2$, $\bar{G}_2 \subset G_1$. By the local convexity of $\Phi_0(E)$ there are points $\{x_i\}_{i=1}^{\infty}$ in X and a C^p partition of unity $\{\mu_i\}_{i=0}^{\infty}$ on X with supp $\mu_0 \subset G_1$ and supp $\mu_i \subset X-\bar{G}_2$ for $i > 0$, such that if

$$g(x) = \mu_0(x)\,\mathrm{D}f(x) + \sum_{i>0} \mu_i(x)h(x_i)$$

then $g(x) \in \Phi_0(E)$ and $g : X \to \Phi_0(E)$ is homotopic to h.

Define $f_1 : X \to E$ by

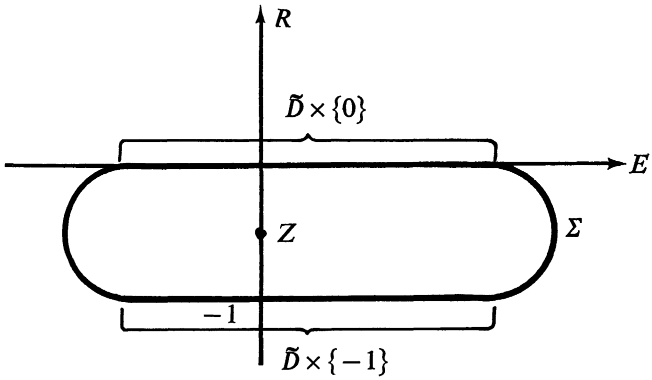
$$f_1(x) = \mu_0(x)f(x) + \sum_{i>0} \mu_i(x)h(x_i)x\,.$$

For each x in X, $Df_1(x)-g(x)$ is an operator of finite rank. Hence f_1 is Φ_0 and $Df_1 : X \to \Phi_0(E)$ is homotopic to h. For the simplest case we may now take $\bar{f} = f_1$.

In the case where $f|\bar{G}_1$ is proper and not surjective take a C^p involution $J_1 : E \to E$ such that $J_1 \circ f_1|\bar{G}_2$ is bounded. This is possible by addendum 2C of OECBM. If $f|G_1$ was bounded take J_1 to be the identity.

Using § A, there is a closed pseudo-disc $\tilde{D}$ in E, with centre 0 and C^p boundary $\tilde{S}$, containing $J_1f_1(\bar{G}_2)$ in its interior. Take a C^p map $\lambda : X \to [0, \infty)$ with $\lambda|G \equiv 0$ such that $f_2 : X \to E\times R$ given by $x \mapsto (J_1f_1(x), \lambda(x))$ is proper. This is possible by the local properness of J_1f_1.

Construct a closed, bounded, C^p hypersurface Σ contained in $E\times[-1, 0] \subset E\times R$ having $\tilde{D} \subset \Sigma \cap (E\times\{0\})$ and so that every $\frac{1}{2}$-ray in $E\times R$ emanating from the point $Z = (0, -\frac{1}{2})$ has a unique intersection with it. For example Σ could be a hypersurface of 'revolution'. 'revolved' around $\tilde{S}$ about $\{0\}\times R$, with a section looking like:



The argument at the end of OECBM gives a C^p involution J_2 on $E\times R$, with $J_2|\Sigma = \mathrm{id}_\Sigma$, which turns Σ inside out (there is a C^p diffeomorphism with support inside Σ removing Z, and a C^p automorphism of $E\times R-\{Z\}$ defined by 'reflection' in Σ). The composition $J_2 f_2$ has image inside and on Σ. If $Q : E\times R \to E$ is the projection, the map $f_3 : X \to E$ defined by $f_3 = QJ_2 f_2$ will be a proper map.

Finally define $\bar{f}$ by $\bar{f} = J_1 \circ f_3$. Then

$$\bar{f}(x) = J_1 QJ_2 f_2(x) = J_1 QJ_2(J_1 f_1(x), \lambda(x))$$

giving $D\bar{f}(x)v = DJ_1(y_3) \circ Q \circ DJ_2(y_2)(DJ_1(y_1)Df_1(x)v, D\lambda(x)v)$ where $y_i = f_i(x)$, $i = 1, 2, 3$. Since there are homotopies of DJ_1 and DJ_2, in the space of linear isomorphisms, to constant maps A_1, A_2 with $A_1^2 = I$ and $QA_2 = Q$, it follows that $D\bar{f}$ is homotopic in $\Phi_0(E)$ to Df_1. The remaining requirements for $\bar{f}$ are easily seen to be satisfied.//

It is a straightforward exercise to globalise the first part of lemma 8 to the case when X is an E-manifold. This can be done by inductively extending f over a countable, locally finite, atlas of X using the lemma, and yields:

LEMMA 9. *Suppose that X is a C^p manifold over a C^p-smooth Banach space E, $p \geqq 1$, and that V is a neighbourhood of a closed subset A of X. Let $f : V \to E$ be a C^p Φ_0-map such that $(Tf)^* : TV \to V\times E$ extends to a Φ_0 bundle map $h : TM \to M\times E$. Then $f|A$ extends to a C^p Φ_0-map $\bar{f} : M \to E$ with $(T\bar{f})^* : TM \to M\times E$ homotopic to h through Φ_0 bundle maps.//*

The following version of the main integrability result in [15], [16] is now immediate:

THEOREM 10. *Let X be a C^p manifold modelled on the C^p-smooth Banach space E, $p \geqq 1$. Then the map $f \mapsto Tf$ induces a bijection between the homotopy classes of C^p Φ_0-maps $f : M \to E$ and the homotopy classes (through Φ_0 bundle maps) of Φ_0 bundle maps $h : TM \to M\times E$.//*

REMARK. The integrability and concordance classification results in [15], [16] for layer structures on X follows directly, via theorem 0. Note that this method does not require the condition $p \geqq 3$, nor the assumption that TX is parallelisable.

The next proposition shows that a wide class of embeddings can be considered as layer embeddings into an open subset of a Banach space.

PROPOSITION 11. *Let E be the direct sum of Banach spaces E_1 and E_2, with $T : E_1 \to E$ the natural injection. Suppose that the E_1-manifold M and the E-manifold X satisfy the conditions of OECBM and that $j : M \to X$ is the inclusion of M as a C^∞ closed submanifold of X with co-space E_2. Let $\{e_1, e_2, \cdots\}$ be a basis for E_2.*

Then, if we are given any C^∞ layer structure M_1 on M which admits *a proper bounded, C^∞, $L(I_{E_1})$-map into E_1 and has a layer trivialisation of its tangent bundle which extends to a trivialisation of TX, there is a C^∞ open embedding $q : X \to E$ such that $q \circ j : M_1 \to E$ is an $L(T)$-map, strong with respect to $\{e_1, e_2, \cdots\}$.*

Proof. Taking an open embedding we may assume that X is an open subset of E. A trivialisation $\tau : TX \to X \times E$, say, which restricts to a trivialisation of TM_1, is then an automorphism of $X \times E$ and has the form:

$$\tau(x, v) = (x, A(x)v) \quad \text{where} \quad A : X \to GL(E).$$

Since it restricts to a trivialisation of TM, we can write $E = E_1 \times E_2$ in such a way that $\tau^{-1}(M \times \{0\} \times E_2)$ is a transverse bundle to TM in TX. Thus there is an $r : M \to R(> 0)$ such that the map

$$\xi : M \times rE_2 \to X \quad \text{given by} \quad \xi(m, v) = j(m) + A(m)^{-1}v$$

determines a tubular neighbourhood of M in X, with image G_0, say.

Consider M as an open subset of E_1 and take a C^∞ proper bounded $L(I_{E_1})$-map $g : M_1 \to E_1$. Since τ gives a layer trivialisation of TM_1 the maps $M \to \Phi_0(E_1)$ given by $m \mapsto A(m) \circ D_j(m)$ and by Dg differ only by a map into the operators of finite rank and are therefore homotopic. It follows that if we define $\eta : M \times E_2 \to E_1 \times E_2$ by $\eta(m, v) = (g(m), v)$ and $f : G_0 \to E$ by $f = \eta \circ \xi^{-1}$ we obtain a Φ_0-map with $Df : G_0 \to \Phi_0(E)$ homotopic to $A|G_0$.

Next take $r_1, r_2 : M \to R(> 0)$ with $r_2 < r_1 < r$ and r_1 bounded, so that if $G = \xi(M \times r_2 E_2)$ and $G_1 = \xi(M \times r_1 E_2)$ then $\bar{G}$ and $\bar{G}_1$ are closed tubular neighbourhoods of M in X. Then $f|\bar{G}_1$ is proper and bounded and, by the H.E.P., $Df|\bar{G}_1$ extends to a map $h : X \to \Phi_0(E)$ which is homotopic to A.

By lemma 8, $f|\bar{G}$ extends to a C^∞ proper bounded Φ_0-map $\bar{f} : X \to E$ with $D\bar{f} : X \to \Phi_0(E)$ homotopic to h and hence to A. Combine the given basis for E_2 with one for E_1 to obtain a basis $\{a_1, a_2, \cdots\}$ for E. Then theorems 0 and 00, together give a C^∞ open embedding $q : X \to E$ such that $\bar{f} - q$ is locally finite dimensional and has image spanned by the finite linear combinations of the a_i. Since $\bar{f}|M = T \circ g$ it follows that $q \circ j$ is a strong $L(T)$-map as required.//

D. Existence and approximation of embeddings

Let E and F be Banach spaces and X a topological space. Then an E-vector bundle π over X is said to have an *inverse modelled on F* if there is an F-vector bundle π' over X with $\pi \oplus \pi'$ trivial.

An obvious necessary condition for the existence of an immersion of an E-manifold, into a Banach space, with co-space F is that TM has an inverse modelled on F. The next theorem, which is a strengthened version of the result announced in [15], shows that this is often sufficient.

THEOREM 12. *Let M be a separable metrisable manifold of class C^p, $p \geqq 1$, modelled on a Banach space E, and let F be an arbitrary infinite dimensional Banach space. Then if TM has an inverse modelled on a separable Banach space G, with $E \times G$ admitting C^p partitions of unity, there is a bounded closed C^p embedding of M into $E \times G \times F$ with co-space $G \times F$.*
Furthermore:

(i) *any continuous map of M into $E \times G \times F$ can be C^0 approximated by a $1-1$ immersion with co-space $G \times F$;*

(ii) *if F has a basis and X is an open subset of $E \times G \times F$, then for any continuous $f : M \to X$ and $r : M \to R(>0)$ there is a closed C^p embedding $\bar{f} : M \to X$ with co-space $G \times F$ having $\|\bar{f}(x) - f(x)\| < r(f(x))$ for all x in M;*

(iii) *if F has a basis and X is open in $E \times G \times F$ then any proper map of M into X has a C^0_{fine}-approximation by a C^p closed embedding in X with co-space $G \times F$.*

PROOF. Since $E \times G$ is C^p-smooth so is E, and hence so is M. According to § A this means that there is a G-vector bundle of class C^p, $\pi : B \to M$, with $TM \oplus B$ trivial. The tangent bundle TB is equivalent to $\pi^*(TM) \oplus \pi^*(B)$ and is therefore trivial. Also B, being modelled on $E \times G$, is C^p-smooth. Hence, by theorems 0 and 10, B has a C^p layer structure, B_l, modelled on $E \times G$. By corollary 6.2 there is a closed, bounded, C^p $L(T)$-embedding $f : B_l \to E \times G \times F$, where $T : E \times G \to E \times G \times F$ is the injection. Then f restricted to the zero section of π gives the required embedding of M.

Any continuous $f : M \to E \times G \times F$ can be extended to $h : B \to E \times G \times F$ by setting $h = f \circ \pi$. Hence (i) and (ii) follow respectively from proposition 1 and theorem 6, by taking restrictions to the zero section. Part (iii) follows from (ii) as in the proof of corollary 6.1.//

Much work has been done recently on the homotopy type of the general linear groups of the well known Banach spaces, and although there is no general theory yet we can nevertheless make some useful remarks about the existence of inverses to vector bundles:

I(i). Any E-vector bundle π, over a paracompact space X, which is of finite type has an inverse modelled on a finite direct sum of copies of E. A proof is given by Lang [23: Proposition III 9]. This will be true for any vector bundle π if X has finite covering dimension [25: 6.6], [32: 2.4], and hence if X has the homotopy type of such a space. In particular it will be true if X has the homotopy type of a finite dimensional CW complex.

I(ii). If the Banach space E has a split subspace linearly isomorphic to $E \times E$ then any E-vector bundle π with paracompact base X admits an inverse modelled on a space F with $E \times F$ linearly isomorphic to E. This follows from the method of Kuiper and Terpstra-Keppler in [22], after observing that there is a countable locally finite open cover $\{U_i\}$ of X with each $\pi | U_i$ trivial, (by a theorem of E. Michael, or A. H. Stone for X pseudo-metrisable, any open cover of X has a σ-discrete open refinement, [37: 5.28, 4.21] or [32: lemma 2.4]).

An important special case of this is that if E is linearly isomorphic to its square $E \times E$ then any E-vector bundle with paracompact base has an inverse modelled on E, (add the trivial E-bundle to the inverse F-bundle already mentioned).

I(iii). Any E-vector bundle π with paracompact base admits an inverse π' modelled on any of the infinite direct sums $l_p(E)$, $1 \leqq p < \infty$, or $c_0(E)$. This follows by the same proof as I(i), by taking a countable trivialising cover as in I(ii). In fact by considering the infinite Whitney sum $\oplus_{l_p} (\pi \oplus \pi')$ or $\oplus_{c_0}(\pi \oplus \pi')$ we see that the trivial $l_p(E)$ and $c_0(E)$ bundles furnish inverses.

Note that according to Bonic and Frampton [3: page 881], if E is C^r-smooth then so is $c_0(E)$.

I(iv). If E is the quasi-reflexive space of R. C. James [19] whose natural embedding in its second dual space has codimension one then E does not satisfy the conditions of I(ii). This is discussed in detail in the Appendix where an example of a C^1-smooth manifold is given which admits no C^1 embedding into any finite direct sum of its model space.

I(v). Kuiper's proof that the general linear groups of infinite dimensional Hilbert spaces are contractible has been extended to more general sequence spaces, including l_p, $1 \leqq p < \infty$, and c_0, by Arlt and Neubauer (see [31]). More recently it has been extended to some function spaces, including $C[0, 1]$, and Lp, $1 \leqq p \leqq \infty$, [11], [26], [27].

From these considerations: by I(iii):

COROLLARY 12.1. *Every C^p separable metrisable manifold M, modelled on a C^p-smooth Banach space has a closed bounded C^p embedding into a C^p smooth Banach space, $p \geqq 1$.//*

By I(ii):

COROLLARY 12.2. *Suppose that E is a C^p-smooth Banach space, $p \geqq 1$, linearly isomorphic to its square and that M is a separable metrisable C^p*

E-manifold. Then M has a closed bounded C^p embedding into E, with co-space E.

If also E has a split subspace with a basis then any continuous map $f: M \to X$, of M into an open subset X of E, is homotopic to a C^p closed embedding into X, with co-space E. If f is proper then it can be C^0_{fine}-approximated by such an embedding.//

By I(i):

COROLLARY 12.3. *Any separable metrisable C^p-smooth E-manifold which has the homotopy type of a finite dimensional CW complex has a closed bounded C^p embedding into a finite direct sum of copies of E, $p \geqq 1$.*//

A consequence of the existence of a closed C^p embedding of a manifold M into a space with a C^p norm is that there exists a complete C^p Finsler structure on M. A question which arises in this context is: which manifolds admit a C^p Finsler structure whose induced metric is both complete and bounded?

Before closing the discussion of the existence of embeddings we quote the main result of [22] since it has a much more straightforward proof than that of theorem 12 [1]:

THEOREM (N. Kuiper and Besseline Terpstra-Keppler). *If there exists a closed linear split injection, for the C^k-smooth Banach space E, of $E \times E$ into E, then any C^k E-manifold M, $k \geqq 1$, has a closed embedding into E.*

E. Ambient isotopy of embeddings

Suppose that E is the direct sum of the Banach spaces E_1 and E_2, and that M is an E_1-manifold and X a parallelisable E-manifold. We will say that two immersions $f_i : M \to X$, $i = 0, 1$, are *tangentially homotopic* if for some trivialisation of TX the induced maps $(Tf_i)^* : TM \to M \times E$ are homotopic through split vector bundle injections. This condition, together with the existence of a homotopy of f_0 with f_1, is the obvious necessary condition for the existence of a regular homotopy between f_0 and f_1. We shall also use another condition: if f_0 is an embedding it will be called a *flat embedding* provided there is a trivialisation τ of TX which restricts to a trivialisation of the tangent bundle of $f_0(M)$ i.e.:

$$\tau \circ Tf_0 : TM \to f_0(M) \times E_1 \times E_2$$

is an isomorphism onto $f_0(M) \times E_1$. This is a necessary condition for the existence of an open embedding $f: X \to E$ with $f(X) \cap E_1 = f(f_0(M))$, (see Corollary 25.2 below).

[1] J. P. Penot also has a general embedding theorem in his thesis, Paris 1970.

THEOREM 14. *Suppose that E is the direct sum of two Banach spaces E_1 and E_2 which are infinite dimensional, C^∞-smooth, and possess Schauder bases. Let M be a C^∞ parallelisable E_1-manifold and X a C^∞ E-manifold, both separable and metrisable. Suppose that X_0 and X_1 are open subsets of X with $\overline{X}_1 \subset X_0$ and that f_0 and f_1 are closed C^∞ embeddings of M into X having co-space E_2 and image in X_1. Then if:*

(i) *f_0 and f_1 are homotopic as maps into X_1,*

(ii) *f_0 and f_1 are tangentially homotopic,*

(iii) *$f_0(M)$ is a flat submanifold of X_0,*

there is a C^∞ isotopy $\Phi : R\times X \to R\times X$ with support in X_0 such that $\Phi_0 = \mathrm{id}_X$ and $\Phi_1 \circ f_0 = f_1$.

PROOF. The method of proof will be to reduce to the layer situation and obtain an isotopy from Corollary 6.3. This will be done using proposition 11. We will therefore start by arranging for the conditions of this proposition to be satisfied.

Assume first that $f_0(M)$ and $f_1(M)$ are disjoint. Throughout the proof we will confuse trivialisations of trivial bundles with the corresponding maps into the general linear groups. Condition (iii) ensures that TX_0 is trivial and so M and X_0 satisfy the conditions of OECBM and may therefore be considered as open subsets of their models.

Since $f_0 : M \to X_0$ is flat there is a trivialisation τ of TX_0, $\tau : X_0\times E \to X_0\times E$ such that $\tau \circ Tf_0$ maps TM isomorphically onto $f_0(M)\times E_1$. This gives a trivialisation $\tau_0 : TM \to M\times E_1$.

By theorem 0 and lemma 8 there is a C^∞ layer structure M_l on M induced by a proper bounded Φ_0-map $g : M \to E_1$, with $Dg : M \to \Phi_0(E_1)$ homotopic to τ_0. Then, it follows (e.g. from [16]) that there is a $t : M \to GL(E_1)$, with each $Dg(m)-t(m)$ of finite rank, and with t homotopic in $GL(E_1)$ to τ_0. The map t induces a layer trivialisation of TM_l.

Set $M_i = f_i(M)$, $i = 0, 1$, with the layer structures determined by M_l. We next construct a trivialisation τ' of TX_0 which restricts to the trivialisation of $M_0 \cup M_1$ determined by t. To do this take a homotopy $f : I\times M \to X$ between f_0 and f_1. Then τ induces a trivialisation

$$\tau(f) : f^*(TX_0) \to I\times M\times E.$$

Since f_0 and f_1 are tangentially homotopic there is a vector bundle injection $\alpha : I\times TM \to f^*(TX_0)$ with α_0, α_1 induced by Tf_0 and Tf_1.

The composition

$$\beta = \tau(f) \circ \alpha \circ (\mathrm{id}_I\times t^{-1}) : I\times M\times E_1 \to I\times M\times E_1\times E_2$$

is an injection whose image has a transverse E_2-bundle. Since β_0 is isotopic to the identity we may therefore take an extension of β to an automorphism $\bar{\beta}$ of $I \times M \times E$ which has $\bar{\beta}_0$ isotopic to the identity. Then $\bar{\beta}_1$ is also isotopic to the identity and so, by the H.E.P., there is an $h : X_0 \to GL(E)$ with each $hf_i(m) = \bar{\beta}_i(m)^{-1}$. Taking $\tau' : TX_0 \to X \times E$ to be the composite trivialisation $h\tau$ we can see that $\tau' \circ Tf_i$ is the trivialisation t of TM_l, $i = 0, 1$, as required.

The conditions of proposition 11 are now satisfied in order to have a C^∞ open embedding $q : X_0 \to E$ such that $q \circ f_i : M_l \to E$ is an $L(T)$-map, strong with respect to a basis $\{e_1, e_2, \cdots\}$ of E_2, for each i, where $T : E_1 \to E_1 \times E_2$ is the inclusion. Thus we may apply the layer isotopy result, corollary 6.3, to obtain a C^∞ isotopy Φ' of X_0 sending f_0 to f_1, with support in a suitable neighbourhood of $\bar{X}_1$. This extends to the required isotopy Φ of X.

If $f_0(M) \cap f_1(M) \neq \emptyset$ we first obtain an open embedding q of X_0 as above but with only $q \circ f_0 : M_l \to E$ a strong $L(T)$-map. Using Theorem 6 we obtain a third C^∞ closed embedding $f_2 : M \to X_0$ with $f_2(M) \subset X_1$, and f_2 homotopic in X_1 to f_0, which is disjoint from $f_1(M)$ and $f_2(M)$ and has $q \circ f_2$ an $L(T)$-map of M_l, strong for the fixed basis of E_2. Then there is an isotopy sending f_0 to f_2. Hence f_2 is tangentially homotopic to f_0, and therefore to f_1. Also $f_2(M)$ must be flat in X_0 so we may apply the proof above to get an isotopy sending f_2 to f_1.//

The weakest point in theorem 14 appears to be the insistence that $f_0(M)$ is flat in X_0. This implies that f_0 and f_1 have trival normal bundles, which is necessary in order to apply the layer theory. We will however successively modify theorem 14, in 14B, 14B.1 and theorem 24, below, in an attempt to avoid the dependence on flatness. Unfortunately we can say nothing about the C^r case for $1 \leqq r < \infty$.

THEOREM 14B. *The conclusion of theorem* 14 *remains true when condition* (iii) *is replaced by*:

(iii)B: $f_0(M)$ *has trivial normal bundle*

together with one of

(iii)1: $GL(E)$ *is contractible*,

(iii)2: $f_0(M)$ *is a retract of* X_0 *and* TX_0 *is trivial*,

(iii)3: *there exists a closed* C^0 *embedding of* $I \times M$ *in* X *extending* f_0 *and* f_1, (*with the obvious modification for some extra embedding of* M *if* f_0 *and* f_1 *do not have disjoint images*).

When $GL(E)$ *is contractible we may also replace* (ii) *by the condition that* $f_1(M)$ *has trivial normal bundle.*

PROOF. The statements for $GL(E)$ contractible are immediate. Also (iii)B together with (iii)2 implies (iii), since (iii)2 ensures that any trivialisation of $TX_0|f_0(M)$ extends to a trivialisation of TX_0.

In the remaining case let $f: I\times M \to X$ be such an embedding of $I\times M$. Then $TX|f(I\times M)$ is trivial, since $TX|f_0(M)$ is trivial by (iii)B. Since $f(I\times M)$ is closed in X there is an open neighbourhood X_0 of $f(I\times M)$ in X with TX_0 trivial. Since $I\times M$ is an ANR we may choose X_0 so that $f(I\times M)$ is a retract of it. But then, since $f_0(M)$ is a retract of $f(I\times M)$, we are in the situation covered by (iii)2.//

CCROLLARY 14B.1. *The conclusion of theorem* 14 *remains true when condition* (iii) *is merely replaced by the assumption that* $f_0(M)$ *has trivial normal bundle.*

PROOF. This is because results on C^0 Fréchet manifolds show that (iii)3 of 14B is always satisfied. One way of seeing this is to use the homeomorphism extension theorem of Anderson and McCharen [2]. Assume that $f_0(M)$ and $f_1(M)$ are disjoint and set $Y = f_0(M)\cup f_1(M)$. Using Corollary 12.2 we can construct C^∞ Hilbert manifolds X', M' homotopy equivalent to X, M with closed C^∞, homotopic, infinite co-dimensional embeddings $f_0', f_1' : M' \to X'$ having disjoint images and such that there is a homotopy equivalence of pairs $h: X, Y \to X'$, $f_0'(M')\cup f_1'(M')$. According to Anderson and McCharen this can be assumed to be a homeomorphism of pairs. Because we are dealing with Hilbert manifolds f_0' and f_1' are flat embeddings and so the proof of theorem 14 gives an extension of $f_0'\cup f_1'$ to a closed embedding $f': M\times I \to X'$. Then $f'\circ h^{-1}$ gives the required C^0 extension of $f_0\cup f_1$.//

REMARKS (i). Since the proof of corollary 14B.1 uses deep theorems on C^0 topology the proofs of which require a completely different approach from the C^∞ theory we shall not use the corollary in the sequel. However when it can be applied to avoid the assumption of flatness we shall write this assumption in parentheses.

(ii). The proof of the corollary suggests a method for proving ambient isotopy theorems in the C^0 category.

ADDENDUM 14B.2. *The conclusions of theorems* 14B, 14, *and* 14B.1 *remain true without the assumption that M is infinite dimensional.*

PROOF. It is only necessary to modify the proof of theorem 14; and this also requires a modification of Proposition 11.

We can assume $f_0(M)$ and $f_1(M)$ are disjoint and set Y equal to their union. As in the proof of 14 there is a trivialisation $\tau' : TX_0 \to X\times E$ which restricts to a trivialisation of TY. For a basis $\{e_1, e_2, \cdots\}$ for E take a closed bounded strong layer embedding f_1 of Y into E. This extends to a proper bounded Φ_0-map, $f: G \to E$ taking a closed tubular neighbourhood G of Y in X to a closed tubular neighbourhood of $f_1(Y)$, and which is tangentially isotopic through Φ_0-bundle maps to $\tau'|TG$. We can therefore apply lemma 8 and obtain a proper bounded Φ_0-map $\bar{f}: X \to E$ which restricts to f_1 on Y and is tangentially isotopic to τ'. This allows us to apply theorem 00 and Corollary 6.3 as before.//

Even when both X and M are parallelisable and M has infinite co-dimension in X, we cannot assume that $f_0(M)$ automatically has a trivial normal bundle, as the following example shows.

EXAMPLE. An infinite co-dimensional embedding of $S^1 \times L_p$ into L_p, $1 < p \neq 2 < \infty$, with non-trivial normal bundle.

CONSTRUCTION. Pełcynski's extension of Douady's example [10] shows that $GL(l_2 \times l_p)$ is not connected if $p \neq 2$. It follows that there is a non-trivial C^∞ $l_2 \times l_p$-bundle, ξ, over S^1. Since $l_p(L_p)$ is isomorphic to L_p, and $L_p \times l_2 \times l_p \approx L_p$, $p > 1$, [33], remark I(iii) in section D above implies that the Whitney sum $\underline{E} \oplus \xi$ of ξ with the trivial L_p bundle is trivial. Thus the total space of $\underline{E} \oplus \xi$ is diffeomorphic to $S^1 \times L_p$ and so has a closed co-dimension one embedding in $R^2 \times L_p \approx L_p$. The induced embedding of the total space $S^1 \times L_p$ of $\underline{E}$ has co-space $l_2 \times l_p$ but non-trivial normal bundle.//

F. Stability and diffeomorphisms

We shall use the following form of Kuiper and Burghelea's ambient uniqueness theorem for tubular neighbourhoods:

THEOREM (Kuiper and Burghelea). *Let X be a C^∞ closed submanifold of the C^∞-smooth manifold M and $\rho : X \times F \to R(\geqq 0)$ a Finsler-like function on a product Banach space bundle over X, which is C^∞ except on the zero section. Let $D_t = \rho^{-1}[0, t)$ and, for $0 < \alpha < \beta < \gamma$ suppose that $f_i : D_\gamma \to M$, $i = 0, 1$, are C^∞ open tubular neighbourhoods of X in M such that each $f_i(\bar{D}_\beta)$ is closed in M. Assume that the compositions with the tangent maps along the fibres, over $X \times \{0\}$,*

$$X \times F \xrightarrow{T_{X\times\{0\}}f_i} TM|_X \xrightarrow{\text{Proj}} TM|_X/TX \qquad i = 0, 1$$

are isotopic through vector bundle isomorphisms.

Then there is a C^∞ isotopy $\Phi : M \times R \to M \times R$, $\Phi(x, t) = (\Phi_t(x), t)$, with $\Phi_0 = \mathrm{id}_M$ and $\Phi_1 \circ f_0|\bar{D}_\alpha = f_1|\bar{D}_\alpha$. This isotopy may be chosen to have support in an arbitrary neighbourhood of $f_0(D_\gamma) \cup f_1(D_\gamma)$.

REMARKS ON THE PROOF. When ρ is a genuine Finsler this is proved for Hilbert manifolds in [7], or in more detail in [20], and this proof applies for any F with a C^∞-norm. When ρ is only Finsler-like then D_α, D_β, D_γ

can nevertheless be separated sufficiently by a radial multiplication, using ρ, to reduce to the case of disc neighbourhoods again. The assumption of a smooth norm can be avoided as usual by the substitution of norm-like functions.//

Our proof of stability will require yet another version of Bessaga's theorem:

LEMMA 15. *Let* $\rho : X\times E \to R(\geqq 0)$ *be a Finsler-like function on the product of the* C^p*-smooth manifold* X *with the Banach space* E; ρ *is assumed* C^p *except on the zero section. Let* D, D' *denote* $\rho^{-1}[0, 1)$, $\rho^{-1}[0, 2)$ *and* S, S' *their boundaries*; H *will be a maximal hyperplane of* E. *Then*:

(i) S *is* C^p *diffeomorphic to* $X\times H$ *by a fibre preserving map*,

(ii) *given a linear isomorphism* s *of* E *onto the product of* E *with a Banach space* F *there is a* C^p, *fibre preserving, diffeomorphism* $h : X\times E, D', D \to X\times E\times F, D'\times F, D\times F$ *such that the map of* X *into* Lis $(E, E\times F)$, *determined by the tangent map to* h *at* $X\times\{0\}$ *along the fibres, is homotopic to the constant map into* s.

PROOF. Let $q : E \to R\times H$ be a splitting, so that $q|H$ is the natural inclusion, and set $H_1 = q^{-1}(R\times 0)$. There is a continuous linear injection $i_0 : H \to l_2$ onto a dense subspace of l_2 (e.g. see OECBM § 6A), and so there is a similar injection $i : E \to l_2$ which has $i(H_1)$ orthogonal to $i(H)$. Set $j = \mathrm{id}_X\times i : X\times E \to X\times l_2$. Since ρ was Finsler-like and $S = \rho^{-1}(1)$ there is a C^p map $r : X \to R(> 0)$ such that $j(S) \subset X\times rl_2$. Let $\Sigma \subset X\times E$ be the inverse image under j of the boundary of $X\times rl_2$. Since both Σ and S are transverse to every ray from 0 in the fibres of $X\times E$ we can project S radially onto Σ to get a C^p diffeomorphism. We will compose this with radial multiplication by r^{-1} which gives a diffeomorphism of Σ onto Σ_0, where Σ_0 is the inverse image under i of the unit sphere in l_2.

Note that the existence of the Finsler-like map ρ on $X\times E$ implies that E is C^p-smooth. Hence if P is one point of $H_1 \cap \Sigma_0$ we may apply Bessaga's theorem [OECBM § 6A] to delete P from Σ_0, and then project stereographically onto H from P to obtain a C^p diffeomorphism of Σ_0 onto H. Thus, by composition, we obtain the diffeomorphism required for (i).

For part (ii), observe that there is a linear isomorphism $s_1 : H \to H\times F$ such that the composition

$$E \xrightarrow{q} R\times H \xrightarrow{\mathrm{id}\times s_1} R\times H\times F \xrightarrow{q^{-1}\times \mathrm{id}_F} E\times F$$

is linearly isotopic to s. We define h as the composition

$$X \times E \xrightarrow{\beta} X \times (E - \{0\}) \xrightarrow{\varphi} R(>0) \times S \xrightarrow{\mathrm{id} \times \theta} R(>0) \times X \times H \longrightarrow$$
$$\xrightarrow{\mathrm{id} \times s_1} R(>0) \times X \times H \times F \xrightarrow{\mathrm{id}_R \times \theta^{-1} \times \mathrm{id}_F} R(>0) \times S \times F \longrightarrow$$
$$\xrightarrow{\varphi^{-1} \times \mathrm{id}_F} X \times (E - \{0\}) \times F \xrightarrow{\beta^{-1} \times \mathrm{id}_F} X \times E \times F,$$

where:

β is a diffeomorphism obtained via Bessaga's theorem, with support in $e^{-1}[0, \frac{1}{2})$;

φ is defined by $\varphi(x, v) = (\rho(x, v), \bar{v})$ where $\bar{v}$ is the 'projection' of v onto S;

θ is the diffeomorphism of part (i).

Noting that the tangent map at $X \times \{0\}$ to β along the fibres is homotopic to 1 in $GL(E)$ and that the corresponding map for $(\mathrm{id}_R \times \theta) \circ \varphi$ at any $X \times \{v\}$, for v in $E - \{0\}$, is homotopic to q in Lis $(E, R \times H)$, we find that h satisfies our requirements.//

If E and F are Banach spaces we will say that E is *F-stable* if E is linearly isomorphic to $E \times F$, and that E is *strongly F-stable* if also the natural inclusion $GL(E) \to GL(E \times F)$, $T \mapsto T \times \mathrm{id}_F$, is a homotopy equivalence. In most of what follows we will require our model spaces to be F-stable for some infinite dimensional space F, and often they will have to be strongly F-stable for the same F; so here are some examples:

S1: Each space L_p is L_p, l_2, and l_p-stable, $1 < p < \infty$ [33];

S2: $C[0, 1]$ and c_0 are both c_0-stable;

S3: When $GL(E)$ is contractible F-stability trivially implies strong F-stability, therefore, using the references in 1(v) § E, there is also strong stability in the examples of S1, S2;

S4: I know of no example where F-stability does not imply strong F-stability. This seems to be one of the basic problems involved in studying the homotopy type of general linear groups, particularly in the special case of the strong E-stability of a space E which is isomorphic to its square.

S5: Douady's results [10] show that $c_0 \times l_2$ is strongly c_0, l_2, and $c_0 \times l_2$-stable. Edelstein and Mitjagin's [27] show that James' space J is strongly l_2-stable (see the Appendix).

The use of filtrations as in the proof of the next proposition, in order to prove stability, was suggested to me by J. Eells.

PROPOSITION 16. *Let E be a C^∞-smooth Banach space with a basis, which is F-stable for some Banach space F. Then for any open subset X of E there is a C^∞ diffeomorphism of X onto $X \times F$.*

PROOF. Take a basis $\{e_i\}_{i=1}^{\infty}$ for E and a norm on E which makes it monotone [8]. Set

$$E_n = \mathrm{Sp}\{e_1, \cdots, e_n\},\ E^n = \overline{\mathrm{Sp}}\{e_{n+1}, e_{n+2}, \cdots\},\ \text{and } X_n = X \cap E_n.$$

Then $E = E_n \times E^n$ and we have corresponding projections $p_n : E \to E_n$, $p^n : E \to E^n$. Using lemma 2 there is a continuous bounded map $r : X \to R(> 0)$ such that if Z_n, Z_n' are the standard open tubes about X_n in E of radius $r|X_n$, $2r|X_n$ then $\overline{Z_n'} \subset X$ and $X = \bigcup_n Z_n$.

For positive integers n and m let $Z_n(m)$ be the standard open tube about X_n of radius $(2-1/m)r|X_n$, and set $V_n(m) = \bigcup_{j \leqq n} Z_j(m)$. Then

$$Z_n \subset V_n(m) \subset \overline{V_n(m)} \subset V_n(m+1)$$

and

$$V_n(m) \subset V_{n+1}(m).$$

The sets $V_n(m)$ will be considered as neighbourhoods of X_n in the bundle $X_n \times E^n \to X^n$, and we show next that their fibres are radial subsets of E^n.

To do this note that $x \in V_n(m)$ iff $\|p^k x\| < (2-1/m)r(p_k x)$ for some $k \leqq n$. Suppose $x_t = (x_n, tx^n) \in E_n \times E^n$, $t > 0$. Then $x_t = tx_1 + (1-t)p_n x_1$, whence, for $j \leqq n$:

$$p_j(x_t) = p_j(x_1)$$

and

$$\begin{aligned} \|p^j x_t\| &= \|(1-t)p_n p^j x_1 + tp^j x_1\| \\ &\leqq (|1-t|\,\|p_n\| + t)\|p^j x_1\| \\ &= (|1-t| + t)\|p^j x_1\| \qquad \text{(since the base is monotone)} \\ &= \|p^j x\| \text{ if } 0 < t < 1. \end{aligned}$$

Thus if $x_1 \in V_n(m)$ so does x_t for $0 < t < 1$, as required.

The above arguement also shows that, in the sense of fibrewise multiplication,

$$\alpha_m V_n(m) \subset V_n(m+1), \text{ where } \alpha_m = \tfrac{1}{2}\left(1 + \frac{2 - \dfrac{1}{m+1}}{2-1/m}\right) > 1.$$

By remark (i) of § A there is a Finsler-like function $\rho_n : X_n \times E^n \to R(\geqq 0)$ which is C^∞ except on the zero section and has

$$\overline{V_n(n)} \subset \rho_n^{-1}[0, 1) \subset \rho_n^{-1}[0, 2] \subset V_n(n+1)$$

Set $D_n = \rho_n^{-1}[0, 1)$ and $D_n' = \rho_n^{-1}[0, 2)$. We shall inductively construct a sequence of diffeomorphisms $d_n : D_n', D_n \to D_n' \times F, D_n \times F$ which are homotopic to the inclusions and make the following diagram commute:

$$\begin{array}{ccc} D'_{n+1} & \xrightarrow{d_{n+1}} & D'_{n+1} \times F \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i \times id_F} \\ D_n & \xrightarrow[d_n|D_n]{} & D_n \times F \end{array}$$

Here, and hereafter, i denotes an inclusion.

Since $i(D_n) \subset D_{n+1}$ for each n, and $X = \bigcup_n D_n$, the sequence $\{d_n\}$ induces a C^∞ diffeomorphism $d_\infty : X \to X \times F$, as required.

To obtain $\{d_n\}$ first use induction to obtain a sequence of linear isomorphisms $\{s_n\}_{n=1}^{\infty}$, $s_n : E^n \to E^n \times F$, for which the following diagram commutes up to isotopy:

$$\begin{array}{ccc} E^n & \xrightarrow{s_n} & E^n \times F \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ E^{n+1} & \xrightarrow[s_{n+1}]{} & E^{n+1} \times F \end{array}$$

Next, using the s_n, take fibre preserving diffeomorphisms

$$h_n : D'_n,\ D_n \to D'_n \times F,\ D_n \times F$$

as in lemma 15. Set $d_1 = h_1$ and assume $d_1, \cdots, d_N$ constructed with the additional property that they are isotopic to $h_1, \cdots, h_N$. We have two homotopic closed embeddings of the finite dimensional manifold X_N in $D'_{N+1} \times F$ given by the compositions

$$j_0 : X_N \xrightarrow{\subset} D'_N \xrightarrow{i} D'_{N+1} \xrightarrow{h_{N+1}} D'_{N+1} \times F$$

$$j_1 : X_N \subset D'_N \xrightarrow[d_N]{} D'_N \times F \xrightarrow{i \times id_F} D'_{N+1} \times F.$$

Since $\overline{D'_N} \subset V_N(N+1) \subset V_{N+1}(N+1) \subset D_{N+1}$ and $d_N(D_N) = D_N \times F$, and $h_{N+1}(D_{N+1}) = D_{N+1} \times F$, the images of j_0 and j_1 lie in $D_{N+1} \times F$. Therefore, by applying addendum 14B.2, there is a C^∞ isotopy $\Phi_t : D'_{N+1} \times F \to D'_{N+1} \times F$, having support in $D_{N+1} \times F$, with $\Phi_0 = \mathrm{id}$ and $\Phi_1 \circ j_0 = j_1$.

We now have two tubular neighbourhoods of $j_1(X_N)$ in $D'_{N+1} \times F$, namely the compositions:

$$f_0 : D'_N \xrightarrow{i} D'_{N+1} \xrightarrow{h_{N+1}} D'_{N+1} \times F \xrightarrow{\Phi_1} D'_{N+1} \times F$$

and

$$f_1 : D'_N \xrightarrow{d_N} D'_N \times F \xrightarrow{i \times id_F} D'_{N+1} \times F.$$

The closures of the images of f_0 and f_1 lie in $D_{N+1} \times F$, and both $f_i(\rho_N^{-1}[0, \frac{2}{3}])$ are closed in $D'_{N+1} \times F$, $i = 0, 1$. The tangent maps along the fibres to f_0, f_1 over $X \times \{0\}$ are isotopic because of the construction of the h_n and the relationships between the s_n, both tangent maps being isotopic, loosely speaking, to $\mathrm{id} \times s_N : X_N \times E^N \to X_N \times E^N \times F$. We can therefore apply the tubular neighbourhood theorem to obtain an isotopy $\Theta_t : D'_{N+1} \times F \to D'_{N+1} \times F$ with $\Theta_0 = \mathrm{id}$ and $\Theta_1 \circ f | D_N = f_1 | D_N$, and also having support in $D_{N+1} \times F$.

If we set $d_{N+1} = \Theta_1 \circ \Phi_1 \circ h_{N+1} : D'_{N+1} \times F \to D'_{N+1} \times F$, since both Θ_1 and Φ_1 have support in $D_{N+1} \times F$, we see that it sends $D_{N+1} \times F$ onto itself. It certainly satisfies the other requirement, and so the induction is complete.//

REMARKS. (i) It seems essential here to be working in the C^∞ category. This is because the proof of Kuiper and Burghelea's tubular neighbourhood theorem for the C^r case, $r < \infty$, would give a drop in the differentiability class of the isotopy Φ, and we have to apply this theorem an infinite number of times.

The finite dimensional approach to this would be to use theorems on the approximation of C^p diffeomorphisms by C^{p+k} diffeomorphisms, and these seem to require a good theory of C^1 approximation. Some work has been done on this for infinite dimensional spaces, particularly by Nicole Moulis in her thesis [29] where she obtains good theorems for l_p spaces and for c_0; however the results of J. Wells in [36] suggest that there may be fundamental difficulties in extending these.

(ii) For open subsets of l_2 each neighbourhood $V_n(m)$ is a standard tube about X_n and the use of Finsler-like functions is not needed. This means that the proof for this case can be considerably simplified. However this does not seem to be true for subsets of c_0.

(iii) We only used the ambient isotopy theorem for finite dimensional submanifolds and so it should be possible to make this proof independent of the main discussion in §§ A–F.

We next give another extension of Mazur's tangential equivalence theorem, proved for C^∞ Hilbert manifolds in [7]. In order to avoid a drop in differentiability class, as mentioned in Remark (i) above, we will use an infinite dimensional version of the Lemma 2 in Hirsch's paper [18]:

LEMMA 17. *For a C^p-smooth vector bundle $\pi : B \to X$, let $\rho : B \to R$ ($\geqq 0$) be a Finsler-like function which is C^p off the zero section $\zeta(X)$, $p \geqq 1$. Suppose $0 < \alpha < \beta < \gamma$ and let $f : \rho^{-1}[0, \beta] \to \rho^{-1}[0, \gamma]$ be a C^p closed interior embedding as a tubular neighbourhood of $\zeta(X) \subset \rho^{-1}[0, \gamma]$.*

Assume either: (i) $p = \infty$ *or*: (ii) *B is a trivial bundle whose fibre* *admits a C^p norm. Then $f|\rho^{-1}[0, \alpha]$ extends to a C^p diffeomorphism of $\rho^{-1}[0, \gamma]$ onto itself.*

PROOF. The proof outlined by Hirsch is easily seen to still apply for the case of trivial bundles with C^p normed fibres. Since the differentials of a norm-like function on a space F may not be bounded in F^* (although the differentials of a smooth norm would lie on the unit sphere) it is not clear that this method applies more generally. When $p = \infty$ we can simply apply Kuiper and Burghelea's tubular neighbourhood theorem (§ F).//

PROPOSITION 18. *Let M and N be parallelisable C^p-smooth E-manifolds, $p \geqq 1$, and F an infinite dimensional Banach space. Suppose either $p = \infty$ or F has a C^p-norm. Then, given a homotopy equivalence $h : M \to N$, there is a C^p diffeomorphism $\bar{h} : M \times F \to N \times F$ homotopic to $h \times \mathrm{id}_F$.*

PROOF (c.f. [7], [18]). Using theorems 0 and 10 of § C there are C^p layer structures M_1, N_1 on M, N modelled on E, with trivial layer tangent bundles. Let H be a maximal hyperplane in F, and $T : E \to E \times H$ the inclusion. By proposition 1 there are closed C^p $L(T)$-embeddings of M_1 and N_1 into $E \times H$. Using the trivial spray on $E \times H$ these extend to open layer tubular neighbourhood embeddings of neighbourhoods of the zero sections in $M_1 \times H$, $N_1 \times H$. These can be taken to be of class C^p by using the existence of a C^p normal bundle proved in § A. By remark (i) in § A there are Finsler-like functions $\rho_0 : M \times F \to R(\geqq 0)$, $\rho_1 : N \times F \to R(\geqq 0)$, which are C^p off the zero sections, such that these embeddings restrict to closed embeddings of $(M \times H) \cap \rho_0^{-1}[0, 2]$ and $(N \times H) \cap \rho_1^{-1}[0, 2]$. Set $U(n) = (M \times H) \cap \rho_0^{-1}[0, 2-1/n)$, $V(n) = (N \times H) \cap \rho_1^{-1}[0, 2-1/n)$. These are layer manifolds which are layer diffeomorphic to open subsets of $E \times H$.

Since H contains an infinite dimensional closed subspace with a basis [8a], according to lemma 5 there is a closed C^p $L(T)$-embedding $f : M \to V(1)$ homotopic to h and with image contained in $V(\frac{2}{3})$. For each natural number n take a layer tubular neighbourhood of $f(M)$, and after a radial compression, if necessary, obtain an extension of f to a closed C^p embedding $\tilde{f}_n : \overline{U(n+1)} \to V(n+2)$ with image in $\underline{V(n)}$. Using a homotopy inverse to h obtain similar embeddings $\tilde{g}_n : \overline{V(n+1)} \to U(n+2)$, $n = 1, 2, \cdots$.

We now have homotopic embeddings of M into $U(n+1)$ namely the inclusion $\zeta : M \to U(n+1)$ as zero section and the restriction $\tilde{g}_{n+1} \circ \tilde{f}_n|\zeta(M)$. These are both closed $L(T)$-embeddings: assume their images are disjoint. The method in Corollary 6.3 together with proposition 1 gives us a $1-1$ C^p immersion $\varphi : M \times R \to U(n+1)$ with $\varphi|M \times \{0\} =$

$\tilde{g}_{n+1} \circ \tilde{f}_n \circ \zeta$ and $\varphi | M \times \{1\} = \zeta$. This can be modified in a standard way to give a closed embedding $\bar{\varphi} : M \times R \to U(n+3) \times R$ with image in $U(n+2) \times R$ which restricts to φ on $M \times \{0, 1\}$. We can now apply Hirsch's (first) lemma (see § A) to obtain a C^p isotopy $\Phi_t^n : U(n+3) \times R \to U(n+3) \times R$, with support in $U(n+2) \times R$, which satisfies $\Phi_0^n = \mathrm{id}$ and $\Phi_1^n \circ \tilde{g}_{n+1} \circ \tilde{f}_n \circ \zeta = \zeta$. If the embeddings did not have disjoint images we can take a third embedding as usual and still obtain the isotopy Φ_t^n.

Write $f_n = \tilde{f}_n \times \mathrm{id}_R$ restricted to give a map $f_n : U(n) \times R \to V(n+1) \times R$, and similarly for $g_n : V(n) \times R \to U(n+1) \times R$. Then $\Phi_1^n \circ g_n \circ f_n : U(n) \times R \to U(n+2) \times R$ is a C^p tubular neighbourhood of $\zeta(M)$ which extends to a closed interior embedding of $\overline{U(n+1)} \times R$ into $\overline{U(n+2)} \times R$. The method of lemma 15 shows that there is a C^p diffeomorphism over M, sending the pair $\overline{U(n+1)} \times R$, $\overline{U(n)} \times R$ onto the pair $\rho^{-1}[0, n+1]$, $\rho^{-1}[0, n]$. We can therefore apply Hirsch's second lemma, lemma 17, to extend $\Phi_1^n \circ g_n \circ f_n$ to a C^p diffeomorphism $\varphi_n : U(n+2) \times R \to U(n+2) \times R$.

We can now use the usual arguments: the sequence of open embeddings

$$U(1) \times R \xrightarrow{f_1} V(2) \times R \xrightarrow{g_2} U(3) \times R \xrightarrow{f_3} V(4) \times R \to \cdots$$

embeds in the commutative diagram

$$\begin{tikzcd}
U(1) \times R \arrow[r, "f_1"] \arrow[d, "\mathrm{id}"] & V(2) \times R \arrow[r, "g_2"] & U(3) \times R \arrow[r, "f_3"] \arrow[d, "\varphi_1^{-1} \circ \Phi_1^1"] & V(4) \times R \arrow[r, "g_4"] & U(5) \times R \arrow[r] \arrow[d, "\varphi_3^{-1} \circ \Phi_1^3"'] & {} \\
U(1) \times R \arrow[rr, "\subset"'] & & U(3) \times R \arrow[rr, "\subset"'] & & U(5) \times R \arrow[r] & {}
\end{tikzcd}$$

and its limit is therefore C^p diffeomorphic by $\varinjlim \varphi_n^{-1} \circ \Phi_1^n$ to $M \times H \times R \cong M \times F$. Similarly it is diffeomorphic to $N \times F$.//

Recall that a *tangential equivalence* (f, α) between manifolds M and N is a homotopy equivalence $f : M \to N$ together with a C^0 vector bundle isomorphism $\alpha : TM \to f^*(TN)$ over N. This notion allows us a version of proposition 18 for non-parallelisable manifolds:

Corollary 18.1. *Let E be a Banach space, which is isomorphic to its square, $E \times E$. Then, if either $p = \infty$ and E is C^∞-smooth or $p \geqq 1$ and E has a C^p norm:*

(i) *Every C^p E-manifold M has $M \times E$ C^p-diffeomorphic to the total space of a C^p E-vector bundle over an open subset of E.*

(ii) *If M, N are tangentially equivalent C^p E-manifolds there is a C^p diffeomorphism of $M \times E$ onto $N \times E$.*

PROOF. (i) By corollary 12.2 there is a C^p closed embedding of M into E with co-space E and by § A this has a C^p tubular neighbourhood. Thus there is a C^p E-vector bundle $\nu_M : \nu(M) \to M$ over M whose total space is C^p diffeomorphic to an open subset of E. It is easy to see that the total space of the Whitney sum $\nu(M) \oplus TM$ is diffeomorphic to the total space of the pull back $\nu_M^*(TM)$ of TM over $\nu(M)$. Since $\nu(M) \oplus TM \approx M \times E$ result (i) follows.

(ii) With the corresponding notation we have also $N \times E$ diffeomorphic to the total space $\nu_N^*(TN)$ of the pull back of TN over a bundle $\nu_N : \nu(N) \to N$. The tangential equivalence is seen to induce a homotopy equivalence $f : \nu(M) \to \nu(N)$ together with a bundle equivalence $\alpha : \nu_M^*(TM) \to f^* \nu_N^*(TN)$. According to the proposition $f \times \mathrm{id}_E$ is homotopic to a C^p diffeomorphism of $\nu(M) \times E$ onto $\nu(N) \times E$. It follows that $\nu_M^*(TM) \times E$ is C^p diffeomorphic to $\nu_N^*(TN) \times E$, giving (ii), since $E \approx E \times E$.//

We next have one of our main results. When E is a Hilbert space, part (i) was first proved using the combined results of [7], [28], [13], and a proof of part (ii) for this case has been given independently by D. Burghelea, [6], using handle decompositions.

THEOREM 19. *Let M and N be separable metrisable C^∞ E-manifolds, with trivial tangent bundles. Assume that E is C^∞-smooth, possesses a Schauder base, and is F-stable for some infinite dimensional Banach space F. Then*:

(i) *Any homotopy equivalence $f : M \to N$ is homotopic to a C^∞ diffeomorphism*;

(ii) *If E is also strongly F-stable and F has a basis, then for any tangential equivalence $(f, \alpha) : M \to N$ there is a C^∞ diffeomorphism $d : M \to N$ tangentially homotopic to (f, α).*

(iii) *If F has a basis any two C^∞ diffeomorphisms $d_0, d_1 : M \to N$ which induce homotopic tangential equivalences are C^∞ isotopic.*

PROOF. Both M and N satisfy the conditions of OECBM and so can be considered as open subsets of E. By proposition 16 there are C^∞ diffeomorphisms $M \cong M \times F$, $N \cong N \times F$. Part (i) follows by proposition 18.

For part (iii), by considering $d_1^{-1} \circ d_0 : M \to M$, we may assume that $M = N$ and $d_1 = \mathrm{id}_M$. Also, by part (i), we may assume M has the form $X \times F$ where X is a parallelisable C^∞ E-manifold. Then we have two embeddings of X in $X \times F$, namely $i : X \to X \times F$, $i(x) = (x, 0)$ and $d_0 \circ i : X \to X \times F$. These satisfy the conditions of Theorem 14, since d_0 is tangentially homotopic to id_M. There is therefore an isotopy $\Phi_t : X \times F \to X \times F$ with $\Phi_0 = \mathrm{id}$ and $\Phi_1 \circ d_0 \circ i = i$. The diffeomorphism $\Phi_1 \circ d_0 : X \times F \to X \times F$ is then a tubular neighbourhood of $X \times \{0\}$ in $X \times F$. The standard tubular neighbourhood uniqueness theorem, as in Lang [23], shows that this is isotopic to a diffeomorphism of the form $(x, v) \mapsto (x, T(x)v)$ where $T : X \to GL(F)$. Since d_0 was tangentially homotopic to the identity the map $\hat{T} : X \to GL(E \times F)$ given by $x \mapsto \mathrm{id}_E \times T(x)$ is homotopic to the identity.

There is an open subset U of F which is homotopy equivalent to X. Part (i) shows that X is C^∞ diffeomorphic to $U \times E$. The map $\Phi_1 \circ d_0$ can therefore be represented as the map $U \times E \times F \to U \times (E \times F)$ given by $(x, v, w) \mapsto (x, \hat{T}(x)(v, w))$ which is isotopic to the identity.

To prove (ii) first observe, by (i), that M is diffeomorphic to N and that this diffeomorphism may be taken to be homotopic to any homotopy equivalence. It therefore suffices to consider the case where $M = N$ and $f \sim \mathrm{id}_M$. We may also represent M as $U \times E$ where U is open in F, and α as a map $U \times E \to GL(F \times E)$ with $\alpha(x, v) = \alpha(x, 0)$. By hypothesis $x \mapsto \alpha(x, 0)$ is homotopic to a map $\hat{\beta} : U \to GL(F \times E)$ with the form $\hat{\beta}(x) = \mathrm{id}_F \times \beta(x)$. The map $\beta : U \to GL(E)$ can be chosen to be C^∞ and we obtain the required diffeomorphism $d : U \times E \to U \times E$ by defining $d(x, v) = (x, \beta(x)v)$.//

Let $\pi_0 \operatorname{Diff}^r[M, N]$ denote the set of C^r isotopy classes of C^r diffeomorphisms between manifolds M and N, and $[M, N]^*$ the set of homotopy classes of homotopy equivalences. The following examples follow from the theorem using the smoothness results of Bonic and Frampton [3], and the properties of the general linear groups mentioned in I(v), § D, and in Douady's paper [10] and also remarks S1–S5 above. Here KO denotes the representable functor of real K-theory.

EXAMPLES. a) Let E be isomorphic to one of the spaces L_{2n}, l_{2n}, for $n = 1, 2, \cdots$, or to c_0. Then if M and N are C^∞ metrisable E-manifolds the natural map gives a bijection:

$$\pi_0 \operatorname{Diff}^\infty [M, N] \leftrightarrow [M, N]^*.$$

b) Let M and N be parallelisable C^∞ metrisable $(c_0 \times l_2)$-manifolds. Then the tangential homotopy class induces a bijection (depending on parallelisations of M and N)

$$\pi_0 \operatorname{Diff}^\infty [M, N] \leftrightarrow [M, N]^* \times KO(M).$$

Our next result will be strengthened later, and partially superceded by Theorem 23 below. If N_i is a submanifold of M_i, $i = 1, 2$, a tangential equivalence of pairs $(f, \alpha) : M_1, N_1 \to M_2, N_2$ is a homotopy equivalence of pairs $f : M_1, N_1 \to M_2, N_2$ together with a vector bundle isomorphism $\alpha : TM_1 \to f^*(TM_2)$ which restricts to an isomorphism $TN_1 \to f^*(TN_2)$.

Lemma 20. *For $i = 1, 2$, let M_i be an E-manifold and N_i an H-manifold embedded as a closed C^∞, (flat), submanifold of M_i with infinite dimensional co-space H'. Assume that each M_i, N_i satisfies the conditions of OECBM and that H' has a base, and also that the following holds for some infinite dimensional Banach space F:*

(0) *H is F-stable*

and either

(i) *H and E are strongly F-stable and F has a basis*

or

(ii) *the natural inclusion $GL(H) \to GL(H \times H')$, $T \mapsto T \times \mathrm{id}_{H'}$, is nullhomotopic.*

Then, if $(f, \alpha) : M_1, N_1 \to M_2, N_2$ is a tangential equivalence of pairs, f is homotopic to a C^∞ diffeomorphism $h : M_1 \to M_2$ which restricts to a diffeomorphism of N_1 onto N_2 homotopic to $f|N_1 : N_1 \to N_2$. If case (i) *holds then h may be chosen so that it is tangentially homotopic to (f, α) and $h|N_1$ is tangentially homotopic to $(f|N_1, \alpha|TN_1)$. If $f|N_1$ is already a C^∞ diffeomorphism onto N_2 and $\alpha|TN_1 = T(f|N_1)^*$ we may choose $h|N_1 = f|N_1$. In this case we only require E to be F-stable in* (0) *and do not need the condition on H in* (i).

Proof. According to theorem 19 there are C^∞ diffeomorphisms $h_0 : M_1 \to M_2$ and $h_1 : N_1 \to N_2$ homotopic respectively to f and $f|N_1$. These give rise to two homotopic embeddings of N_1 into M_2, namely $h_0|N_1$ and h_1.

In case (i) we may take h_0 and h_1 to be tangentially homotopic to (f, α) and $(f|N_1, \alpha|TN_1)$. The two embeddings will then be tangentially homotopic and we can apply theorem 14 to obtain a C^∞ isotopy $\Phi_t : M_2 \to M_2$ with $\Phi_0 = \mathrm{id}$ and $\Phi_1 \circ h_0|N_1 = h_1$. We may then set $h = \Phi_1 \circ h_0$. In case (ii) the two embeddings are necessarily tangentially homotopic and so the same procedure works.

Next we consider manifolds with boundary. First note the observation of D. Burghelea that for any infinite dimensional Banach space E the natural map $GL(E) \to GL(E \times R)$ is a homotopy equivalence (see [21]). It follows that if M and N are infinite dimensional manifolds with boundaries ∂M, ∂N then a homotopy equivalence of pairs $f : M, \partial M \to N, \partial N$ together with a vector bundle isomorphism $\alpha : TM \to f^*(TN)$ induces an isomorphism $T\partial M \to f^*(T\partial N)$ which is uniquely determined up to isotopy. Such a pair (f, α) can therefore be called a tangential equivalence of manifolds with boundary, $(f, \alpha) : M, \partial M \to N, \partial N$. Another consequence is that if TM is trivial then so is $T\partial M$.

Lemma 21. *Suppose that M is an E-manifold with boundary ∂M, satisfy-*

ing the conditions of OECBM. Assume that E is strongly F-stable for some infinite dimensional Banach space F which has a base, and let D be a closed C^∞ *pseudo-disc about* 0 *in F. Then there is a manifold* $\tilde{M}$, *diffeomorphic to* int M, *which contains a copy of* ∂M *as a* C^∞, *closed, flat submanifold having a closed tubular neighbourhood* $i: X \times D \to \tilde{M}$ *for which* $\tilde{M} - i(X \times \operatorname{int} D)$ *is diffeomorphic to M.*

PROOF. Take any copy X of ∂M. Then $X \times D$ is diffeomorphic to ∂M by theorem 19. We may therefore glue $X \times D$ to M along the boundaries to obtain a C^∞ manifold $\tilde{M}$ having $X \times D$ as a closed tubular neighbourhood of X and M diffeomorphic to $\tilde{M} - (X \times \operatorname{int} D)$. By the theorem 19 (ii) the strong F-stability of E enables this to be done in such a way that $\tilde{M}$ is parallelisable and X is flat in $\tilde{M}$. Since $X \times \partial D$ is a deformation retract of $X \times D$, M is a deformation retract of $\tilde{M}$ and int M is homotopy equivalent to $\tilde{M}$. Thus $\tilde{M}$ is diffeomorphic to int M.//

In [7], Kuiper and Burghelea give an extension of Mazur's tangential equivalence theorem which proves our next result for the case of stable Hilbert manifolds with boundary $M \times E$, $\partial M \times E$ (see also [6]).

THEOREM 22. *Consider two* C^∞ *manifolds with boundary,* M, ∂M *and* N, ∂N, *which are separable, metrisable, and parallelisable. Assume*:

(i) *E is* C^∞ *smooth*;

(ii) *E is strongly F-stable for some infinite dimensional space F*;

(iii) *E and F have Schauder bases.*

Then if $(f, \alpha): M, \partial M \to N, \partial N$ *is a tangential equivalence there is a* C^∞ *diffeormorphism* $d: M, \partial M \to N, \partial N$ *which is tangentially homotopic to* (f, α) *and has* $d|\partial M$ *homotopic to* $f|\partial M$.

PROOF. Apply lemma 21, together with its notation, to obtain manifold pairs $\tilde{M}$, X and correspondignly $\tilde{N}$, Y. The equivalence (f, α) induces an equivalence of $\tilde{M} - (X \times \operatorname{int} D)$ with $\tilde{N} - (Y \times \operatorname{int} D)$ which extends, by (ii), to a tangential equivalence of pairs $(\tilde{f}, \tilde{\alpha}): \tilde{M}, X \to \tilde{N}, Y$. Applying lemma 20 we get a diffeomorphism $h: \tilde{M}, X \to \tilde{N}, Y$ which is tangentially homotopic to $(\tilde{f}, \tilde{\alpha})$ and has $h|X$ tangentially homotopic to $(\tilde{f}|X, \tilde{\alpha}|TX)$: $X \to Y$. By composing with an isotopy of $\tilde{N}$ obtained from the tubular neighbourhood theorem, if necessary, we may assume there is a C^∞ map $k: X \to (0, 1)$ such that $h(X \times kD)$ is contained in $Y \times \operatorname{int} D$ and is radial there. The map h restricts to a diffeomorphism of $\tilde{M} - \operatorname{int}(X \times kD)$ with $\tilde{N} - h[\operatorname{int}(X \times kD)]$. Taking collars of ∂M and ∂N and deforming them in $\tilde{M}$, $\tilde{N}$ we see that this induces the required diffeomorphism d.//

We can now return and give the promised strengthening of lemma 20:

THEOREM 23. *For* $i = 1, 2$, *consider pairs* M_i, N_i *where* N_i *is a* C^∞

H-manifold embedded as a closed infinite co-dimensional C^∞ submanifold of the E-manifold M_i. Assume that each M_i, and N_i, is separable, metrisable, and parallelisable and that:

(i) *E is C^∞-smooth*

(ii) *H and E are strongly F-stable for the same infinite dimensional space F*

(iii) *E, H, and F have Schauder bases.*

Then if $(f, \alpha) : M_1, N_1 \to M_2, N_2$ is a tangential equivalence of pairs, there is a C^∞ diffeomorphism of pairs $d : M_1, N_1 \to M_2, N_2$ with (f, α) tangentially homotopic to d and $(f|N_1, \alpha|TN_1)$ tangentially homotopic to $d|N_1$. If $f|N_1$ is already a diffeomorphism tangentially homotopic to $(f|N_1, \alpha|TN_1)$ we may choose d with $d|N_1 = f|N_1$, and in this case the condition on H in (ii) is unnecessary.

Proof. Use theorem 19 to obtain a diffeomorphism $h : N_1 \to N_2$ in the required tangential homotopy class and construct tubular neighbourhoods $g_i : B_i \to M_i$ of N_i, $i = 1, 2$, where $\pi_i : B_i \to N_i$ is a vector bundle. The equivalence $\alpha|N_1$ induces a C^∞ vector bundle isomorphism $\hat{h} : B_1 \to B_2$ over h. For a C^∞ Finsler-like function on B_1 there is a closed pseudo-disc neighbourhood D_1 of N_1 in B_1 such that both $g_1(D_1)$ and $g_2(\hat{h}D_1)$ are closed tubular neighbourhoods Z_1 and Z_2, say, of N_1, N_2 in M_1, M_2. The composition $g_2\hat{h}g_1^{-1}$ restricts to a diffeomorphism $\bar{h} : Z_1, \partial Z_1 \to Z_2, \partial Z_2$ of manifolds with boundary. Since N_i is homotopy negligible in M_i, (f, α) induces a tangential equivalence of manifolds with boundary $(f', \alpha') : M_1 - \operatorname{int} Z_1, \partial Z_1 \to M_2 - \operatorname{int} Z_2, \partial Z_2$. This is homotopic to a diffeomorphism h' as in Theorem 22.

We now have two diffeomorphisms $\partial Z_1 \to \partial Z_2$ given by the restrictions of h' and $\bar{h}$. These are easily seen to be tangentially homotopic, and hence isotopic by theorem 19. After a modification in tubular neighbourhoods of ∂Z_1 and ∂Z_2 they can therefore be combined to give the required diffeomorphism d.//

Remarks. (i) So far, in the discussion of manifolds with boundary and of manifold pairs the diffeomorphisms d we have obtained have not been shown to be homotopic to the original homotopy equivalences f through maps of pairs. In order to get a diffeomorphism tangentially homotopic through equivalences of pairs to (f, α) we could proceed as follows: First use Theorem 22 to show that any M, ∂M which satisfies the conditions of that theorem is diffeomrophic to some $N \times E$, $\partial N \times E$ where N, ∂N is an F-manifold with boundary. Next use the extension of Mazur's theorem for stable F-manifolds with boundary following Kuiper and

Burghelea in [7], mentioned above. This will give a diffeomorphism $N_1 \times F, \partial N_1 \times F \to N_2 \times F, \partial N_2 \times F$ homotopic to any given homotopy equivalence, as maps of pairs. Then use the method of theorem 19 (ii) to get the required result for manifolds with boundary. This in turn gives enough information about the map h' in the proof of theorem 23 to give the strengthened result for manifold pairs.

(ii) The case of finite co-dimensional submanifolds (with Hilbert space models) is considered by Kuiper in [21] using the same method as in our proof of theorem 23. In this situation there are extra homotopy invariants to consider (since $M_i - N_i$ need no longer be homotopy equivalent to M_i): for example Kuiper shows that not all closed embeddings of the Hilbert sphere into Hilbert space with co-dimension 2 give diffeomorphic pairs. Nevertheless these invariants still determine the diffeomorphism type.

G. More on isotopy and embeddings

The first result in this section follows immediately from Theorem 23 and 19 (iii). It is a version of the isotopy theorem, Theorem 14, with the assumption of trivial normal bundles (in particular of flatness) replaced by mild conditions on the Banach spaces and loss of information about the support of the isotopy. The remarks after Theorem 23 show that we could obtain similar, but more complicated, isotopy theorems for finite co-dimensional submanifolds using essentially the same methods.

THEOREM 24. *Let M and X be C^∞ manifolds modelled on Banach spaces H and E, both manifolds being separable, metrisable, and parallelisable, and both model spaces being C^∞-smooth and having Schauder bases. Suppose that $f_i : M \to X$, $i = 0, 1$ are closed C^∞ embeddings with the same infinite dimensional co-space and assume*:

(i) *f_0 and f_1 are homotopic*;

(ii) *f_0 and f_1 are tangentially homotopic*;

(iii) *E is strongly F-stable for some infinite dimensional space F which has a base.*

Then there is a C^∞ isotopy $\Phi : R \times X \to R \times X$ with $\Phi_0 = \mathrm{id}_X$ and $\Phi_1 \circ f_0 = f_1$.//

Next we have some results on open embeddings. The theorem gives a strengthening of the main result in OECBM and the first part is similar to theorem 8.4 of [7]. From it we go on to deduce corollaries concerning manifolds with boundary and relative open embeddings of manifold pairs.

THEOREM 25. *Let M and N be C^∞ E-manifolds which satisfy the condi-*

tions of OECBM and let F be an infinite dimensional Banach space with a base.

(i) *If E is F-stable then any map $f: M \to N$ is homotopic to an open embedding. This embedding may be chosen to lie in an arbitrary given neighbourhood of $\overline{f(M)}$. Furthermore if E is strongly F-stable it may be also chosen to be tangentially homotopic to a given equivalence $\alpha: TM \to f^*TN$.*

(ii) *Let W be a closed parallelisable C^∞ submanifold of M, modelled on E_1 and with co-space E_2. Suppose that $f: M \to N$ is continuous and restricts to a closed C^∞ embedding of W in N with co-space E_2 and that $\alpha: TM \to f^*TN$ is an isomorphism extending $t(f|N)$. Assume that E_1 and E_2 have bases and that E_2 is F-stable and also that either E is strongly F-stable or that the inclusion $GL(E_1) \to GL(E_1 \times E_2)$ is nullhomotopic. Moreover suppose that W and $f(W)$ are flat in M and N. Then $f|W$ extends to a C^∞ open embedding of M into N which is homotopic to f.*

PROOF. The proof of (i) follows easily from the open embedding theorem in OECBM together with our proposition 16 and theorems 12 and 19 (ii), suing the existence of tubular neighbourhoods.

To prove (ii) first take diffeomorphisms of pairs $h_0: M, W \to M \times F, W \times \{0\}$ and $h_1: N, f(W) \to N \times F, f(W) \times \{0\}$. This is possible by lemma 20 (it is only here that we use the unnecessarily restrictive assumption that W and $f(W)$ are flat submanifolds). Set $f' = h_1 f h_0^{-1}$. Then using α, extend $f'|W \times \{0\}$ to a diffeomorphism f'' of an open tubular neighbourhood U of $W \times \{0\}$ in $M \times \{0\}$ onto a tubular neighbourhood $f''(U)$ of $f(W) \times \{0\}$ in $N \times \{0\}$. We may take N to be an open subset of E and choose an open neighbourhood U_1 of W in $M \times \{0\}$ with $\overline{U}_1 \subset U$. According to lemma 8, $f''|\overline{U}_1$ extends to a C^∞ Φ_0-map f''' : $M \times \{0\} \to E$ whose tangent map is isotopic to α through Φ_0 bundle maps. By theorem 0 of § C this induces a parallelisable C^∞ layer structure $\{M, f'''\}$ on M modelled on E such that $f''|\overline{U}_1$ is a closed $L(T)$-embedding into $N \times F$, $T: E \to E \times F$ being the inclusion.

Next take a basic sequence $\{f_1, f_2, \cdots\}$ in F and also an extension of $f''|\overline{U}_1$, $f^4: M \times \{0\} \to N \times F$, which is homotopic to f' and has $f^4(M \times \{0\} - \overline{U}_1) \subset N \times (F - \{0\})$. By Theorem 6, f^4 is homotopic to a closed $L(T)$-embedding $f^5: M \times \{0\} \to N \times F$ which agrees with f'' on W. Since $T\{M, f'''\}$ was trivial this embedding has trivial normal bundle and therefore extends to an open embedding f^6 of $M \times F$ as a tubular neighbourhood. The composition $h_1^{-1} f^6 h_0$ is the required open embedding of M in N.//

The first corollary follows immediately from part (i):

COROLLARY 25.1. *Let M be a separable metrisable C^∞ E-manifold, where E is C^∞-smooth, has a Schauder base, and is strongly F-stable for some infinite dimensional Banach space F. Then any trivialisation of TM is isotopic to an integrable one (in fact to one which may be realised by a single chart).*//

REMARK. In Corollary 25.1 the condition that E be F-stable may be replaced by assuming that $GL_c(E)$ is contractible in $GL(E)$. The proof in this case comes directly from OECBM (see theorems 0, 00 and 10 in § C of this paper) using the exactness at $GL(E)$ of the homotopy sequence $[X, GL_c(E)] \to [X, GL(E)] \to [X, \Phi_0(E)]$ for paracompact X, [15], [16].

COROLLARY 25.2. *Let W be a closed flat infinite dimensional C^∞ submanifold of M, modelled on E_1 and with infinite dimensional co-space E_2. Assume that W and M satisfy the conditions of OECBM and that E_1 and E_2 satisfy the conditions of part* (ii) *of the theorem. Moreover, suppose that either E_1 is strongly H-stable for some infinite dimensional H with a basis or that $GL_c(E_1)$ is contractible in $GL(E_1)$. Then there exists a C^∞ open embedding $h : M \to E_1 \times E_2$ such that $h(W) = h(M) \cap (E_1 \times \{0\})$.*

PROOF. Since W is flat in M there is a trivialisation $\alpha : TM \to M \times E$ sending TN to $N \times E_1$. According to 25.1 or the remark above there is an open embedding $h_0 : W \to E_1$ with Th_0 isotopic to $\alpha|TN$. Set $Z = E_1 - h_0(W)$ and $N = E_1 \times E_2 - Z \times \{0\}$. Then $h_0(W)$ is a closed flat submanifold of N and h_0 extends to a continuous map $h_1 : M \to N$. Moreover α is isotopic to an isomrophism of TM with $h_1^*(TN)$ which extends Th_0. Part (ii) of the theorem therefore gives an open embedding of M into N extending h_0 and hence an open embedding into $E_1 \times E_2$ extending h_0, as required.//

COROLLARY 25.3. *Let M be a separable metrisable C^∞ parallisable E-manifold with boundary ∂M. Assume that E is C^∞-smooth, that E is strongly F-stable for some infinite dimensional space F, and that both E and F have bases. Then there is an open subset U of E such that the union of U together with certain of its boundary components is a C^∞ submanifold with boundary of E which is diffeomorphic to M.*

PROOF. We use lemma 21 and its notation. There is an open embedding h of $\tilde{M}$ into E which restricts to a closed embedding of X. This can be seen, for example, from the proof of 25.2 since there N is diffeomorphic to $E_1 \times E_2$ because Z is homotopy negligible in $E_1 \times E_2$, [14]. Take a closed pseudo-disc neighbourhood V of $h(x)$ in E contained in $h(\tilde{M})$ and set $U = h(\tilde{M}) - V$. It is easy to see that M is diffeomorphic to $U \cup \partial V$ as required.//

REMARKS. (i) The conclusion of 25.3 is true for any separable metrisable C^∞ submanifold with boundary modelled on one of the spaces L_{2n}, l_{2n}, $n = 1, 2, \cdots$, or c_0; the conclusion of 25.2 is true for any C^∞ submanifold with infinite dimension and codimension of a separable metrisable C^∞ manifold modelled on l_{2n}, $n = 1, 2, \cdots$, or c_0. This is because of the contractibility of the general linear groups of these spaces and the fact that all infinite dimensional split subspaces of l_{2n} and c_0 are isomorphic to their ambient space, [33].

(ii) Corollary 25.3 begs the question as to when $M, \partial M$ is diffeomorphic to an open subset U of E together with all of its boundary. There are homotopy considerations involved in this: for example a necessary condition is that it must be possible to glue a space with the homotopy type of a CW complex to M along ∂M and obtain a contractible space. This sort of problem is discussed by Burghelea in [5].

Corollary 25.2 shows that the differential structure of a wide class of flatly embedded submanifolds can be realised by a single chart. We next prove the corresponding result for manifolds with boundary:

THEOREM 26. *Suppose that the Banach space E satisfies the following conditions*:

(i) *E is C^∞-smooth.*

(ii) *E is strongly F-stable for some infinite dimensional space F.*

(iii) *E and F have Schauder bases.*

Let E_λ^+ be a closed half-space of E, i.e. $\lambda \in E^*$ and $E_\lambda^+ = \{x \in E : \lambda(x) \geqq 0\}$. Then any parallelisable, separable metrisable, C^∞ E-manifold with boundary, $M, \partial M$ is C^∞ diffeomorphic to an open subset of E_λ^+.

PROOF. According to theorem 22, $M, \partial M$ is C^∞ diffeomorphic to $M \times F, \partial M \times F$. There is also a diffeomorphism of E_λ^+ with $[0, \infty) \times \partial E_\lambda^+$, and a linear isomorphism of ∂E_λ^+ with $\partial E_\lambda^+ \times F$. Let $T : \partial E_\lambda^+ \to \partial E_\lambda^+ \times F$ be the natural embedding.

Give M a C^∞ parallelisable layer structure modelled on E (for example let $\tilde{M}$ be the double of M and give $\tilde{M}$ such a structure using theorems 0 and 10, ∂M then has the naturally induced structure of a layer submanifold since it has finite co-dimension in $\tilde{M}$, [16]). According to theorem 6 there is a closed, C^∞, $L(T)$-embedding $f : \partial M \to \partial E_\lambda^+ \times F$. This can be extended trivially to a closed $L(T)$-embedding of a closed collar of ∂M into $\partial E_\lambda^+ \times F \times [0, \infty)$. Theorem 6 then shows that a restriction of this to a smaller collar can be extended to a closed $L(T)$-embedding of M into $\partial E_\lambda^+ \times F \times [0, \infty) = E_\lambda^+ \times F$. This has trivial normal bundle and so extends to a tubular neighbourhood embedding of $M \times F$ in $E_\lambda^+ \times F$, the diffeomorphisms then give the required embedding of M in E_λ^+.//

After the results of this section one would expect a positive solution to the following problem. It is stated in terms of Hilbert manifolds for simplicity.

PROBLEM. Let $f: M \to N$ be a C^∞ submersion of l_2-manifolds with infinite dimensional fibres. Does there exist a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{d} & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi|W} \\ N & \xrightarrow[d_0]{} & U \end{array}$$

with d, d_0 diffeomorphisms onto cpen sets W, U of l_2 and $\pi: l_2 \to l_2$ a bounded linear surjection?

H. Open subsets of Banach spaces with smooth factors

We will first consider some facts about direct sums of Banach spaces with respect to a base, introduced by Pelczynski in [33]. Let W be a Banach space with a fixed normalized orthogonal base $\{w_i\}_{i=1}^\infty$: i.e. $\{w_i\}$ is a Schauder base with $||w_i||_W = 1$ such that if $\{\alpha_i\}$, $\{\beta_i\}$ are real sequences with $||\beta_i|| \leqq ||\alpha_i||$ for each i and $||\Sigma\alpha_i w_i||_W < \infty$ then $||\Sigma\beta_i w_i||_W \leqq ||\Sigma_1 \alpha_i w_i||_W$. Then for any sequence $\{H_i, ||\ ||_i\}_i$ of Banach spaces we can form the direct sum, $\Sigma_W H_i$, of this sequence with respect to W. It is defined to be the set of those sequences $\{x_i\}$ with $x_i \in H_i$ such that $\Sigma_i ||x_i|| w_i < \infty$. Given the norm $||\{x_i\}|| = ||\Sigma_i ||x_i||_i w_i||_W$ it becomes a Banach space under coordinate-wise addition and scalar multiplication. The examples $l_p(E) = \Sigma_{l_p}(E)$ and $c_0(E) = \Sigma_{c_0}(E)$ have already been used in § D. Set $H = \Sigma_W H_i$ and define $\pi_n : H \to H$ by $\pi_n(\{x_i\}) = (x_1, \cdots, x_n, 0, 0, \cdots)$; the proof of the following lemma is straightforward:

LEMMA 26. *For each n, $H_1 \times \cdots \times H_n$ can be considered as a closed subspace of H and π_n is a continuous projection onto this subspace. Furthermore $||\pi_n|| = 1$ and $\{\pi_n\}$ converges strongly to id_H. In particular $\bigcup_{n=1}^\infty H_1 \times \cdots \times H_n$ is dense in H.//*

The basic result of this section concerns the stability of open subsets of certain Banach spaces which are not assumed to admit differentiable partitions of unity:

THEOREM 27. *Let E be a separable Banach space and F an infinite dimensional split subspace of E. Assume that E is the direct sum $H \times G$ of two infinite dimensional subspaces which satisfy the following conditions:*

(i) *F and G have Schauder bases*;

(ii) *H is C^∞-smooth*;

(iii) *H is an infinite direct sum with respect to a basis, $H = \Sigma_W H_i$, where each H_i is infinite dimensional and has a base, and*

(iv) *for each n, $H_1 \times \cdots \times H_n$ is strongly F-stable.*

Then if X is any open subset of E, X is C^∞ diffeomorphic to $X \times F$.

PROOF. Let $\{g_1, g_2, \cdots\}$ be a basis for G. We shall be considering the following subspaces of E:

$$G_n = \operatorname{Sp}\{g_1, \cdots, g_n\},\ E_n = H_1 \times \cdots \times H_n \times G_n,$$
$$E^n = \overline{\operatorname{Sp}}\{g_i : i > n\} \times H_{n+1} \times H_{n+2} \times \cdots,\ E_n^{n+1} = H_{n+1} \times Rg_{n+1}.$$

Let π_n' be the projection of G onto G_n determined by the basis. Then E splits into $E_n \times E^n$ with corresponding projections $p_n : E \to E_n, p^n : E \to E^n$ where $p_n = \pi_n \times \pi_n' : H \times G \to (H_1 \times \cdots \times H_n) \times G_n$. We may renorm G if necessary so that $\{g_1, g_2, \cdots\}$ is a monotone base and then take the norm $\|\ \|_H + \|\ \|_G$ on $H \times G$. This will ensure that $\|p_n\| = 1$ for each n. Note also that $\{p_n\}_{n=1}^\infty$ converges strongly to id_E.

Set $X_n = X \cap E_n$. Because of these properties of $\{p_n\}$ we can use the proof of proposition 16 to obtain a sequence of open tubular neighbourhoods $\{V_n(m)\}_{m=1}^\infty$ of X_n in X, for each n, such that:

a) $X = \bigcup_n V_n(1)$;

b) $V_n(m) \subset V_{n+1}(m)$;

c) $V_n(m)$ is a bounded radial neighbourhood of $X_n \times \{0\}$ in $X_n \times E^n$;

d) $V_n(m) \subset [V_n(m) \cap (X_n \times E_{n+1}^n)] \times E^{n+1}$;

e) Using fibrewise multiplication in $X_n \times E^n$, there is an $\alpha_m > 1$ with $\alpha_m V_n(m) \subset V_n(m+1)$.

Take a splitting $s_1 : E_1 \to E_1 \times F$ and extend it by the identity to obtain splittings $s_n : E_n \to E_n \times F$ for each n. We shall inductively construct a sequence of C^∞ diffeomorphisms $\{d_n\}_{n+1}^\infty$, $d_n : X_n \to X_n \times F$, which are in the tangential homotopic class determined by s_n and the natural inclusion of X_n in $X_n \times F$. These extend trivially by the identity map on the fibres, to diffeomorphisms $\bar{d}_n : V_n(n) \to V_n(n) \times F$, and we further require that these make the following ladder commutative:

$$\begin{tikzcd}
V_1(1) \arrow[r, "i_1"] \arrow[d, "\bar{d}_1"] & V_2(2) \arrow[r, "i_2"] \arrow[d, "\bar{d}_2"] & V_3(3) \arrow[r] \arrow[d, "\bar{d}_3"] & \cdots \\
V_1(1) \times F \arrow[r, "i_1 \times id_F"'] & V_2(2) \times F \arrow[r, "i_2 \times id_F"'] & V_3(3) \times F \arrow[r] & \cdots
\end{tikzcd}$$

Here $i_n : V_n(n) \to V_{n+1}(n+1)$ denotes the inclusion. In the limit this will give the diffeomorphism $\bar{d}_\infty : X \to X \times F$ which we seek.

To construct $\{d_n\}$ first note that X_n is modelled on $H_1 \times \cdots \times H_n \times R^n$ and is therefore C^∞ smooth. Also E_n is strongly F-stable by the discussion before lemma 2.1. Thus theorem 19 gives us d_1. Assume therefore that $d_1, \cdots, d_n$ have been constructed as required. Set $U = V_n(n) \cap (X_n \times E^n_{n+1})$ and $V = V_n(n+1) \cap (X_n \times E^n_{n+1})$. These are open tubular neighbourhoods of X_n in X_{n+1}, bounded and radial about X_n in $X_n \times E^n_{n+1}$; also $\alpha_n U \subset V$ for some $\alpha_n > 1$. It follows from § 0 that there is a smooth Finsler-like function ρ on $X_n \times E^n_{n+1}$ with $U \subset \rho^{-1}[0, 1] \subset V$. Write $D = \rho^{-1}[0, 1]$ and $\tilde{X} = X_{n+1} - \text{int } D$. Then $\partial\tilde{X} = \partial D$.

The diffeomorphism $d_n : X_n \to X_n \times F$ extends by the identity on the fibres of D to a diffeomorphism $d : D, \partial D \to D \times F, \partial D \times F$. This is homotopic to the inclusion and tangentially homotopic to s_{n+1}. By theorem 22 there is also a diffeomorphism $d' : \tilde{X}, \partial D \to \tilde{X} \times F, \partial D \times F$ in the corresponding tangential homotopy class. According to theorem 19, d and d' are isotopic. They may therefore be glued together near ∂D to give a diffeomorphism $d_{n+1} : X_{n+1} \to X_{n+1} \times F$ with $d_{n+1}|U = d|U = \bar{d}_n|U$. Clearly d_{n+1} is in the right tangential homotopy class. Also, by condition d), we see that $\bar{d}_{n+1}$ fits into the commutative diagram as required.//

The spaces l_2, c_0 are isomorphic to the infinite sums $\Sigma_{l_2} l_2$, $\Sigma_{c_0} c_0$ and are C^∞ smooth. Also $C[0, 1]$ is c_0-stable and L_p is l_2-stable, $p > 1$, [33]. Thus the following examples, where '$\cong$' denotes C^∞ diffeomorphism are direct applications of the theorem (using a repeated application for (iii)).

EXAMPLES 27.1. Let X be an open subset of E. Then if:

(i) $E = C[0, 1]$, $X \cong X \times c_0$

(ii) $E = L_p$, $1 < p < \infty$, $X \cong X \times l_2$

(iii) $E = l_1 \times c_0 \times l_2$, $X \cong X \times c_0 \times l_2$.//

Unfortunately the theorem gives no information about open subsets of spaces which do not have infinite dimensional C^∞-smooth factors, for example l_p when p is not an even integer [3]. However the method can be extended to a non-separable space and used to show that any open subset X of non-separable Hilbert space of dimension aleph one is C^∞ diffeomorphic to $X \times l_2$. This is done by considering an uncountable filtration of X by separable Hilbert submanifolds X_n, and the proof becomes a simple exercise in transfinite induction. Hardly anything is known about the differential topology of non-separable Banach spaces, basically because it is not known whether any of them admit differentiable partitions of unity. (*Added in proof*: Hilbert spaces do, see [36a].)

Our final theorem comes directly from theorem 27 and proposition 18:

THEOREM 2.8. *Let $f: X \to Y$ be a homotopy equivalence between open subsets of a separable C^r-smooth Banach space E, $r \geqq 1$. If E satisfies the conditions of theorem* 27, *then f is homotopic to a C^r diffeomorphism $\bar{f}: X \to Y$.*//

COROLLARY 28.1. *The proof of theorem* 19 (ii) *yields: If also E is strongly B-stable for an infinite dimensional Banach space B then $\bar{f}$ may be chosen in any given tangential homotopy class.*//

For positive p denote by $\bar{p}$ the largest integer strictly less than p except when p is an even integer in which case set $\bar{p} = \infty$. Then, according to Bonic and Frampton [3], if $2 \leqq p < \infty$, L_p is $C^{\bar{p}}$-smooth. Thus, using 27.1, theorem 28 gives:

EXAMPLES 28.2. Suppose $E = L_p$, $2 \leqq p < \infty$. Then homotopy equivalent open subsets of E are $C^{\bar{p}}$ diffeomorphic. In particular any open subset of E is $C^{\bar{p}}$ diffeomorphic to $U \times E$ where U is some open subset of l_2.//

We can now generalise Nicole Moulis' theorem [28] on the existence of an m-function on open subsets of L_2 to the case of open subsets of L_p:

COROLLARY 28.3. *Let X be an open subset of L_p, $2 \leqq p < \infty$. Then there is a complete Finsler metric μ on X together with a $C^{\bar{p}}$ function $f: X \to R$ which satisfies the following conditions*:

(i) *f is bounded below*;

(ii) *f satisfies condition C of Palais and Smale with respect to μ*;

(iii) *All the critical points of f are weakly non-degenerate in the sense of Karen Uhlenbeck* [35] *and have finite index. In particular they are isolated.*

PROOF. We can write $X = U \times L_p$ where U is open in l_2, by 28.2. The theorem of Nicole Moulis shows that there is a complete Riemannian metric on U together with a C^∞ function $f_1 : U \to R(> 1)$ which satisfies (i), (ii), (iii). On the other hand if $f_2 : L_p \to R$ is defined by $f_2(y) = (||y||_{L_p})^p$ the computation in [3] shows that f_2 is $C^{\bar{p}}$ and satisfies (i), (ii), (iii) with respect to the natural Finsler on L_p. If we take μ to be the product of these two metrics on $U \times E$ and set $f(x, y) = f_1(x)(1 + f_2(y))$ for $(x, y) \in U \times E$ we obtain a function as required.//

Corollary 28.3 is best possible in the sense that there can exist no C^r function $f: X \to R$ which satisfies (i) and (ii) and has isolated critical points when $r > \bar{p}$. To see this assume that X is connected. Then such an f would attain its minimum on X at some point x_0 in X. Say $f(x_0) = m$. If W is a closed bounded neighbourhood of x_0 in L_p contained in X it is shown in [3] that $\inf\{f(x) : x \in \partial W\} = m$. Condition C would then imply that f has a critical point on ∂W, which shows that x_0 is not an isolated critical point. A similar argument using the results of J. Wells, [36], shows that if X is open in c_0 there is no C^2 function on X satisfying (i) and (ii) which has isolated critical points (c.f. § A Remark (ii)).

APPENDIX

Manifolds with quasi-reflexive models

Let $j : E \to E^{**}$ denote the canonical embedding of the Banach space E into its second dual space. Recall that E is said to be *quasi-reflexive* if $E^{**}/j(E)$ is finite dimensional. The dimension of $E^{**}/j(E)$ is then called the *order* of E. Recall also that a bounded linear map $T : E \to F$ of Banach spaces is weakly compact if and only if the second adjoint T^{**} of T sends E^{**} into $j(F)$ i.e. $T^{**}(E^{**}) \subset j(F)$, [34, page 250]. The space of such maps will be denoted by $W(E, F)$, or $W(E)$ if $E = F$. Then $W(E)$ is a closed ideal of $L(E)$. We will write $GL_W(E)$ for the subset of $GL(E)$ consisting of those elements of the form $1+w$ where $w \in W(E)$. This is a closed invariant subgroup of $GL(E)$.

Write $\hat{E} = E^{**}/j(E)$. If $T \in L(E)$, since $T^{**}(j(E)) \subset j(E)$, T^{**} induces an element $\hat{T} \in L(\hat{E})$. In this way we get a continuous algebra homomorphism $Q : L(E) \to L(\hat{E})$ by defining $Q(T) = \hat{T}$. The kernel of this map is easily seen to be precisely $W(E)$. Restricting Q to $GL(E)$ we obtain a group homomorphism $Q_0 : GL(E) \to GL(\hat{E})$ with kernel $GL_W(E)$.

Suppose that E is quasi-reflexive of order one. Let E^n denote the direct sum of n-copies of E, $E^n = E \times \cdots \times E$. Then E^n is quasi-reflexive of order n. Elements of $L(E^n)$ may be regarded as $n \times n$-matrices with coefficients in $L(E)$. The map $L(R^n) \to L(E^n)$ given by $(a_{ij}) \mapsto (a_{ij} I_E)$ defines a continuous algebra homomorphism $\psi : L(\hat{E}^n) \to L(E^n)$ which restricts to a group homomorphism $\psi_0 : GL(\hat{E}^n) = GL(n) \to GL(E^n)$. These are sections of Q, Q_0 respectively. It follows that Q and Q_0 are trivial bundles over $L(R^n)$, $GL(n)$ and therefore $L(E^n)$ is linearly isomorphic to $W(E^n) \times L(R^n)$ and $GL(E^n)$ is *analytically diffeomorphic* to $GL_W(E^n) \times GL(n)$. In particular $GL(E^n)$ is not connected.

Let F be some infinite dimensional separable C^∞-smooth Banach space. There is an open subset B of F which is homotopy equivalent to real infinite dimensional projective space, $RP(\infty)$, and over B there is a universal line bundle $\xi : U \to B$ say, of class C^∞. If E is quasi-reflexive of order one let $\xi(E) : U(E) \to B$ be the C^∞ E-bundle over B associated to ξ by the section ψ_0. The bundle $\xi(E)$ is canonical embedded in its second dual bundle $\xi(E)^{**}$ and the corresponding quotient bundle $\widehat{\xi(E)}$ is just ξ again. Since ξ has not got a finite dimensional inverse bundle (for example by looking at the Stiefel-Whitney classes), it follows that $\xi(E)$ *has not got an inverse modelled on E^n for any finite n*: for if η were such an inverse, $\hat{\eta} = \eta^{**}/\eta$ would be a finite dimensional inverse for ξ.

The C^∞ manifold $U(E)$ is modelled on the quasi-reflexive space $E \times F$ and has tangent bundle isomorphic to the Whitney sum of the trivial F-bundle with the pull back of $\xi(E)$ over itself. Since F is reflexive the above method shows that $T(U(E))$ has not got an inverse modelled on a finite direct sum of copies of $E \times F$. In fact, since any closed subspace of a quasi-reflexive space of order n is quasi-reflexive of order at most n, the method shows that $T(U(E))$ cannot be embedded as a subbundle of a trivial $(E \times F)^n$ bundle for any finite n.

If E is separable then so is E^{**} and hence E^* is also. It follows that E is C^1-smooth, [3]. Thus in this case, *$U(E)$ is a separable, C^∞, $E \times F$-manifold which is C^1-smooth but admits no C^1 immersion into any finite direct sum of copies of $E \times F$.*

The standard quasi-reflexive space was constructed by R. C. James [19]. It ie of order one, and will be denoted by J. It consists of those real sequences $x = (a_1, a_2, \cdots)$ for which $\lim a_n = 0$ and

$$||x|| = \sup \left[\sum_{i=1}^{n} |a_{p_{2i-1}} - a_{p_{2i}}|^2 + |a_{p_{2n+1}}|^2\right]^{\frac{1}{2}}$$

is finite, where the supremum is over all finite increasing sequences of integers $\{p_1, p_2, \cdots, p_{2n+1}\}$, $n = 0, 1, 2, \cdots$.

James showed that the subspace J_0 of J consisting of those sequences $(a_1, a_2, \cdots)$ of J with $a_{2i} = 0$ for each i, is linearly isomorphic to l_2. I am grateful to G. Jameson for pointing out to me that this subspace splits in J: the mapping $(a_1, a_2, \cdots) \mapsto (a_1 - a_2, 0, a_3 - a_4, 0, \cdots)$ being a projection of J onto J_0. It follows that J is isomorphic to $J \times l_2$. Therefore if we take $F = l_2$ the manifold $U(J)$ defined above is modelled on J. Our discussion yields the following:

There is a separable Banach space J which is C^1-smooth and a C^∞ separable metrisable J-manifold which admits no C^1 immersion into any finite direct sum of copies of J.

REMARKS.

(i) I do not know whether J is C^r-smooth for $r > 1$. However, apparently V. Meshkov has shown that it does not admit a compatible C^2 norm.

(ii) Mitjagin and Edelstein [27] have shown that $Q_0 : GL(J^n) \to GL(n)$

is a homotopy equivalence for each n. In particular each $GL_W(J^n)$ is contractible. This is not true for general $GL_W(E)$ when E is an arbitrary Banach space: for example if E is reflexive $GL_W(E) = GL(E)$, on the other hand if E contains no infinite dimensional closed reflexive subspaces e.g. $E = l_1$, it is easy to see that every element of $W(E)$ is strictly singular ('semi-compact' in the terminology of [34]) and therefore $GL_W(E)$ is homotopy equivalent to $GL(\infty)$, [16]. The same holds when $E = C[0, 1]$ by Theorem C II 8.5 of [34]. The result of Herman and Whitley [17] that every infinite dimensional closed subspace of J contains a subspace isomorphic to l_2 means that each element in $L(l_p, J)$ and $L(J, l_p)$ is strictly singular, $p \neq 2$. Hence Pełcynski's extension of Douady's method in [10] shows that for $p \neq 2$ and $1 < p < \infty$, the quasi-reflexive space $J \times l_p$ does not have $GL_W(J \times l_p)$ connected. A quasi-reflexive space E with an even more complicated general linear group constructed by Mitjagin and Edelstein is described in [21]. For this E the map $\varphi_0 : GL(E) \to GL(\hat{E})$ is not surjective.

(iii) For any Banach space E, an immersion of an E-manifold M into a Banach space F with reflexive co-space induces a reduction of TM to $GL_W(E)$. In the separable C^r-smooth case, $r > 1$, the methods of § C can be used to show that any such reduction is equivalent to an integrable reduction. The only essential change needed in the proof is that operators in $GL(E) + W(E)$ must be used instead of Φ_0-operators.

It follows that a necessary condition for a manifold modelled on a quasi-reflexive space E to immerse in E is that it admits an integrable $GL_W(E)$-structure.

(iv) Since J is isomorphic to $J \times l_2$, theorem 28 shows that every open subset of J^n is C^1 diffeomorphic to the product of J^n with some open subset of l_2. One would expect that every J-manifold is diffeomorphic to a J-vector bundle over some such open subset. Remark (ii) shows that J is strongly l_2-stable, so there is no obstruction in the tangential homotopy type.

REFERENCES

R. D. Anderson
[1] Homeomorphisms of infinite dimensional manifolds, I.C.M. Nice 1970.

R. D. Anderson
[1a] Proper homotopies in l_2 manifolds, Comp. Mat. 23(1971) 155–157.

R. D. Anderson and J. D. McCharen
[2] On extending homeomorphisms to Fréchet manifolds, Proc. A.M.S. 25 (1970) 283–289.

R. Bonic and J. Frampton
[3] Smooth functions on Banach manifolds, J. of Math. and Mech. 15 (1966) 877–898.

R. Bonic, J. Frampton and A. J. Tromba
[4] Λ-manifolds, Journal of Functional Analysis 3 (1969).

D. Burghelea
[5] (i) Imbedding Hilbert manifolds in finite co-dimension, Séminaire de mathématiques supérieures, Montréal, 1969.
(ii) Imbedding Hilbert manifolds with given normal bundle, Math. Ann. 187 (1970) 207–219.

D. Burghelea
[6] Diffeomorphisms for Hilbert manifolds and handle decompositions, Bulletin A.M.S. 76 (1970) 352–357.

D. Burghelea and N. H. Kuiper
[7] Hilbert manifolds, Ann. of Math. 90 (1969) 379–417.

M. Day
[8] Normed linear spaces, Ergebn. Math. Heft 21 (1958).

M. Day
[8a] On the basis problem in Banach spaces, Proc. A.M.S. 13 (1962) 655–658.

A. Douady
[9] Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, Ann. Inst. Fourier, Grenoble 16 (1966) 1–95.

A. Douady
[10] Un espace de Banach dont le groupe linéare n'est pas connexe, Nederl. Akad. v. W. A68 = Ind. Math. 27 (1965) 787–789.

I. Edelstein, B. Mitjagin and E. Semenov
[11] The Linear groups of C and L_1 are contractible, Bull. Acad. Pol. Sc. Ser. Math., 18 No. 1, 27–33 (1970).

J. Eells
[12] A setting for global analysis, Bull. A.M.S. 72 (1966) 751–807.

J. Eells and K. D. Elworthy
[13] (OECBM) Open embeddings of certain Banach manifolds, Ann. of Math. 91 (1970) 465–485.

J. Eells and N. H. Kuiper
[14] Homotopy negligible subsets, Comp. Math. 21 (1969) 155–161.

K. D. Elworthy
[15] Fredholm maps and $GL_c(E)$ structures, Bull A.M.S. 74 (1968) 582–586.

K. D. Elworthy and A. J. Tromba
[16] Fredholm maps and differential structures on Banach manifolds. Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) 45–94 A.M.S. (1970).

R. Herman and R. Whitley
[17] An example concerning reflexivity, Studia Math. 28 (1967) 289–294.

M. Hirsch
[18] On tangential equivalence of manifolds. Ann. of Math. 83 (1966) 211–217.

R. C. James
[19] Bases and reflexivity of Banach spaces, Ann. of Math. 52 (1950) 518–527.

N. H. Kuiper
[20] Les variétés hilbertiens, Séminaire de Mathématiques Supérieures, Conférence Montréal 1969.

N. H. Kuiper
[21] The differential topology of separable Banach manifolds, ICM Nice 1970.

N. H. Kuiper and Besseline Terpstra
[22] Differentiable closed embeddings of Banach manifolds, Conference in honour of G. de Rham, Springer Verlag (1970) 118–125.

S. Lang
[23] Introduction to differentiable manifolds, Interscience, New York 1962.

E. Michael
[24] Continuous selections I, Ann. of Math. 63 (1956) 361–382.

J. Milnor
[25] Der Ring der Vektorraumbündel eines topologischen Raumes, Bonn 1959 (lecture notes).

B. S. Mitjagin
[26] Contractibility of the Linear group of a function Banach space, Uspechi Mat. Nauk 26 (1970).

B. S. Mitjagin and I. S. Edelstein
[27] The homotopy type of the linear groups of two classes of Banach spaces, Functional Analysis 4 (1970).

Nicole Moulis
[28] Sur les variétés hilbertiennes et les fonctions non dégénérées, Nederl. Akad. v. W. A71 = Ind. Math. 30 (1968) 497–511.

Nicole Moulis
[29] Approximation de fonctions différentiables sur certains espaces de Banach. Thèse, Paris, 1970 (to appear).

Nicole Moulis
[30] Structure Fredholm sur les variétés Banachiques, Notes of lectures by K. D. Elworthy. Séminaire de mathématiques supérieures, Montréal, 1969.

G. Neubauer
[31] On a class of sequence spaces with contractible linear group (Mimeographed notes Berkeley 1967).

R. S. Palais
[32] Homotopy theory of infinite dimensional manifolds, Topology 5 (1966) 1–16.

A. Pelczynski
[33] Projections in certain Banach spaces, Studia Math. 19 (1960) 209–228.

D. Przeworska-Rolewicz and S. Rolewicz
[34] Equations in linear spaces, Monografie Matematyczne 47, PWN, Warsaw 1968.

Karen Uhlenbeck
[35] Integrals with non-degenerate critical points, Bull. A.M.S. 76 (1970) 125–128.

J. Wells
[36] Differentiable functions on c_0, Bull. A.M.S. 75 (1969) 117–118.

J. Wells
[36a] C^1 partitions of unity on non-separable Hilbert space, Bull. A.M.S. 77 (1971) 804–807.

J. L. Kelley
[37] General Topology, Van Nostrand 1955.

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