COMPOSITIO MATHEMATICA

EMANUEL G. CALYS

The radius of univalence and starlikeness of some classes of regular functions

Compositio Mathematica, tome 23, nº 4 (1971), p. 467-470

<http://www.numdam.org/item?id=CM_1971__23_4_467_0>

© Foundation Compositio Mathematica, 1971, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

COMPOSITIO MATHEMATICA, Vol. 23, Fasc. 4, 1971, pag. 467–470
Wolters-Noordhoff Publishing
Printed in the Netherlands

THE RADIUS OF UNIVALENCE AND STARLIKENESS OF SOME CLASSES OF REGULAR FUNCTIONS

by

Emanuel G. Calys

1.

The writing of this paper has been motivated by recent results of A. E. Livingston [7] and S. D. Bernardi [2].

Let S denote the class of functions $f(z) = z+\sum_2^\infty a_n z^n$ which are regular and univalent in $E\{z : |z| < 1\}$ and which map E onto domains $D(f)$. We denote by S^* and K the subclasses of S where $D(f)$ are, respectively, starlike with respect to the origin, and convex. Let P denote the class of functions $p(z)$ which are regular and satisfy $p(0) = 1$, $\operatorname{Re}(p(z)) > 0$, for z in E. In [1] the following theorem was proven.

THEOREM A. *Let $f(z) = z+\sum_2^\infty a_n z^n$ be a member of S^*. Then*

$$F(z) = (c+1)z^{-c}\int_0^z t^{c-1}f(t)dt = z+\sum_2^\infty \left(\frac{c+1}{c+n}\right) a_n z^n \tag{1.1}$$

is also a member of the same class for $c = 1, 2, 3, \cdots$.

Theorem A represents a generalization of the corresponding theorem by R. J. Libera [6] for the case $c = 1$. Solving the relation (1.1) for the inverse function $f(z)$, we have

$$f(z) = \left(\frac{1}{1+c}\right) z^{1-c}[z^c F(z)]'. \tag{1.2}$$

In [2] S. D. Bernardi proved that if $F(z) \in S^*$, then $f(z)$, defined by (1.2), is univalent and starlike for $|z| < r_0$, where

$$r_0 = [-2+(3+c^2)^{\frac{1}{2}}]/(c-1) \text{ for } c = 2, 3, 4, \cdots.$$

This result is sharp. For $c = 1$, $r_0 = \frac{1}{2}$ and this result is due to A. E. Livingston [7].

In this paper we determine the radius of univalence and starlikeness of functions $f(z) = z+a_2z^2+\cdots$ which are regular in E and satisfy

$$F(z) = \frac{2}{z}\int_0^z \frac{f(t)g(t)}{t}\, dt, \tag{1.3}$$

where $F(z) \in S^*$ and (i) $g(z) \in K$, (ii) $g(z) \in S$ and (iii) $g(z)/z \in P$. We shall employ the same techniques used in [7].

2.

THEOREM 1. *If $f(z)$ is regular in E and satisfies* (1.3), *where $F(z) \in S^*$ and $g(z) \in K$, then $f(z)$ is univalent and starlike for $|z| < 2-\sqrt{3}$. This result is sharp.*

PROOF. Since F is in S^*, $\operatorname{Re}\{zF'(z)/F(z)\} > 0$ for all z in E. Hence there exists w, regular in E, such that $|w(z)| \leqq 1$ for z in E and such that

$$\frac{f(z)g(z)-\displaystyle\int_0^z \frac{f(t)g(t)dt}{t}}{\displaystyle\int_0^z \frac{f(t)g(t)dt}{t}} = \frac{zF'(z)}{F(z)} = \frac{1-zw(z)}{1+zw(z)}.$$

Thus

$$f(z)g(z) = \frac{2}{1+zw(z)}\int_0^z \frac{f(t)g(t)dt}{t},$$

and

$$\frac{zf'(z)}{f(z)} = \frac{2-zw(z)-z^2w'(z)}{1+zw(z)} - \frac{zg'(z)}{g(z)}.$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \operatorname{Re}\left\{\frac{2-zw(z)-z^2w'(z)}{1+zw(z)}\right\} - \left|\frac{zg'(z)}{g(z)}\right|. \tag{2.1}$$

Thus $f(z)$ will be univalent and starlike for those values of z for which the right-hand side of (2.1) is positive. Since $g(z)$ is in K,

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{1}{1-|z|} \qquad [4, \text{p. } 13].$$

Therefore the right-hand side of (2.1) will be positive if

$$\operatorname{Re}\left\{\frac{2-zw(z)-z^2w'(z)}{1+zw(z)}\right\} - \frac{1}{1-|z|} > 0. \tag{2.2}$$

Condition (2.2) is equivalent to

$$\operatorname{Re}\{[1+(|z|-2)zw(z)+(|z|-1)z^2w'(z)-2|z|][1+\overline{zw(z)}]\} > 0, \tag{2.3}$$

or

$$\operatorname{Re}\{(1-|z|)z^2w'(z)[1+\overline{zw(z)}]\} < \{1-(1+|z|)\operatorname{Re}(zw(z)) \\ -2|z|+(-2+|z|)|z|^2|w(z)|^2\}. \tag{2.4}$$

Using the well known result

$$|w'(z)| \leqq \frac{1-|w(z)|^2}{1-|z|^2} \qquad [3, \text{p. } 18]$$

and the fact that $\operatorname{Re}(zw(z)) \leqq |z||w(z)|$, we see that (2.4) will be satisfied if

$$(2.5) \qquad (1-|z|)|z|^2 \frac{(1-|w(z)|^2}{1-|z|^2}(1+|z||w(z)|) < \{1-(1+|z|)|w(z)||z|-2|z|+(|z|-2)|z|^2|w(z)|^2\}.$$

Since $|w(z)| \leqq 1$, $(1+|z||w(z)|)/(1+|z|) \leqq 1$ and (2.5) will be satisfied if

$$|z|^2(1-|w(z)|^2) < 1-2|z|-(1+|z|)|z||w(z)|+(|z|-2)|z|^2|w(z)|^2$$

which is equivalent to

$$(2.6) \qquad (|z|^2+2|z|)+(1-|z|)|z|^2|w(z)|^2+(1+|z|)|z||w(z)| < 1.$$

Hence, it suffices to show that (2.6) holds for all functions w, regular in E and satisfying $|w(z)| \leqq 1$, provided $|z| < 2-\sqrt{3}$.

In (2.6) put $a = |z|$, $x = |w(z)|$ and consider the function

$$p(x) = a^2+2a+a(1+a)x+a^2(1-a)x^2.$$

Clearly, $p(x)$ is increasing in $[0, 1]$ and $p(1) = 3a+3a^2-a^3$ is less than one for $0 \leqq a < 2-\sqrt{3}$. Condition (2.2) is thus seen to be satisfied if $|z| < 2-\sqrt{3}$. Hence $f(z)$ is univalent and starlike for $|z| < 2-\sqrt{3}$.

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z/(1+z)$. Then $F(z)$ is in S^*, $g(z)$ is in K and $f(z) = (z^2+z)/(1-z)^3$. Thus $f'(z) = (z^2+4z+1)/(1-z)^4$ and $f'(-2+\sqrt{3}) = 0$. Hence $f(z)$ is not univalent in $|z| < r$ if $r > 2-\sqrt{3}$.

THEOREM 2. *If $f(z)$ is regular in E and satisfies* (1.3), *where $F(z) \in S^*$ and $g(z) \in S$, then $f(z)$ is univalent and starlike for $|z| < \frac{1}{5}$. This result is sharp.*

The proof of this theorem is similar to that of Theorem 1. The only essential difference is the estimate

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{1+|z|}{1-|z|} \qquad [4, \text{p. } 5].$$

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z/(1+z)^2$. Then $f(z)g(z) = z^2/(1-z)^3$ and we have

$$\frac{zf'(z)}{f(z)} = \frac{2+z}{1-z} - \frac{zg'(z)}{g(z)} = \frac{2+z}{1-z} - \frac{1-z}{1+z} = \frac{1+5z}{1-z^2} = 0$$

for $z = -\frac{1}{5}$. Thus $f(z)$ is not starlike in $|z| < r$ if $r > \frac{1}{5}$.

REMARK. The above example shows that we cannot improve on the result of Theorem 2 if instead of $g(z)$ in S we assume $g(z)$ in S^*.

THEOREM 3. *If $f(z)$ is regular in E and satisfies* (1.3), *where $F(z) \in S^*$ and $g(z)/z \in P$, then $f(z)$ is univalent and starlike for $|z| < (5-\sqrt{17})/4$. This result is sharp.*

PROOF. Let $h(z) = g(z)/z$. Then

$$\frac{zf'(z)}{f(z)} = \frac{1-2zw(z)-z^2w'(z)}{1+zw(z)} - \frac{zh'(z)}{h(z)},$$

where $w(z)$ is regular in E and $|w(z)| \leqq 1$. Using the estimate

$$\left|\frac{zh'(z)}{h(z)}\right| \leqq \frac{2|z|}{1-|z|^2} \qquad [5]$$

and the techniques of Theorem 1, the result follows.

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z(1-z)/(1+z)$. Then $f(z)h(z) = z/(1-z)^3$, $f(z) = (z^2+z)/(1-z)^4$ and $f'(z) = (2z^2+5z+1)/(1-z)^5 = 0$ for $z = (-5+\sqrt{17})/4$. Hence $f(z)$ is not univalent in any disk $|z| < r$ if $r + (5-\sqrt{17})/4$.

REFERENCES

S. D. BERNARDI

[1] Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969) 429–446.

S. D. BERNARDI

[2] The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 24 (1970) 312–318.

C. CARATHEODORY

[3] *Theory of Functions*, Vol. 2, Chelsea, New York, 1954.

W. K. HAYMAN

[4] *Multivalent Functions*, Cambridge Univ. Press, 1958.

T. H. MACGREGOR

[5] The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14 (1965) 514–520.

R. J. LIBERA

[6] Some classes of regular functions, Proc. Amer. Math. Soc. 16 (1965) 755–758.

A. E. LIVINGSTON

[7] On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966) 352–357.

(Oblatum 20–VII–1970)

Department of Mathematics
Washburn University of Topeka
Topeka, Kansas 66621
U.S.A.