B. NATH

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TOPOLOGIES ON GENERALIZED SEMI-INNER PRODUCT SPACES

by

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Abstract. In the present paper we introduce a straightforward algebraic generalization of semi-inner product (in short GSIP) spaces. We enumerate and derive some fundamental properties of strong topologies in GSIP spaces.

1. Introduction

In his recent paper entitled 'Topologies on generalized inner product spaces' Prugovečki [2] has defined generalized inner product spaces and has introduced different topologies in such spaces in the same fashion as in the case of inner product spaces. In the present paper we define generalized semi-inner product spaces and introduce strong topologies in these spaces an analogous manner and obtain certain theorems. In as much as, an inner product space is also a semi-inner product space and similarly a generalized inner product space is also generalized semi-inner product space. The theorems in this paper contain the corresponding results given by Prugovečki as particular cases.

2. Definitions

DEFINITION 1. [1, p. 31]. Let X be a complex (real) vector space. We shall say that a complex (real) semi-inner product is defined on X, if to any $x, y \in X$ there corresponds a complex (real) number $[x, y]$ and the following properties hold:

(i) $[x+y, z] = [x, z]+[y, z]$
$[\lambda x, y] = \lambda[x, y]$ for $x, y, z \in X$; λ complex (real),

(ii) $[x, x] > 0$ for $x \neq 0$,

(iii) $|[x, y]|^2 \leqq [x, x][y, y]$.

We then call X a complex (real) semi-inner product space.

We introduce the following definition.

DEFINITION 2. A linear space $\mathscr{L}$ is a generalized semi-inner product space if and only if

(i) There is a subset N of $\mathscr{L}$ which is a semi-inner product space;

(ii) There is a non-empty set $\mathscr{A}$ of linear operators on $\mathscr{L}$ which has the following properties:
(a) Each element of $\mathscr{A}$ maps $\mathscr{L}$ into N, i.e., $\mathscr{A}\mathscr{L} \subset N$
(b) If $Ax = 0$ for all $A \in \mathscr{A}$, then $x = 0$.

We denote a generalized semi-inner product space by triple $(\mathscr{L}, \mathscr{A}, N)$. Clearly every semi-inner product space is also a generalized semi-inner product space in trivial sense, i.e., $N = \mathscr{L}$ and $\mathscr{A} = \{1\}$, where 1 denotes the identity operator on $\mathscr{L}$.

Example of GSIP space which is neither GIP space nor semi-inner product space:

Suppose $\mathscr{L}$ is the family of all measurable function on real line. Let N be all equivalence classes of functions that are measurable and p-th power summable on real line, where $2 \leqq p < \infty$. We adopt the semi-inner product in N to be

$$[y, x] = \frac{1}{||x||_p^{p-2}} \int_{-\infty}^{\infty} y(t)|x(t)|^{p-1} \operatorname{sgn} x(t) dt,$$

where

$$||x||_p^{p-2} = \left(\int_{-\infty}^{\infty} |x(t)|^p dt\right)^{(p-2)/p}$$

and sgn is the signum function.

Let $\mathscr{A}$ be the family of operators $E^{p-1}(I)$ such that, for all $x, y \in \mathscr{L}$ and for any scalars α and β,

$$(E^{p-1}(1)(\alpha x+\beta y))(t) = X_I(t)(|x(t)|^{p-2}+|y(t)|^{p-2})(\alpha x(t)+\beta y(t)),$$

where I is the finite-non-degenerate interval, $X_s(t)$ denotes the characteristic function of the set s, $2 \leqq p < \infty$ and $E^1 = E$.

Verification of example:

$\mathscr{L}$ is a linear space by

$$(x+y)(t) = x(t)+y(t), \quad (\alpha x)(t) = \alpha x(t).$$

It is obvious that

$$[y, x] = \frac{1}{||x||_p^{p-2}} \int_{-\infty}^{\infty} y(t)|x(t)|^{p-1} \operatorname{sgn} x(t) dt$$

satisfies semi-inner product properties (i), (ii). We proceed to establish (iii).

Using Hölder's inequality, we have

$$\left|\int_{-\infty}^{\infty} y(t)|x(t)|^{p-1}\operatorname{sgn} x(t)dt\right| \leqq \left(\int_{-\infty}^{\infty} |x(t)|^p dt\right)^{1/q} \left(\int_{-\infty}^{\infty} |y(t)|^p dt\right)^{1/p},$$

where $1/p+1/q = 1$ and the required inequality follows.

It is clear that $\mathscr{A}$ is a set of linear operators. From the definition of $E^{p-1}(I)$,

$$(E^{p-1}(I)x)(t) = \chi_I(t)|x(t)|^{p-2}x(t)$$

and

$$(E(I)y)(t) = \chi_I(t)y(t).$$

Therefore it is obvious that $\mathscr{A}\mathscr{L} \subset N$. It is also clear that

$$Ax = 0, \forall\, A \in \mathscr{A} \Rightarrow x = 0.$$

It can be easily verified that this example is neither *GIP* space nor semi-inner product space.

We define strong topology in *GSIP* space as follows:

DEFINITION 3. Corresponding to each $x \in \mathscr{L}$

$V(x; A_1, \cdots, A_n; \delta) =$
$\{y : [A_1(y-x), A_1(y-x)]^{\frac{1}{2}} < \delta, \cdots, [A_n(y-x), A_n(y-x)]^{\frac{1}{2}} < \delta, y \in \mathscr{L}\}$

for all $\delta > 0, A_1, \cdots, A_n \in \mathscr{A}$ and $n = 1, 2, \cdots$ forms a neighbourhood basis. The topology defined by this neighbourhood basis will be called the strong topology in the generalized semi-inner product space $(\mathscr{L}, \mathscr{A}, N)$.

We define ultra-strong topology in *GSIP* space as follows:

For each $x \in \mathscr{L}$ family of all sets

$$V(x; A_1, A_2, \cdots; \delta) = \bigcap_{k=1}^{\infty} V(x; A_k; \delta),\ A_k \in \mathscr{A},\ \delta > 0,$$

constitutes a neighbourhood basis. The topology defined by this neighbourhood basis will be called the ultra-strong topology in the generalized semi-inner product space $(\mathscr{L}, \mathscr{A}, N)$.

3.

We prove the following theorems.

3.1. *Each* $V(0; A_1, \cdots, A_n; \delta)$ *is balanced and convex.*

3.2. *If in a GSIP space* $(\mathscr{L}, \mathscr{A}, N)$ *a topology is introduced in which* *the sets $V(x; A; \delta)$ are neighbourhood of x for all $\delta > 0$, $A \in \mathscr{A}$, then the resulting topological space is Hausdorff.*

3.3. *In the strong (ultra-strong) topology on the GSIP space $(\mathscr{L}, \mathscr{A}, N)$, the space $\mathscr{L}$ is locally convex Hausdorff linear space.*

3.4. *The GSIP spaces with strong (ultra-strong) topology is metrizable if there is countable subset β of $\mathscr{A}$ which has the property that for any $A \in \mathscr{A}$ there is a B in the linear manifold L_B generated by β such that*

$$[Bx, Bx]^{\frac{1}{2}} \geqq [Ax, Ax]^{\frac{1}{2}} \text{ for all } x \in \mathscr{L}. \tag{3.4.1}$$

Proof of theorem 3.1. Let $x \in V(0; A_1, \cdots, A_n; \delta)$ then $[A_k x, A_k x]^{\frac{1}{2}} < \delta$ for $k = 1, 2, \cdots, n$.

Now consider

$$\begin{aligned}
[A_k(\lambda x), A_k(\lambda x)] &= \lambda [A_k x, A_k(\lambda x)] \\
&= |\lambda| \, |[A_k x, A_k(\lambda x)]| \\
&\leqq |\lambda| \, [A_k x, A_k x]^{\frac{1}{2}} [A_k(\lambda x), A_k(\lambda x)]^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$[A_k(\lambda x), A_k(\lambda x)]^{\frac{1}{2}} \leqq |\lambda| \, [A_k x, A_k x]^{\frac{1}{2}}.$$

Hence, for $|\lambda| \leqq 1$,

$$[A_k(\lambda x), A_k(\lambda x)]^{\frac{1}{2}} < \delta$$

and

$$\lambda x \in V(0; A_1, \cdots, A_n; \delta).$$

This proves that $V(0; A_1, \cdots, A_n; \delta)$ is balanced.

Now we shall prove that $V(0; A_1, \cdots, A_n; \delta)$ is convex.

Let $x_1, x_2 \in V(0; A_1, \cdots, A_n; \delta)$ and $0 \leqq \lambda \leqq 1$.

Consider

$$[A_k(\lambda x_1 + (1-\lambda)x_2), A_k(\lambda x_1 + (1-\lambda)x_2)].$$

We have

$$\begin{aligned}
&[A_k(\lambda x_1 + (1-\lambda)x_2), A_k(\lambda x_1 + (1-\lambda)x_2)] \\
&\quad = [\lambda A_k x_1 + (1-\lambda) A_k x_2, A_k(\lambda x_1 + (1-\lambda)x_2)] \\
&\quad = \lambda [A_k x_1, A_k(\lambda x_1 + (1-\lambda)x_2)] + (1-\lambda)[A_k x_2, A_k(\lambda x_1 + (1-\lambda)x_2)] \\
&\quad \leqq |\lambda| \, [A_k x_1, A_k x_1]^{\frac{1}{2}} [A_k(\lambda x_1 + (1-\lambda)x_2), A_k(\lambda x_1 + (1-\lambda)x_2]^{\frac{1}{2}} \\
&\qquad + |1-\lambda| \, [A_k x_2, A_k x_2]^{\frac{1}{2}} [A_k(\lambda x_1 + (1-\lambda)x_2), A_k(\lambda x_1 + (1-\lambda)x_2)]^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&[A_k(\lambda x_1 + (1-\lambda)x_2), A_k(\lambda x_1 + (1-\lambda)x_2)]^{\frac{1}{2}} \\
&\qquad \leqq |\lambda| \, [A_k x_1, A_k x_1]^{\frac{1}{2}} + |1-\lambda| [A_k x_2, A_k x_2]^{\frac{1}{2}}.
\end{aligned}$$

Since

$$0 \leqq \lambda \leqq 1,$$
$$|\lambda| = \lambda, |1-\lambda| = 1-\lambda.$$

So,

$$\begin{aligned}[A_k(\lambda x_1+(1-\lambda)x_2), A_k(\lambda x_1+(1-\lambda)x_2)]^{\frac{1}{2}} \\ &\leqq \lambda[A_k x_1, A_k x_1]^{\frac{1}{2}}+(1-\lambda)[A_k x_2, A_k x_2]^{\frac{1}{2}} \\ &< \lambda\delta+(1-\lambda)\delta = \delta.\end{aligned}$$

Therefore

$$\lambda x_1+(1-\lambda)x_2 \in V(0; A_1, \cdots, A_n; \delta).$$

This proves that $V(0; A_1, \cdots, A_n; \delta)$ is convex.

PROOF OF THEOREM 3.2.

If the topological space is not a Hausdorff space, then there are at least two elements $x_1, x_2 \in \mathscr{L}$, $x_1 \neq x_2$ for which any two neighbourhoods have common points.

Thus, for any two neighbourhoods $V(x_1; A; 1/n)$ and $V(x_2; A; 1/n)$ there is at least one $y_n \in \mathscr{L}$ such that

$$y_n \in V\left(x_1; A; \frac{1}{n}\right) \cap V\left(x_2; A; \frac{1}{n}\right).$$

Therefore

$$[A(y_n-x_1), A(y_n-x_1)]^{\frac{1}{2}} < \frac{1}{n}$$

and

$$[A(y_n-x_2), A(y_n-x_2)]^{\frac{1}{2}} < \frac{1}{n}.$$

We have

$$\begin{aligned}[A(x_1-x_2), A(x_1-x_2)] &= [A(x_1-y_n+y_n-x_2), A(x_1-x_2)] \\ &= [A(x_1-y_n), A(x_1-x_2)]+[A(y_n-x_2), A(x_1-x_2)] \\ &= -[A(y_n-x_1), A(x_1-x_2)]+[A(y_n-x_2), A(x_1-x_2)] \\ &\leqq |[A(y_n-x_1), A(x_1-x_2)]|+|[A(y_n-x_2), A(x_1-x_2)]| \\ &\leqq [A(y_n-x_1), A(y_n-x_1)]^{\frac{1}{2}}[A(x_1-x_2), A(x_1-x_2)]^{\frac{1}{2}}+[A(y_n-x_2), \\ &\quad A(y_n-x_2)]^{\frac{1}{2}}[A(x_1-x_2), A(x_1-x_2)]^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}[A(x_1-x_2), A(x_1-x_2)]^{\frac{1}{2}} &\leqq [A(y_n-x_1), A(y_n-x_1)]^{\frac{1}{2}} \\ &\quad +[A(y_n-x_2), A(y_n-x_2)]^{\frac{1}{2}} < \frac{1}{n}+\frac{1}{n} = \frac{2}{n}.\end{aligned}$$

Since the above is true for any positive integer n, it follows that $A(x_1-x_2) = 0$.

By second postulate of *GSIP* space,

$$A(x_1-x_2) = 0 \Rightarrow x_1-x_2 = 0 \text{ or } x_1 = x_2,$$

contrary to our assumption.

PROOF OF THEOREM 3.3.

The topology in definition 3 is compatible with the vector operations.

To prove that the operation of the vector summation is continuous, we proceed as follows.

Consider the mapping

$$\psi : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$$
$$\langle x, y\rangle \to x+y.$$

Let $\langle x, y\rangle$ be any point of $\mathscr{L} \times \mathscr{L}$.

Let $V(x_0+y_0; A_i; \delta)$ be an arbitrary basis neighbourhood.

Let $\langle x, y\rangle \in V(x_0; A_i; \frac{1}{2}\delta) \times V(y_0; A_i; \frac{1}{2}\delta)$ be a basis neighbourhood in the product topology.

We now show that

$$(V(x_0; A_i; \tfrac{1}{2}\delta) + V(y_0; A_i; \tfrac{1}{2}\delta)) \subset V(x_0+y_0; A_i; \delta).$$

Since $x \in V(x_0; A_i; \frac{1}{2}\delta)$ and $y \in V(y_0; A_i; \frac{1}{2}\delta)$,

$$[A_i(x-x_0), A_i(x-x_0)]^{\frac{1}{2}} < \tfrac{1}{2}\delta$$

and

$$[A_i(y-y_0), A_i(y-y_0)]^{\frac{1}{2}} < \tfrac{1}{2}\delta.$$

We have

$$\begin{aligned}
&[A_i(x+y-x_0-y_0), A_i(x+y-x_0-y_0)] \\
&\quad = [A_i(x-x_0), A_i(x+y-x_0-y_0)] + [A_i(y-y_0), A_i(x+y-x_0-y_0)] \\
&\quad \leqq [A_i(x-x_0), A_i(x-x_0)]^{\frac{1}{2}}[A_i(x+y-x_0-y_0), A_i(x+y-x_0-y_0)]^{\frac{1}{2}} \\
&\quad + [A_i(y-y_0), A_i(y-y_0)]^{\frac{1}{2}}[A_i(x+y-x_0-y_0), A_i(x+y-x_0-y_0)]^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&[A_i(x+y-x_0-y_0), A_i(x+y-x_0-y_0)]^{\frac{1}{2}} \\
&\quad \leqq [A_i(x-x_0), A_i(x-x_0)]^{\frac{1}{2}} + [A_i(y-y_0), A_i(y-y_0)]^{\frac{1}{2}} \\
&\quad < \tfrac{1}{2}\delta + \tfrac{1}{2}\delta = \delta.
\end{aligned}$$

Hence $x+y \in V(x_0+y_0; A_i; \delta)$.

Thus ψ is continuous at $\langle x, y\rangle$.

Similarly it is easy to show that the operation of multiplication by scalar is continuous.

In the resulting topology, $\mathscr{L}$ is Hausdorff according to theorem 3.2 and is locally convex due to theorem 3.1.

The proof for the ultra-strong topology can be obtained in the same manner.

PROOF OF THEOREM 3.4.

We shall show that the family

$$\left\{V\left(0; B_1, \cdots, B_k; \frac{1}{n}\right); B_1, \cdots, B_k \in \beta, \quad k, n = 1, 2, \cdots\right\} \tag{3.4.2}$$

is a neighbourhood basis of the origin in the strong topology.

For every $A \in \mathscr{A}$ we can find due to (3.4.1), a $B \in L_B$ for which

$$V(0; B; \delta) \subset V(0; A; \delta).$$

As β generates L_B, we have that

$$B = \lambda_1 B_1 + \cdots + \lambda_k B_k, \qquad B_1, \cdots, B_k \in \beta$$

and consequently

$$[Bx, Bx]^{\frac{1}{2}} \leqq |\lambda_1| \, [B_1 x, B_1 x]^{\frac{1}{2}} + \cdots + |\lambda_k| \, [B_k x, B_k x]^{\frac{1}{2}}$$

for all $x \in \mathscr{L}$.

Thus if we choose an integer n such that

$$\frac{1}{n} \leqq \frac{\delta}{k|\lambda_1|}, \cdots, \frac{1}{n} \leqq \frac{\delta}{k|\lambda_k|},$$

we have that

$$V(0; B; \delta) \supset V\left(0; B_1; \frac{1}{n}\right) \cap \cdots \cap V\left(0; B_k; \frac{1}{n}\right)$$
$$= V\left(0; B_1, \cdots, B_k; \frac{1}{n}\right),$$

which shows that family (3.4.2) is a neighbourhood basis of the origin. The remaining portion of the proof is on the same lines as that given in [2, Theorem 3.3].

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