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ON THE MAPS OF ONE FIBRE SPACE INTO ANOTHER

by

I. M. James

1. Introduction

The purpose of this note is to study, in special cases, the Puppe exact sequence of ex-homotopy theory (for details [1], see [4]). We begin by recalling the basic notions of the category of ex-spaces and ex-maps, with respect to a fixed base space B. By an *ex-space* we mean a space X together with a pair of maps

$$B \xrightarrow{\sigma} X \xrightarrow{\rho} B$$

such that $\rho\sigma = 1$. We refer to ρ as the *projection*, to σ as the *section*, and to (ρ, σ) as the *ex-structure*. Let X_i $(i = 0, 1)$ be an ex-space with ex-structure (ρ_i, σ_i). We describe a map $f : X_0 \to X_1$ as an *ex-map* if

$$f\sigma_0 = \sigma_1, \rho_1 f = \rho_0, \tag{1.1}$$

as shown in the following diagram.

$$\begin{tikzcd} & X_0 \arrow[dd, "f"] \arrow[dr, "\rho_0"] & \\ B \arrow[ur, "\sigma_0"] \arrow[dr, "\sigma_1"'] & & B \\ & X_1 \arrow[ur, "\rho_1"'] & \end{tikzcd}$$

In particular, we refer to $\sigma_1 \rho_0$ as the *trivial* ex-map. We describe a homotopy $f_t : X_0 \to X_1$ as an *ex-homotopy* if f_t is an ex-map at every stage. The set of ex-homotopy classes of ex-maps is denoted by $\pi(X_0, X_1)$. Further notions, such as *ex-homeomorphism* and *ex-homotopy equivalence*, are defined in the obvious way.

Let B be a pointed space, with basepoint $e \in B$. A functor Φ can be defined, as follows, from the category of ex-spaces to the category of pointed spaces. If X is an ex-space with ex-structure (ρ, σ), then ΦX is the space $\rho^{-1}e$ with σe as basepoint. If $f : X_0 \to X_1$ is an ex-map, where X_0, X_1 are ex-spaces, then $\Phi f : \Phi X_0 \to \Phi X_1$ is the map determined by

[1] Theories of this type have been developed independently by J. C. Becker and J. F. McClendon, amongst others.

restriction of f. We refer to Φ as the *fibre functor*. Note that Φ determines a function

$$\varphi : \pi(X_0, X_1) \to \pi(\Phi X_0, \Phi X_1),$$

where the codomain means the set of pointed homotopy classes of pointed maps.

In some cases this function φ is both surjective and injective. For example, let A be a pointed space. Regard the wedge-sum $A \vee B$ as an ex-space with section the inclusion and projection constant on A. Then $\Phi(A \vee B) = A$ and we have at once

PROPOSITION (1.2). *Let X be an ex-space with fibre $\Phi X = Y$. Then the function*

$$\varphi : \pi(A \vee B, X) \to \pi(A, Y)$$

is bijective.

By a *fibre ex-space* we mean an ex-space with a fibration as projection. When X_0 and X_1 are fibre ex-spaces there is a useful necessary condition for an element of $\pi(\Phi X_0, \Phi X_1)$ to belong to the image of φ. This condition involves the *brace product*, a pairing of homotopy groups derived from the Whitehead product as follows. Let X be a fibre ex-space with ex-structure (p, s) and fibre Y. Consider the short exact sequence

$$0 \to \pi_*(Y) \underset{i_*}{\to} \pi_*(X) \underset{p_*}{\to} \pi_*(B) \to 0,$$

where $i : Y \subset X$. Given elements $\beta \in \pi_*(B)$, $\eta \in \pi_*(Y)$ we form the Whitehead product $[s_*\beta, i_*\eta]$. This element of $\pi_*(X)$ lies in the kernel of p_*, since $p_* i_* = 0$, and so by exactness there exists a (unique) element $\{\beta, \eta\}$, say, of $\pi_*(Y)$ such that

$$(1.3) \qquad i_*\{\beta, \eta\} = [s_*\beta, i_*\eta].$$

This operation $\{\ ,\ \}$, which we refer to as the brace product, is studied in [5] and [9], where various examples are given. From (1.1), (1.3) and the naturality of the Whitehead product we obtain

PROPOSITION (1.4). *Let X_i $(i = 0, 1)$ be a fibre ex-space with fibre Y_i. If $\alpha \in \pi(Y_0, Y_1)$ belongs to the image of φ then*

$$\alpha \circ \{\beta, \eta\} = \{\beta, \alpha \circ \eta\},$$

for all $\beta \in \pi_*(B)$, $\eta \in \pi_*(Y_0)$.

Here the brace product on the left refers to X_0 while that on the right refers to X_1. In certain cases, as we shall see, the condition is sufficient as well as necessary.

It appears that sphere-bundles play a special role in ex-homotopy theory just as spheres do in ordinary homotopy theory. Let O_q $(q =$

$1, 2, \cdots$) denote the group of orthogonal transformations of euclidean q-space. For $m \geqq q$ we regard the $(m-1)$-sphere S^{m-1} as an O_q-space, in the usual way. Given a principal O_q-bundle over B let E_m denote the associated $(m-1)$-sphere bundle. When $m > q$ we regard E_m as an ex-space by choosing a cross-section of the bundle. When $m > q+1$ we give $\pi(E_m, X)$ a natural group-structure, as described in § 2 below, so that

$$\varphi : \pi(E_m, X) \to \pi(S^{m-1}, Y) = \pi_{m-1}(Y)$$

constitutes a homomorphism. We do not give a group-structure to $\pi(E_{q+1}, X)$.

Now consider the case when B is a sphere, say $B = S^n$ $(n > 1)$. Let E_m $(m = q+1, q+2, \cdots)$ be associated with an O_q-bundle over S^n, as above, and let X be a fibre ex-space over S^n with fibre Y. Let $\iota_r \in \pi_r(S^r)$ $(r = 1, 2, \cdots)$ denote the homotopy class of the identity map and let

$$\psi : \pi_r(Y) \to \pi_{r+n-1}(Y) \qquad (r \geqq q)$$

denote the operator given by

$$\psi(\alpha) = \alpha \circ \{\iota_n, \iota_r\} - \{\iota_n, \alpha \circ \iota_r\} \qquad (\alpha \in \pi_r(Y)). \tag{1.5}$$

Here the brace products are to be interpreted as in (1.4). It follows from (1.7) below that ψ is a homomorphism for $r > q$ but this is not true, in general, for $r = q$. Our aim is to set up an exact sequence containing ψ, as in (1.5), the fibre function φ, and a third operator

$$\theta : \pi_{m+n}(Y) \to \pi(E_{m+1}, X)$$

which can be defined as follows. Recall (see § 3 of [7]) that

$$E_{m+1} = (S^m \vee S^n) \cup e^{m+n}$$

as a cell-complex, where S^m is the fibre and S^n is embedded by the cross-section. If $f : E_{m+1} \to X$ is an ex-map such that Φf is constant then the separation element $d(f, c) \in \pi_{m+n}(X)$ is defined, with respect to this cell-structure, where c denotes the trivial ex-map. Since $pf = pc$ we have $p_* d(f, c) = 0$ and so $d(f, c) = i_* \beta$, by exactness, where $\beta \in \pi_{m+n}(Y)$. Conversely, given β, there exists an ex-map f, as above, such that $d(f, c) = i_* \beta$. We define $\theta(\beta)$ to be the ex-homotopy class of f in $\pi(E_{m+1}, X)$. It is not difficult to check that θ constitutes a homomorphism for $m > q$. Having made the necessary definitions we are now ready to state our main result

THEOREM (1.6). *The sequence*

$$\begin{aligned} \cdots \to \pi(E_{m+1}, X) \xrightarrow{\varphi} \pi_m(Y) \xrightarrow{\psi} \pi_{m+n-1}(Y) \xrightarrow{\theta} \pi(E_m, X) \\ \to \cdots \to \pi_q(Y) \xrightarrow{\psi} \pi_{q+n-1}(Y), \end{aligned}$$

is exact.

It is possible to prove (1.6) by using the methods of Barcus and Barratt [2]. However the proof we give shows, in my opinion, the advantages of exploiting the elementary properties of ex-homotopy theory.

Before we embark on the proof it is convenient to make a few further observations. Suppose that the original q-sphere bundle has classifying element $\beta \in \pi_{n-1}(O_q)$. By (3.7) of [7] the brace product in the case of E_{m+1} is given by

$$\{\iota_n, \iota_m\} = S_*^{m-q} J\beta, \tag{1.7}$$

where S_* denotes the suspension functor and $J\beta \in \pi_{n+q-1}(S^q)$ is defined by the Hopf construction in the usual way. Suppose that $Y = S^r$ $(r \geqq 1)$, regarded as a pointed O_r-space, and that X is the r-sphere bundle with cross-section associated with a principal O_r-bundle. Then $\{\iota_n, \iota_r\} = J\gamma$, similarly, where $\gamma \in \pi_{n-1}(O_r)$ is the classifying element, and hence $\{\iota_n, \alpha\} = J\gamma \circ S_*^{n-1}\alpha$, where $\alpha \in \pi_m(S^r)$. Thus

$$\psi(\alpha) = \alpha \circ S_*^{m-q} J\beta - J\gamma \circ S_*^{n-1}\alpha. \tag{1.8}$$

Where the relevant information on the homotopy groups is available we can calculate the kernel and cokernel of ψ, for a range of values of m, and hence calculate $\pi(E_m, X)$ to within a group extension.

For example, take $n = 2$, $q = 2$. Take E_m to be the $(m-1)$-sphere bundle associated with the Hopf bundle over S^2. Using standard results on the homotopy groups of spheres we find that $\pi(E_8, E_6) \approx Z_2$, in this case. If instead we take E_m to be the trivial $(m-1)$-sphere bundle we find that $\pi(E_8, E_6) \approx Z_2 \oplus Z_2$.

2. The Puppe sequence

Let (K, B) be a CW-pair, such that B is a retract of K, and let $\rho : K \to B$ be a cellular retraction. We regard K as an ex-space with the retraction as projection and the inclusion as section. Let ΣK denote the complex obtained from the union of $K \times I$ and B by identifying $(x, t) \in K \times I$ with $\rho x \in B$ if either $x \in B$ or $t = 0, 1$. A retraction of ΣK on B is given by $(x, t) \mapsto \rho x$. We give ΣK cell-structure, in the obvious way, so that B is a subcomplex and the retraction is cellular. We refer to ΣK as the *suspension* of K, in the ex-category. The r-fold suspension $(r = 1, 2, \cdots)$ is denoted by $\Sigma^r K$. Suppose, for simplicity, that K is locally finite.

Let $\varphi_X K$, for any ex-space X, denote the function-space of ex-maps $K \to X$, with the trivial ex-map as basepoint. By taking adjoints, in the usual way, we identify the homotopy group $\pi_r(\varphi_X K)$ with $\pi(\Sigma^r K, X)$. Thus $\pi(\Sigma^r K, X)$ $(r = 1, 2, \cdots)$ receives a natural group-structure. This group is abelian for $r \geqq 2$ but not, in general, for $r = 1$.

Now suppose that we have a locally finite complex K' containing K as a subcomplex and suppose that ρ can be extended to a cellular retraction $\rho' : K' \to B$. Let K'' denote the complex obtained from K' by identifying points of K with their images under ρ. Let $\rho'' : K'' \to B$ denote the retraction induced by ρ'. We regard K' and K'' as ex-spaces, with the retractions as projections and the inclusions as sections. Then

$$K \xrightarrow{i} K' \xrightarrow{j} K''$$

are ex-maps, where i is the inclusion and j is the identification map. Consider the maps

$$\varphi_X K'' \xrightarrow{j^*} \varphi_X K' \xrightarrow{i^*} \varphi_X K$$

given by functional composition. By a straightforward application of the covering homotopy property we obtain

THEOREM (2.1). *If X is a fibre ex-space then $i^* : \varphi_X K' \to \varphi_X K$ is a fibration.*

Notice that the fibre, over the trivial ex-map, can be identified with $\varphi_X K''$ by means of j^*. Hence the homotopy exact sequence of the fibration can be written in the form

$$\cdots \to \pi(\Sigma^r K'', X) \xrightarrow{j^*} \pi(\Sigma^r K', X) \xrightarrow{i^*} \pi(\Sigma^r K, X) \to \cdots$$

This is an example of the generalization to ex-homotopy theory of the notion of Puppe sequence. An alternative approach (see § 7 of [4]) is to construct a sequence of ex-maps

$$K \to K' \to K'' \to \Sigma K \to \Sigma K' \to \Sigma K'' \to \cdots$$

and apply the functor $\pi(\ , X)$.

The Puppe sequence gives useful information if one of the three domain ex-spaces is ex-contractible. For example we have

COROLLARY (2.2). *Suppose that K'' is ex-contractible. If X is a fibre ex-space then*

$$i^* : \pi(\Sigma^r K', X) \to \pi(\Sigma^r K, X)$$

is bijective, for $r \geqq 1$, and

$$i^* : \pi(K', X) \to \pi(K, X)$$

is monic.

Here the term monic is used to mean that the kernel of i^* is trivial; it is not true that i^* is injective, in general.

In § 1 we are given a principal O_q-bundle over B, and consider the associated $(m-1)$-sphere bundle E_m ($m = q, q+1, \cdots$). We regard E_{m+1} as the fibre suspension of E_m in the usual way (see § 7 of [7]). Let $m > q$.

Then E_m admits a cross-section and so can be regarded as an ex-space. The suspension ΣE_m is defined, as above, and the natural ex-map

$$r : E_{m+1} \to \Sigma E_m$$

is an ex-homotopy equivalence, as shown in § 6 of [4]. We identify $\pi(E_{m+1}, X)$ with $\pi(\Sigma E_m, X)$ under the bijection induced by r, where X is any ex-space. Thus $\pi(E_{m+1}, X)$ receives a group-structure, for $m > q$.

3. The operators in the sequence

Let D^r $(r = 1, 2, \cdots)$ denote the r-ball bounded by S^{r-1}. Choose a map $b_r : D^r \to S^r$ which is constant on S^{r-1} and non-singular [2] on the interior of D^r. Given q, we regard D^m and S^{m-1} as O_q-spaces for $m \geqq q$, in the usual way, and choose b_m to be an O_q-map.

We recall that an O_q-bundle over S^n $(n \geqq 2)$ corresponds to a map $T : S^{n-1} \to O_q$, in the standard classification. Let $m \geqq q$. Write $g(x, y) = T(y) \cdot x$ $(x \in S^{m-1},\ y \in S^{n-1})$ so that $g : S^{m-1} \times S^{n-1} \to S^{m-1}$. Let E_m denote the space obtained from the union of $S^{m-1} \times D^n$ and S^{m-1} by identifying points of $S^{m-1} \times S^{n-1}$ with their images under g, so that the identification map

$$h : (S^{m-1} \times D^n, S^{m-1} \times S^{n-1}) \to (E_m, S^{m-1})$$

is a relative homeomorphism. Write $\pi h(x, y) = b_n y$, where $x \in S^{m-1}$, $y \in D^n$, so that $\pi : E_m \to S^n$. We recall (see § 3 of [7]) that E_m, with this projection, can be identified with the $(m-1)$-sphere bundle corresponding to T, i.e. the $(m-1)$-sphere bundle associated with the given O_q-bundle. If we replace S^{m-1} by D^m, in this construction, we obtain the associated m-ball bundle E'_m with projection π', say. We regard E_m as a subspace of E'_m, in the obvious way, so that $\pi = \pi' u$, where $u : E_m \subset E'_m$.

Let F_m, F'_m denote the spaces obtained from E_m, E'_m by collapsing S^{m-1} to a point. Then $p'u = vp$, as shown below, where v is the inclusion and p, p' are the collapsing maps.

$$\begin{array}{ccc} E_m & \xrightarrow{u} & E'_m \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ F_m & \xrightarrow[v]{} & F'_m \end{array}$$

Write $sy = h(e, y)$, where $y \in D^n$, so that $s : D^n \to E_m$. We regard F_m as an ex-space with projection $\rho = \pi p^{-1}$ and section $\sigma = sb_n^{-1}$. Similarly

[2] Here, and elsewhere, it is unnecessary to specify orientation conventions since the validity of (1.6) is independent of the signs of the operators.

we regard F'_m as an ex-space with ex-structure (ρ', σ') so that v is an ex-map. Using the method described in § 3 of [7] we can endow these spaces with cell-structure so as to satisfy the preliminary conditions of § 2. Consider, therefore, the Puppe sequence associated with the pair (F'_m, F_m). Recall that F''_m, in the notation of § 2, is obtained from F'_m by identifying points of F_m with their images under the projection ρ. Now p determines a homeomorphism $p'' : E''_m \to F''_m$, where E''_m denotes the space obtained from E'_m by identifying points of E_m with their images under the projection π. We endow E''_m with ex-structure so as to make p'' an ex-homeomorphism. The 0_m-map $b_m : D^m \to S^m$ is constant on S^{m-1} and so determines an ex-homeomorphism between E''_m and E_{m+1}. By composing this with the inverse of p'' we obtain an ex-homeomorphism $\beta : F''_m \to E_{m+1}$. Now ΦF_m is a point-space, and $\Phi F'_m = S^m = \Phi E_{m+1}$, where Φ denotes the fibre functor. We use $\Phi\beta$ to identify $\Phi F''_m$ with S^m. Let ω denote the composition

$$F'_m \xrightarrow{j} F''_m \xrightarrow{\beta} E_{m+1},$$

where j is the identification ex-map. Then $\Phi\omega = \Phi$ and hence

(3.1) $$\varphi\omega^* = \varphi,$$

as shown in the following diagram where X is a fibre ex-space with fibre Y.

$$\begin{tikzcd} \pi(E_{m+1}, X) \arrow[rr, "\omega^*"] \arrow[dr, "\varphi"'] & & \pi(F'_m, X) \arrow[dl, "\varphi"] \\ & \pi(S^m, Y) & \end{tikzcd}$$

The fibre of F'_m has been identified with S^m. We embed the base space S^n in F'_m by means of the section. Let $S^m \vee S^n$ denote the union of these two spheres, with the standard ex-structure. If, in F'_m, we identify points of $S^m \vee S^n$ with their images under the projection, we obtain an ex-space which is ex-contractible. Thus (2.2) applies to τ^*, as shown below, where $\tau : S^m \vee S^n \subset F'_m$.

$$\begin{tikzcd} \pi(F'_m, X) \arrow[rr, "\tau^*"] \arrow[dr, "\varphi"'] & & \pi(S^m \vee S^n, X) \arrow[dl, "\varphi"] \\ & \pi(S^m, Y) & \end{tikzcd}$$

By using (1.2) φ is bijective, on the right of the above diagram, and so we obtain

LEMMA (3.2). *The fibre function*

$$\varphi : \pi(F'_m, X) \to \pi(S^m, Y)$$

is bijective for $m > q$, monic for $m = q$.

The next step is to set up a bijection between $\pi(F_m, X)$ and $\pi(S^{m+n-1}, Y)$. Certainly F_m and $S^{m+n-1} \vee S^n$, as spaces, have the same homotopy type; however in general they do not, as ex-spaces, have the same ex-homotopy type. Situations of this kind can be dealt with as follows. Consider the induced fibration $\rho^* X$ over F_m. The section of X over S^n determines a cross-section of $\rho^* X$ over S^n. The extensions over F_m of this partial cross-section correspond to the ex-maps of F_m into X. Similarly the vertical homotopies of cross-sections, rel S^n, correspond to ex-homotopies. By standard theory (see [1]) such cross-sections are classified by elements of $\pi_{m+n-1}(Y)$. The corresponding result, in our situation, is that

$$\xi : \pi(F_m, X) \to \pi_{m+n-1}(Y)$$

is bijective, where ξ is given as follows. Let $f : F_m \to X$ be an ex-map of class $\gamma \in \pi(F_m, X)$, and let $c : F_m \to X$ denote the trivial ex-map. Let $i : Y \subset X$. Then

$$i_* \xi(\gamma) = d(f, c), \tag{3.3}$$

where the separation element is defined with respect to the pair (F_m, S^n). Let $l : S^{m+n-1} \to F_m$ be a map [3] of degree 1 such that $\rho l : S^{m+n-1} \to S^n$ is nul-homotopic. By consideration of the induced fibration $l^* \rho^* X$ over S^{m+n-1} we obtain the relation

$$i_* \xi(\gamma) = \gamma_*(\lambda), \tag{3.4}$$

where $\lambda \in \pi_{m+n-1}(F_m)$ denotes the homotopy class of l.

We identify S^{m+n-1} with the boundary of $D^m \times D^n$ in the usual way. Let

$$h' : (D^m \times D^n; D^m \times S^{n-1}, S^{m-1} \times D^n) \to (E'_m; D^m, E_m)$$

denote the identification map used in the construction of the m-ball bundle E'_m. By restricting $p'h'$ to the boundary of $D^m \times D^n$ we obtain a map

$$H : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to F'_m.$$

Notice that H is constant on $S^{m-1} \times S^{n-1}$. Also H maps $D^m \times S^{n-1}$ into S^m and $S^{m-1} \times D^n$ into F_m. Let $\kappa \in \pi_{m+n-1}(S^m)$ denote the class of the map

[3] We use the phrase which follows to mean that l determines a homeomorphism when $S^n \subset F_m$ is collapsed to a point.

$$k : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to S^m$$

which agrees with H on $D^m \times S^{n-1}$ and is constant on $S^{m-1} \times D^n$. Also let $\lambda \in \pi_{m+n-1}(F_m)$ denote the class of the map

$$l : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to F_m$$

which agrees with H on $S^{m-1} \times D^n$ and is constant on $D^m \times S^{n-1}$. By (3.9) of [10] the class of H in $\pi_{m+n-1}(F'_m)$ is equal (with suitable conventions) to

$$j_*(\kappa)+v_*(\lambda)-\tau_*[\iota_n, \iota_m],$$

where τ, j, v are the inclusion maps and where $\iota_m, \iota_n \in \pi_*(S^m \vee S^n)$ are the classes of the inclusion maps of S^m, S^n respectively. But H is nul-homotopic, since h' extends over $D^m \times D^n$, and so we conclude that

$$v_*(\lambda) = \tau_*[\iota_n, \iota_m]-j_*(\kappa). \tag{3.5}$$

It is easy to check that l, as above, is a map of degree 1, and that

$$\rho l : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to S^n$$

is nul-homotopic. Moreover it follows from the basic theory of the Hopf construction (see [10]) that

$$\pm_\kappa = S_*^{m-q} J\alpha = \{\iota_n, \iota_m\},$$

where $\alpha \in \pi_{n-1}(O_q)$ denotes the homotopy class of T and the brace product is defined with respect to E_m. We arrange our orientation conventions so that $\kappa = \{\iota_n, \iota_m\}$.

The next step in the proof of our main theorem is to establish that

$$\psi\varphi = \xi v^*, \tag{3.6}$$

as shown in the following diagram, where ψ is defined by means of the brace product, as in (1.5).

$$\begin{tikzcd}
\pi(F'_m, X) \arrow[r, "v^*"] \arrow[d, "\varphi"'] & \pi(F_m, X) \arrow[d, "\xi"] \\
\pi(S^m, Y) \arrow[r, "\psi"'] & \pi(S^{m+n-1}, Y)
\end{tikzcd}$$

Let $\gamma' \in \pi(F'_m, X)$. Write $\varphi\gamma' = \eta \in \pi_m(Y)$, $v^*(\gamma') = \gamma \in \pi(F_m, X)$. By naturality $\gamma'_* \tau_*[\iota_n, \iota_m] = [s_* \iota_n, i_* \eta] = i_*\{\iota_n, \eta\}$, by (1.3), where $s : S^n \to X$ denotes the section. Also $\gamma'_* j_*(\kappa) = i_* \eta_*(\kappa) = i_* \eta_* \{\iota_n, \iota_m\}$, as we have just seen. Now compose both sides of (3.5) with γ'_* and we obtain the relation

$$i_* \psi\phi(\gamma') = \gamma'_* v_*(\lambda) = \gamma_*(\lambda) = i_* \xi(\gamma),$$

by (3.4). Since i_* is injective this proves (3.6).

It follows at once from (3.3) by naturality that

$$\theta\xi = p^*, \tag{3.7}$$

as shown in the following diagram, where θ is defined by means of the separation element as in § 1.

$$\begin{tikzcd} \pi(F_m, X) \arrow[rr, "p^*"] \arrow[dr, "\xi"'] & & \pi(E_m, X) \\ & \pi_{m+n-1}(Y) \arrow[ur, "\theta"'] & \end{tikzcd}$$

Now we are ready to complete the proof of (1.6). As we have seen, the Puppe sequence associated with the pair (F'_q, F_q) can be written in the form

$$\cdots \to \pi(E_{m+1}, X) \xrightarrow{\omega^*} \pi(F'_m, X) \xrightarrow{v^*} \pi(F_m, X) \xrightarrow{p^*} \pi(E_m, X) \to \cdots.$$

We use φ and ξ to replace the terms in the middle by homotopy groups of Y. We recall that φ is monic for $m = q$, bijective for $m > q$; also ξ is bijective for $m \geqq q$. Hence and from (3.1), (3.6) and (3.7) we obtain the exact sequence of § 1.

4. Proper cross-sections

The above theory can be used to give an alternative proof of the main result of [3], which concerns the following problem. Let B be a pointed space. Let X_i $(i = 0, 1)$ be an ex-space of B with fibre $\Phi X_i = Y_i$, say. Suppose we have an ex-map $p : X_1 \to X_0$ which, as an ordinary map, constitutes a fibration with fibre Z, say, over the basepoint of $Y_0 \subset X_0$. Write $\Phi p = q$. Then $q : Y_1 \to Y_0$ can be regarded as a fibration with fibre Z so that we have the situation indicated in the following diagram, where u, v and u_i are the inclusions.

$$\begin{tikzcd} Z \arrow[r, "1"] \arrow[d, "v"'] & Z \arrow[d, "u"'] & \\ Y_1 \arrow[r, "u_1"] \arrow[d, "q"'] & X_1 \arrow[r, "\rho_1"] \arrow[d, "p"'] & B \arrow[d, "1"'] \\ Y_0 \arrow[r, "u_0"] & X_0 \arrow[r, "\rho_0"] & B \end{tikzcd}$$

Under what conditions does the fibration p admit a cross-section? As in [3] we describe such a cross-section $f : X_0 \to X_1$ as *proper* if

$$(4.1) \qquad f\sigma_0 = \sigma_1.$$

Note that proper cross-sections are ex-maps, since p is an ex-map. If f is proper then $g : Y_0 \to Y_1$ constitutes a cross-section of q, where $g = \Phi f$. As in [3] we describe g, or its vertical homotopy class, as the *type* of f. We approach our problem by asking, for each cross-section g of q, whether p admits a proper cross-section of type g.

Now let B be a (pointed) CW-complex. Suppose that (X_i, Y_i) is a CW-pair, and that the section σ_i embeds B as a subcomplex. Then $(X_i, B \vee Y_i)$ forms a CW-pair, and the answer to our question is independent of the choice of g in its class, by the homotopy lifting property. Consider the homomorphism

$$\theta : \pi_r(B) \oplus \pi_r(Y_0) \to \pi_r(X_0)$$

given by σ_{0*} on the first summand and τ_{0*} on the second. Certainly θ is an isomorphism for $r \geqq 2$, since σ_0 is a right inverse of ρ_0. Suppose that θ is also an isomorphism for $r = 1$. This is the case, for example, if $\pi_1(X_0)$ is abelian, or if $\pi_1(B)$ is trivial. Under this hypothesis we prove

LEMMA (4.2). *Let $f: X_0 \to X_1$ be an ex-map such that $\Phi f : Y_0 \to Y_1$ is a cross-section of q. Then there exists a proper cross-section f' of p such that $\Phi f' = \Phi f$.*

Since $pf : X_0 \to X_0$ maps $B \vee Y_0$ identically we have $(pf)_* \theta = \theta$, where θ is as above and

$$(pf)_* : \pi_*(X_0) \to \pi_*(X_0).$$

Hence $(pf)_* = 1$, since θ is an isomorphism, and so pf is a homotopy equivalence, by Theorem 1 of [11]. Hence pf determines a homotopy equivalence of the pair $(X_0, B \vee Y_0)$ with itself, by (3.1) of [6], since pf maps $B \vee Y_0$ identically. Therefore there exists an inverse homotopy equivalence $k : X_0 \to X_0$, say, which also maps $B \vee Y_0$ identically. Write $f'' = fk : X_0 \to X_1$. Then f'' agrees with f on $B \vee Y_0$. Let $l_t : X_0 \to X_0$ be a homotopy of pf'' into the identity such that $l_t(B \vee Y_0) \subset B \vee Y_0$. By composing with $h = f|B \vee Y_0$ we lift $l_t|B \vee Y_0$ to a homotopy $l'_t : B \vee Y_0 \to X_1$ such that $l'_0 = h = l'_1$. We extend l'_t over X_0 so as to cover l_t, using the homotopy lifting property, and thus deform f'' into f', say. Then f' is a proper cross-section of p and $\Phi f' = \Phi f'' = \Phi f$, as required.

To apply (4.2) we return to the situation considered in § 1, where E_{m+1} is an m-sphere bundle over S^n. The fundamental group is abelian, since $n > 1$. We apply (4.2) with $(X_0, Y_0) = (E_{m+1}, S^m)$ and write $(X_1, Y_1) = (X, Y)$, so that $p : X \to E_{m+1}$ and $q : Y \to S^m$. Hence and from the exactness of our sequence we obtain

COROLLARY (4.3). *Let $\gamma \in \pi_m(Y)$ be the class of a cross-section of q. Then p admits a proper cross-section of type γ if and only if*

$$\gamma \circ \{\iota_n, \iota_m\} = \{\iota_n, \gamma\}.$$

In particular, suppose that Y is a pointed O_m-space, and that q is a pointed 0_m-map. Suppose that X and E_{m+1} share the same principal 0_m-bundle and that p is the map associated with q. To obtain the main result of [3] from (4.3) it is only necessary to convert the brace products into the kind of product used in [3], as shown in [5].

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