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by

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Introduction

In [7] Monna and Springer introduced an integration theory in which the scalar field was a field K with a non-archimedean valuation. In [10] a non-archimedean Fourier theory has been developed, mostly for abelian groups. This paper deals with continuous representations of locally compact groups G into non-archimedean Banach spaces.

It is an easy exercise to show that in order that G has sufficiently many of such representations G must be 0-dimensional. Like in the 'classical' case, a general treatment appears to be extremely difficult. As a first step, however, this paper tries to show that the n.a. representation theory for *compact* groups is at least as smooth as the 'classical' theory. One can prove non-archimedean analogues of the Peter-Weyl Theorem and the Tannaka Duality Theorem. A disadvantage is the lack of Hilbert spaces in the non-archimedean situation. In this paper we do introduce a sort of a non-archimedean Hilbert space (0.2), but still some work is involved to show that an invariant subspace has an *invariant* complement. (Lemma 4.2). On the other hand, the generalized Fourier transformation is an *isometry*, which phenomenon does not occur in the classical situation. (4.10). Since a 0-dimensional compact group G is projective limit of finite groups, it will not come as a surprise to the reader that our basic tools are Maschke's Theorem and the orthogonality relations of representations of finite groups. In fact, we will show that 'unitary' representations of G into n.a. Hilbert spaces over K have very intimate relationships with representations of finite groups into vector spaces over the residue class field of K (5.1 and 5.2). As to a treatment of representations for locally compact groups, the only result is in fact Theorem 3.1 which shows that there is a 1–1 correspondence between representations of G and of $L(G)$. The next step should be some sort of a Gelfand-Raikov Theorem. The classical proof involves positive functionals and the Krein-Milman Theorem and a direct translation of this proof seems to be impossible. For the 'classical' theory of representations we refer to [2].

0. Preliminaries

Let K be a complete non-archimedean valued field. The valuation is supposed to be non-trivial. A (K-) *Banach-space* is a K-vector space E, together with a real-valued function $\|\ \|$, defined on E, such that

(i) $\|\xi\| > 0$; $\|\xi\| = 0$ if and only if $\xi = 0$

(ii) $\|\lambda\xi\| = |\lambda|\ \|\xi\|$

(iii) $\|\xi+\eta\| \leqq \max(\|\xi\|, \|\eta\|)$

for all $\xi, \eta \in E$, $\lambda \in K$, and such that E is complete. By $\|E\|$ and $|K|$ we mean $\mathrm{Im}\ \|\ \|$, resp. $\mathrm{Im}\ |\ |$.
A collection $(e_\alpha)_{\alpha\in I}$ in E is called *orthonormal* if for all finite subsets $J \subset I$ we have

$$\|\sum_{\alpha\in J} \lambda_\alpha e_\alpha\| = \max_{\alpha\in J} |\lambda_\alpha| \qquad (\lambda_\alpha \in K).$$

An orthonormal set $(e_\alpha)_{\alpha\in I}$ is called an *orthonormal basis* in case its linear hull is dense in E. Then every $\xi \in E$ can be written as a convergent sum

$$\xi = \sum_{\alpha\in I} \lambda_\alpha e_\alpha \qquad (\lambda_\alpha \in K,\ \lim \lambda_\alpha = 0)$$

where

$$\|\xi\| = \max |\lambda_\alpha|$$

Two subspaces S, T are called *orthogonal* in case for every $\xi \in S$, $\eta \in T$ we have

$$\|\xi+\eta\| = \max(\|\xi\|, \|\eta\|).$$

In 4.7 we need the following

0.1 LEMMA. *Let E be a Banach space, let $(e_\alpha)_{\alpha\in I}$, $(f_\alpha)_{\alpha\in I}$ be subsets of E and E^* respectively. Suppose $\|e_\alpha\| \leqq 1$, $\|f_\alpha\| \leqq 1$ for all α and*

$$f_\alpha(e_\beta) = d_\alpha \delta_{\alpha\beta}$$

where $|d_\alpha| = 1$. Then (e_α) and (f_α) are orthonormal.

PROOF. Let $f = \Sigma\, \lambda_\alpha f_\alpha (\alpha \in K)$. Clearly $\|f\| \leqq \max |\lambda_\alpha|$.
Conversely $\|f\| \geqq \|f(e_\alpha)\|/\|e_\alpha\| \geqq \|f(e_\alpha)\| = |\lambda_\alpha|$ for all α. A similar proof works for the e_α's.

The following concept of non-archimedean Hilbert spaces, due to van der Put is different from that of Kalisch [5]. We give the statements without offering the proofs. Most of them are unpublished results of van der Put. Some proofs can be found in [8].

0.2. DEFINITION. *Let E be a Banach space with $\|E\| = |K|$. E is called a n.a. Hilbert space if one of the following equivalent conditions is satisfied.*

(i) *Every closed subspace has a normorthogonal complement.*

(ii) *For every closed subspace $S \subset E$ there is a projection P of E onto S with $||P|| = 1$.*

(iii) *Every orthonormal set can be extended to an orthonormal basis.*

(iv) *Every maximal orthonormal set is an orthonormal basis.*

Before giving a classification of n.a. Hilbert spaces we introduce a 'residue class space' as follows. Let E be a n.a. Hilbert space. Then $\bar{E} = \{\xi \in E : ||\xi|| \leqq 1\}/\{\xi \in E : ||\xi|| < 1\}$ is in the obvious way a vector space over the residue class field k of K. A K-linear map $\phi : E \to F$ with $||\phi|| \leqq 1$ defines in a natural way a k-linear map $\bar{\phi} : \bar{E} \to \bar{F}$. The assignment $E \mapsto \bar{E}$ is functorial. A set $(e_\alpha)_{\alpha \in I} \subset E$ is orthonormal if and only if $(\bar{e}_\alpha)_{\alpha \in I}$ forms an independent set. From this we can see that finite dimensional K-vector spaces with an orthonormal basis are n.a. Hilbert spaces. It is also easy to verify that every K-Banach space E with $||E|| = |K|$ over a discrete valued field is a n.a. Hilbert space. Without proof we mention that every n.a. Hilbert space is one of the above types. Note that subspaces, quotient spaces, and finite direct sums of n.a. Hilbert spaces are n.a. Hilbert spaces. In 5 we need the following lemma.

0.3 LEMMA. *Let E,F be n.a. Hilbert spaces. Let $\phi : E \to F$ be a K-linear map with $||\phi|| \leqq 1$. Then*

(i) $\bar{\phi} = 0$ *iff* $||\phi|| < 1$,

(ii) $\bar{\phi}$ *is bijective iff* ϕ *is a surjective isometry.*

PROOF. Left to the reader.

A n.a. *Banach algebra* over K is a Banach space A over K, together with a multiplication such that A becomes a K-algebra and such that $||fg|| \leqq ||f||\,||g||$ for all $f, g \in A$. We will say that a net $(u_\alpha)_{\alpha \in I}$ is an *approximate identity* for A if $||u_\alpha|| \leqq 1$ for all α and $\lim f u_\alpha = \lim u_\alpha f = f$ for all $f \in A$.

1. G-modules and A-modules

Throughout this paper G is a locally compact 0-dimensional group with identity e and K is a complete non-archimedean valued field. The valuation is supposed to be non-trivial.

1.1. DEFINITION. *A G-module is a Banach space E, together with a map $\tau : G \times E \to E$ (written as $(x, \xi) \mapsto x\xi$) such that*

(i) *τ is separately continuous and linear in ξ*

(ii) $x(y\xi) = (xy)\xi$; $e\xi = \xi$

(iii) *there is* $M > 0$ *with* $||x\xi|| \leqq M||\xi||$,

for all $x, y \in G$, $\xi \in E$.

If we put $U_x(\xi) = x\xi$, we obtain a bounded and a strongly continuous representation $x \mapsto U_x$ of G with E as a respresentation space. Conversely, a bounded and strongly continuous representation of G with representation space E induces on E a structure of a G-module. Both viewpoints are equivalent. From (i), (ii), (iii) it follows easily that the structure map τ is jointly continuous. If G is compact, condition (iii) follows from (i) and (ii): for $\xi \in E$, the map $x \mapsto ||x\xi||$ is bounded, the uniform boundedness principle implies that the maps $\xi \mapsto x\xi (x \in G)$ are uniformly bounded.

1.2. It is clear what one should mean by (continuous) homomorphisms of G-modules and G-submodules. If S is a G-submodule of a G-module E, then E/S with the obvious structure map and the quotient norm is a G-module. Let E, F be G-modules and let $\phi : E \to F$ be a homomorphism. Then Ker ϕ and $\overline{\text{Im } \phi}$ are G-modules. E and F are called *topologically equivalent* (notation $E \sim F$) in case there exists a homomorphism $\phi : E \to F$ that is an isomorphism of topological vector spaces.

If, in addition, ϕ can be chosen to be an isometry we call E and F *isometrically equivalent* (notation $E \simeq F$).

Let E, F be G-modules. The space $E \oplus F$ with the norm

$$||(\xi, \eta)|| = \max(||\xi||, ||\eta||) \qquad (\xi \in E, \eta \in F)$$

and the structure map

$$x.(\xi, \eta) = (x\xi, x\eta) \qquad (x \in G, \xi \in E, \eta \in F)$$

is a G-module, called the *direct sum* of E and F.

A G-module is called *simple* (*irreducible*) in case it has only trivial submodules.

1.3. *Isometrical G-modules.*

Let E be a non-zero G-module. Define

$$n(E) = \sup\left\{\frac{||x\xi||}{||\xi||} : x \in G, \;\; \xi \in E, \;\; \xi \neq 0\right\}.$$

It is an easy exercise to show that $n(E) \geqq 1$. We call E an *isometrical G-module* in case $n(E) = 1$. This is equivalent to $||x\xi|| = ||\xi||$ for all $x \in G, \xi \in E$. Every G-module E is topologically equivalent to an isometrical G-module: define on E the new norm $||\xi||' = \sup_{x\in G} ||x \cdot \xi||$, and observe that $||\;||' \sim ||\;||$ and $||x \cdot \xi||' = ||\xi||' \quad (x \in G, \;\; \xi \in E)$.

For isometrical G-modules we define arbitrary direct sums as follows. let $(E_\alpha)_{\alpha\in I}$ be a collection of isometrical G-modules. The space $\{(\xi_\alpha)_{\alpha\in I} \in$

$\Pi_\alpha E_\alpha : \lim_\alpha ||\xi_\alpha|| = 0\}$ together with the norm

$$||(\xi_\alpha)_{\alpha\in I}|| = \sup_\alpha ||\xi_\alpha||$$

and the structure map defined via

$$x(\xi_\alpha)_{\alpha\in I} = (x\xi_\alpha)_{\alpha\in I}$$

is easily seen to be an isometrical G-module, called the *direct sum* of the E_α, notation $\oplus_{\alpha\in I} E_\alpha$. (In fact, it is the direct sum in the category of isometrical G-modules, the morphisms being G-module homomorphisms with norm $\leqq 1$).

A *Hilbert G-module* is an isometrical G-module E, where E is a n.a. Hilbert space (See 0.2.).

1.4 *Finite dimensional G-modules.*
Let E be a finite dimensional G-module. Its *character* χ_E is defined via

$$\chi_E(x) = \operatorname{tr}(\xi \mapsto x\xi) \qquad (x \in G;\ \xi \in E).$$

χ_E is a continuous function: $G \to K$; $\chi_E(xy) = \chi_E(yx)$ for all $x, y \in G$; if E, F are finite dimensional then $\chi_{E\oplus F} = \chi_E + \chi_F$, if $E \sim F$ then $\chi_E = \chi_F$. The proofs are classical.

1.5 Definition. *Let A be a K-Banach algebra with approximate identiy. An A-module is a Banach space E, together with a map $\rho : A \times E \to E$ (written as $(f, \xi) \mapsto f\xi$) such that*

(i) *ρ is bilinear and continuous*
(ii) *$(fg)\xi = f(g\xi)$ for all $f, g \in A$, $\xi \in E$*
(iii) *$\overline{\operatorname{Im}\rho} = E$.*

Write $T_f(\xi) = f\xi$. The map $f \mapsto T_f$ is a non-degenerate continuous representation of A with representation space E. Conversely, a non-degenerate continuous representation of A with representation space E induces on E the structure of an A-module. Let $(u_\alpha)_{\alpha\in I}$ be an approximate identity of A. Conditions (i), (ii), (iii) are equivalent to (i), (ii) and

(iii)′: $u_\alpha \xi \to \xi$ for all $\xi \in E$.

Proof: Clearly (iii)′ implies (iii). To show that (iii)′ follows from (i), (ii), (iii), consider the set $E_0 = \{\xi \in E : u_\alpha \xi \to \xi\}$. Since $u_\alpha f\xi \to f\xi$ for all $f \in A$, E_0 is dense in E. We claim that E_0 is closed: let $\xi_n \in E_0$, $\xi_n \to \xi$. Then there is $C > 0$ such that

$$\begin{aligned} ||\xi - u_\alpha \xi|| &\leqq \max(||\xi - \xi_n||,\ ||\xi_n - u_\alpha \xi_n||,\ ||u_\alpha \xi_n - u_\alpha \xi||) \\ &\leqq \max(C||\xi - \xi_n||,\ ||\xi_n - u_\alpha \xi_n||). \end{aligned}$$

By choosing first n properly and then α, we can make $||\xi - u_\alpha \xi||$ small.

1.6. The remarks in 1.2 through 1.4 can be made for A-modules as well, of course with the obvious modifications. To find the counterpart of an isometrical G-module, define for a non-zero A-module E:

$$q(E) = \sup \left\{ \frac{||f\xi||}{||f|| \, ||\xi||} : f \in A, \ \xi \in E, f \neq 0, \ \xi \neq 0 \right\}.$$

Again, we have $q(E) \geqq 1$. If $q(E) = 1$ we call E a *normalized A-module.* Every A-module E is topologically equivalent to a normalized A-module. (Define $||\xi||' = \sup \{||f\xi||/||f|| : f \in A, f \neq 0\}$). It is also clear how one should define infinite direct sums of normalized A-modules. If E is finite dimensional, define

$$\Phi_E(f) = \operatorname{tr}(\xi \mapsto f\xi).$$

The properties of Φ_E are similar to those mentioned in 1.4 for characters.

2. Vector-valued Haar measure

Let E be a K-Banach space. By $C_\infty(G \to E)$ we mean the space of continuous functions : $G \to E$ that vanish at infinity. With the supremum norm $C_\infty(G \to E)$ is a Banach space. Define $\theta : C_\infty(G \to K) \otimes E \to C_\infty(G \to E)$ via

$$\theta(f \otimes \xi)(x) = \xi f(x) \qquad (f \in C_\infty(G \to K), \ \xi \in E; \ x \in G).$$

If one puts on $C_\infty(G \to K) \otimes E$ the greatest cross-norm (π-norm), θ becomes an isometry with dense image, hence θ can be extended to an isomorphism of Banach spaces (again called θ)

$$\theta : C_\infty(G \to K) \hat{\otimes} E \to C_\infty(G \to E).$$

Let $m_K : C_\infty(G \to K) \to K$ be a continuous, non zero, left invariant linear function (i.e., m_K is a left Haar integral). Define $m_E : C_\infty(G \to E) \to E$ by

$$m_E = (m_K \hat{\otimes} 1) \circ \theta^{-1}.$$

Then m_E is continuous, non-zero, and left invariant. Further, let $\phi : E \to F$ be a continuous linear map. Then

$$\phi(m_E(f)) = m_F(\phi \circ f) \qquad (f \in C_\infty(G \to E)).$$

The proofs are straightforward and are omitted. The above observations lead to the following

2.1. Definition. *K is called suitable with respect to G in case for each*

K-Banach space E there exists a continuous map $m_E : C_\infty(G \to E) \to E$ *such that*

(i) m_E *is continuous and non zero*

(ii) m_E *is left invariant*

(iii) *for every continuous linear map* $\phi : E \to F(E, F$ *Banach spaces*) *we have* $\phi(m_E(f)) = m_F(\phi \circ f)$.

2.2. THEOREM. *If* m_E *and* m'_E *satisfy* (i), (ii), (iii) *of Definition* 2.1. *then there is* $\lambda \in K$ *such that* $m_E = \lambda m'_E$ *for all E. Further, K is suitable with respect to G if and only if a K-valued left Haar integral on* $C_\infty(G \to K)$ *exists.*

PROOF. Let $\xi \in E$. Define $\phi : K \to E$ by $\phi(\lambda) = \lambda\xi (\lambda \in K)$. Then we have

$$\phi(m_K(f)) = m_E(\phi \circ f) \qquad (f \in C_\infty(G \to K))$$

Now $(\phi \circ f)(x) = f(x)\xi = \theta(f \otimes \xi)$. So $m_E(\phi \circ f) = (m_K \hat{\otimes} 1)(f \otimes \xi) = m_K(f)\xi$. Similarly, $m'_E(\phi \circ f) = m'_K(f)\xi$. Since the K-valued Haar measure is unique (see [7]), there is $\lambda \in K$ with $m_K = \lambda m'_K$. Hence $m_E = \lambda m'_E$ on functions of the type $\theta^{-1}(f \otimes \xi)$ hence for all functions. The second part of the theorem has been proved at the beginning of this section.

2.3 In the sequel Γ_G will be the collection of the compact open subgroups of G. The characteristic of a field L is denoted by $\chi(L)$, the residue class field of K by k.

A necessary and sufficient condition for K to be suitable with respect to G is given in [7]. For reasons of simplicity we shall work mostly with pairs G, K that have the following property: either $\chi(k) = 0$ or G is p-free whenever $\chi(k) = p \neq 0$. (Here p-free means that for every $H_1, H_2 \in \Gamma_G$, $H_1 \subset H_2$, the index $(H_2 : H_1)$ is not divisible by p). Then K is suitable with respect to G and moreover we can choose m_K such that $|m_K(H)| = 1$ for all $H \in \Gamma_G$. For more details we refer to [10]. In [7], a Fubini Theorem is proved for K-valued integrals. Using the definition of m_E it is not very hard to prove a Fubini Theorem for the vector-valued Haar integral.

3. G-modules and $L(G)$-modules

In this section we assume that G admits a left Haar integral m_K with $|m_K(H)| = 1$ for all $H \in \Gamma_G$. Instead of $m_E(f)$ we sometimes write $\int f(x)dx$ $(f \in C_\infty(G \to E))$. It is shown in [10], that the 'integral norm' on $C_\infty(G \to K)$

$\|f\|_{m_K} = \sup\left\{\dfrac{|m_K(fg)|}{\|g\|_\infty} : g \in C_\infty(G \to K),\ g \neq 0\right\}$, $(f \in C_\infty(G \to K))$ is

equal to the supremum norm, and that $|m_K(f)| \leqq \|f\|_\infty$ $(f \in C_\infty(G \to K))$. It is not very hard to show that also $\|m_E(f)\| \leqq \|f\|_\infty (f \in C_\infty(G \to E))$. Further, it is also shown in [10], that the space $C_\infty(G \to K)$ forms a Banach algebra $L(G)$ under convolution and that $\{u_H : H \in \Gamma_G\}$ forms an approximate identity for $L(G)$. Here $H_1 \leqq H_2$ iff $H_1 \supset H_2$ and u_H is defined as follows

$$u_H(x) = \begin{cases} m_K(H)^{-1} & \text{if } x \in H \\ 0 & \text{elsewhere.} \end{cases}$$

Let $f : G \to K$ and $x \in G$. We define f_x by $f_x(y) = f(x^{-1}y)$, $(y \in G)$.

3.1. THEOREM *Let E be a G-module. The map $(f, \xi) \mapsto f\xi$ given by*

$$f\xi = \int x\xi f(x)\,dx \qquad (\xi \in E, f \in L(G))$$

makes E into an $L(G)$-module. Conversely, if E is an $L(G)$-module, then the map $(x, \xi) \mapsto x\xi$, given by

$$x \cdot \xi = \lim_H (u_H)_x \xi \qquad (\xi \in E, x \in G).$$

makes E into a G-module. The above constructions yield a $1-1$ correspondence between the G-module structures and the $L(G)$-module structures on the Banach space E.

PROOF. Since for each ξ, f the function $x \mapsto x\xi f(x)$ is in $C_\infty(G \to K)$, $f\xi$ is a well-defined element of E. Further, $\|f\xi\| = \|\int_G x\xi f(x)dx\| \leqq \leqq \sup_{x\in G} \|x\xi f(x)\| \leqq \sup_{x\in G} \|x \cdot \xi\| \cdot \|f\|_\infty$. Finally, for $f, g \in L(G \to K)$ and $\xi \in E : (f * g)\xi = \iint x\xi f(xy)g(y^{-1})dy\,dx = \iint x\xi f(y)g(y^{-1}x)\,dy\,dx = \iint x\xi f(y)g(y^{-1}x)\,dx\,dy = \iint yx\xi f(y)g(x)\,dx\,dy$. On the other hand, $f(g\xi) = \int x(g\xi)f(x)dx = \iint xy\xi g(y)f(x)\,dy\,dx$.

Now let E be an $L(G)$-module, and let $x \in G$. The set $E_0 = \{\xi \in E$: $\lim (u_H)_x\xi$ exists$\}$ is a linear space. For $f \in C_\infty(G \to K)$ we have: $(u_H)_x(f\xi) = ((u_H)_x * f)\xi = (u_H * f)_x\xi \to f_x\xi$. Hence E_0 is dense in E. Now let $\xi_n \in E_0$, $\lim \xi_n = \xi$. Let H, H' be compact open subgroups. Then

$$\|(u_H)_x\xi - (u_{H'})_x\xi\| \leqq \max\left(q(E)\|\xi - \xi_n\|, \|(u_H)_x\xi_n - (u_{H'})_x\xi_n\|\right).$$

By choosing first n large enough and then H and H' sufficiently small we can make the left hand expression arbitrarily small. Hence E_0 is closed and $E_0 = E$, so $x\xi$ is well defined. Next,

$$||x\xi|| = \lim_H ||(u_H)_x \xi|| \leqq q(E)||\xi||.$$

Finally we only need to check $(xy)\xi = x(y\xi)$ where ξ is of the form $f\eta$ $(f \in L(G),\ \eta \in E) : (xy)f\eta = \lim\ (u_H)_{xy}(f\eta) = \lim\ ((u_H)_{xy} * f)\eta = f_{xy}\eta$, whereas $x(yf\eta) = x(f_y\eta) = (f_y)_x\eta = f_{xy}\eta$. The 1–1 correspondence between the G– and $L(G)$-module structures will follow from the following formulas

(1) $\quad x\xi = \lim \int y\xi(u_H)_x(y)\,dy$ for E as a G-module

(2) $\quad f\xi = \int \lim\,(u_H)_x\,\xi f(x)\,\mathrm{d}x$ for E as an $L(G)$-module.

To prove (1), let $\xi \in E$, $x \in G$ and $\varepsilon > 0$. For $H \in \Gamma_G$ we have: $\int y\xi(u_H)_x(y)dy = \int y\xi u_H(x^{-1}y)dy = \int xy\xi u_H(y)dy$. Choose H such that $||xy\xi - x\xi|| < \varepsilon$ for all $y \in H$. Then

$$\left|\left|\int y\xi(u_H)_x(y)dy - x\xi\right|\right| = \left|\left|\int (xy\xi - x\xi)u_H(y)\mathrm{d}y\right|\right| < \varepsilon.$$

In order to show (2), it suffices to establish the formula for the case where ξ has the form $g\eta$ where $g \in L(G)$, $\eta \in E$. So we have to check

(2)′ $\qquad fg\eta = \int g_x\eta f(x)dx \qquad (f, g \in L(G);\ \eta \in E).$

Now the left hand expression is equal to $(f * g)\eta = (\int f(x)g_x dx)\eta = (m_{L(G)}(h))\eta$, where h is the function $x \mapsto f(x)g_x$ of $C_\infty(G \to L(G))$. Define a map $\phi : L(G) \to E$ via

$$\phi(t) = t\eta \qquad (t \in L(G)).$$

By Definition 2.1 (iii) we have

$$(m_{L(G)}(h))\eta = \phi(m_L(G)(h)) = m_E(\phi \circ h) = \int \phi(h(x))dx = \int h(x)\eta\,dx =$$

$$= \int g_x\eta f(x)\,\mathrm{d}x.$$

Example. The left regular representation. $L(G)$ is an $L(G)$-module under the map $(f, g) \mapsto f * g$. The corresponding G-module structure is given by $xf = f_x$ $(f \in L(G),\ x \in G)$.

3.2. Theorem. *Let E, F be G-modules and let $\tilde{E}$, $\tilde{F}$ be the corresponding $L(G)$-modules in the sense of Theorem* 3.1 *respectively. Then we have*

(i) $n(E) = q(\tilde{E})$. *In particular, E is an isometrical G-module if and only if $\tilde{E}$ is a normalized $L(G)$-module.*

(ii) $\phi : E \to F$ *is a G-module map if and only if* $\phi : \tilde{E} \to \tilde{F}$ *is an $L(G)$-module map.*

(iii) *E and $\tilde{E}$ have the same submodules.*

(iv) $E \sim F$ *iff* $\tilde{E} \sim \tilde{F}$; $E \simeq F$ *iff* $\tilde{E} \simeq \tilde{F}$.

(v) *E is simple if and only if $\tilde{E}$ is simple.*

PROOF. (i) follows from the proof of Theorem 3.1. (iii), (iv), (v) follow from (ii). To prove (ii), let $\phi : E \to F$ be a G-module map. By Definition 2.1 (iii) and Theorem 3.1 we have for $f \in L(G)$, $\xi \in E$

$$\phi(f\xi) = \phi\left(\int x\xi f(x)dx\right) = \int \phi(x\xi f(x))dx = \int x\phi(\xi)f(x)dx = f\phi(\xi).$$

Conversely, let $\phi : \tilde{E} \to \tilde{F}$ be an $L(G)$-module map. Then for $x \in G$, $\xi \in E$ we have

$$\phi(x\xi) = \phi(\lim (u_H)_x \xi) = \lim \phi((u_H)_x \xi) = \lim (u_H)_x \phi(\xi) = x\phi(\xi).$$

3.3. THEOREM. *Let E be a finite dimensional G-module. Let χ_E and Φ_E be the trace functions defined in* 1.4 *and* 1.6 *respectively. Then we have*

$$\Phi_E(f) = \int f(x)\chi_E(x)\,\mathrm{d}x$$

$$\chi_E(x) = \lim \Phi_E((u_H)_x).$$

PROOF. Write $U_x(\xi) = x\xi$ and $T_f(\xi) = f\xi (x \in G, f \in L(G), \xi \in E)$. Then $\Phi_E(f) = \mathrm{tr}\,(\xi \mapsto f\xi) = \mathrm{tr}\,(T_f) = \mathrm{tr}\,(\int U_x f(x)dx) = \int \mathrm{tr}\,(U_x)f(x)dx = = \int f(x)\chi_E(x)dx$. Next, $\chi_E(x) = \mathrm{tr}\,(U_x) = \mathrm{tr}\,(\lim T_{(u_H)_x}) = \lim \mathrm{tr}\,(T_{(u_H)_x}) = \lim \Phi_E((u_H)_x)$.

3.4 THEOREM. *Let E be a finite dimensional G-module. Then* Ker $E =$ $\{x \in G : x\xi = \xi$ *for all* $\xi \in E\}$ *is an open normal subgroup of G.*

PROOF. Ker E is a normal subgroup of G. Consider the corresponding $L(G)$- module structure on E and write $T_f(\xi) = f\xi (f \in L(G), \xi \in E)$. By 1.5 we have $\lim u_H \xi = \xi$ for all ξ, hence T_{u_H} converges to the identity operator. Since u_H is idempotent T_{u_H} is a projection and $H_1 \supset H_2$ implies $\mathrm{Im}\, T_{u_{H_1}} \subset \mathrm{Im}\, T_{u_{H_2}}$. So there is an $H_0 \in \Gamma_G$ with $u_{H_0}\xi = \xi$. Let $x \in H_0$. Then $x\xi = \lim (u_H)_x u_{H_0}\xi = \xi$. Hence Ker E is open.

3.5 THEOREM. *Let G be compact and let E be a simple G-module. Then* $\dim E < \infty$.

PROOF. Let $\xi \in E$, $\xi \neq 0$. Since $u_H \xi \to \xi$ there is $H \in \Gamma_G$ such that $u_H \xi \neq 0$. The linear space $\{fu_H \xi : f \in L(G)\}$ is non-zero, finite dimensional (every function of the type $f * u_H$ is constant on left cosets of H), and an $L(G)$-submodule, hence a G-submodule. Since E is simple, $\dim E < \infty$.

4. Compact groups

In this section we prove a non-archimedean variant of the Peter-Weyl theorem. A *supernatural number* is a formal product $\prod p^{n_p}$ where p runs through the set of the primes and where n_p is an integer $\geqq 0$ or ∞. It is clear how to define products, l.c.m., g.c.d. of elements of a set of supernatural numbers. We define the order $(G : 1)$ of G as l.c.m. $\{(G/H : 1) : H \in \Gamma_G\}$. The *index* $(G : 1)_c$ is defined similarly.

4.1. THEOREM. *Let G be compact and let $\chi(K) \not| \ (G : 1)$ whenever $\chi(K) \neq 0$. Then G has suffciently many continuous irreducible representations.*

PROOF. Let $x \in G$ $(x \neq e)$. There is $H \in \Gamma_G$, H normal, with $x \notin H$. Thanks to Maschke's Theorem the left regular respresentation of G/H (which is faithful) is completely reducible, since $\chi(K) \not| \ (G/H : 1)$. Hence there is an irreducible representation V of G/H with $V_{x \bmod H} \neq I$. V induces an irreducible representation U of G with $U_x \neq 1$.

The next lemma is very useful when trying to decompose representations into irreducible ones. *From now on in* 4 *and* 5 *we suppose that G is compact and that a K-valued Haar measure m_K exists on G with $|m_K(H)| = 1$ for all $H \in \Gamma_G$. We take $m_K(G) = 1$.*

4.2 LEMMA. *Let E, F be isometrical G-modules and let $\phi : E \to F$ be a continuous linear operator. Define ϕ' by*

$$\phi'(\xi) = \int x^{-1}\phi(x\xi)\, dx.$$

Then we have

(i) *ϕ' is a G-module map* $: E \to F$, $||\phi'|| \leqq ||\phi||$,

(ii) *if $E = F$ and ϕ is a projection onto a G-submodule S of E, then ϕ' is a projection, $\phi'(E) = S$,* Ker *ϕ' is a G-submodule.*

PROOF. (i) The continuity of $x \mapsto (x^{-1}, x) \mapsto (x^{-1}, x\xi) \mapsto (x^{-1}, \phi(x\xi)) \mapsto x^{-1}\phi(x\xi)$ yields no problems. Hence $\phi'(\xi)$ is well-defined and linear in ξ. Next,

$$||\phi'(\xi)|| \leqq \sup_{x\in G} ||x^{-1}\phi(x\xi)|| \leqq ||\phi|| \sup_{x\in G} ||x\xi|| = ||\phi||\, ||\xi||,$$

whence $||\phi'|| \leqq ||\phi||$. Further, for $y \in G : \phi'(y\xi) = \int x^{-1}\phi(xy\xi)dx = = \int yx^{-1}\phi(x\xi)dx = y\phi'(\xi)$.

(ii) For $\xi \in E : \phi(x\xi) \in S$, hence $x^{-1}\phi(x\xi) \in S$. Thus $\phi'(E) \subset S$. If $\xi \in S$, then $x^{-1}\phi(x\xi) = x^{-1}x\xi = \xi$. So we have

$$\phi'(\xi) = \xi \int 1\, dx = \xi.$$

That Ker ϕ' is a G-submodule follows from (i).

Let E be a finite dimensional G-module. Let $e_1, \cdots, e_n$ be a basis of E. The functions

$$x \mapsto e_i^*(xe_j)$$

span a finite dimensional space of continuous K-valued functions, called $\mathscr{R}_E(G)$. It is clear that another choice of the basis doesn't change $\mathscr{R}_E(G)$, and that $E \sim F$ implies $\mathscr{R}_E(G) = \mathscr{R}_F(G)$.

Define $\mathscr{R}(G)$ to be the K-linear space generated by $\{\mathscr{R}_E(G) : E$ is a simple G-module$\}$.

4.3. Lemma. *The following sets are equal.*

(1) *The K-linear span of* $\{\mathscr{R}_E(G) : E$ *is a simple Hilbert G-module*$\}$.

(2) $\mathscr{R}(G)$.

(3) *The K-linear span of* $\{\mathscr{R}_E(G) : \dim E < \infty\}$.

(4) *The space of the locally constant functions.*

(5) $\{f \in L(G) : (f_s)_{s \in G}$ *spans a finite dimensional space*$\}$.

Proof. We show (1) $\subset$ (2) $\subset$ (3) $\subset$ (4) $\subset$ (5) $\subset$ (1).

(3) $\subset$ (4) follows from Theorem 3.4. The rest, except (5) $\subset$ (1), is obvious. Let $f \in (5)$. The left ideal $I \subset L(G)$ generated by f is finite dimensional. Considering I as a G-module under the left regular representation, we can apply Maschke's theorem and find that $I = \oplus I_i$ (in the algebraic sense), where the I_i are simple. Thus we may assume that I is simple. I is a n.a. Hilbert space, and I is an isometrical G-module. Let $\phi \in I^*$ be the function $f \mapsto f(e) (f \in I)$. Then $f(x) = \phi(f_x)$, hence $f \in \mathscr{R}_I(G)$.

4.4. Lemma. *Let E, F be simple G-modules and let* $E \nsim F$, $f \in \mathscr{R}_E(G)$, $g \in \mathscr{R}_F(G)$. *Then*

$$\int f(x)g(x^{-1})\, dx = 0.$$

Proof. Let $\phi \in E^*$ and $\eta \in F$. By Lemma 4.2 the map $\phi' : E \to F$ defined via

$$\phi'(\xi) = \int x^{-1}\phi(x\xi)\eta\, dx$$

is a G-module map, hence $\phi' = 0$. So for all $\xi \in E$, $\psi \in F^*$ we have:

$$\psi\left(\int x^{-1}\eta\phi(x\xi)\,dx\right) = 0 \text{ or}$$

$$\int \psi(x^{-1}\eta)\phi(x\xi)\,dx = 0.$$

4.5. Corollary. *Every simple G-module is topologically equivalent to a simple Hilbert G-module.*

Proof. Let E be a simple G-module and let $f \in \mathscr{R}_E(G)$. If for every Hilbert module F we had $E \not\sim F$ then $\int f(x)g(x^{-1})dx = 0$ for all $g \in \mathscr{R}_F(G)$. By Lemma 4.3 we then had $\int f(x)g(x^{-1})dx = 0$ for all $g \in \mathscr{R}(G)$. Since $\mathscr{R}(G)$ is dense, $f = 0$. Contradiction.

4.6. Definition. *Let G be a compact group, and let k be the residue class field of K. K is called a splitting field for G in case $\xi^n - 1 = 0$ has n different roots in k whenever $n \mid (G:1)_c$.*

In the terminology of [1] our definition is equivalent to: 'k is a splitting field for G/H for each normal $H \in \Gamma_G$'.

4.7. Theorem (Peter-Weyl) *Let K be a splitting field for G. Let $(E_\sigma)_{\sigma\in\Sigma}$ be a complete set (modulo $\sim$) of simple Hilbert G-modules. For each σ, let $e_1^\sigma, \cdots, e_{d_\sigma}^\sigma$ be an orthonormal basis of E_σ. Define for $\sigma \in \Sigma$, $1 \leqq i, j \leqq d_\sigma$*

$$u_{ij}^\sigma(x) = (e_j^\sigma)^*(xe_i^\sigma) \qquad (x \in G).$$

Then

$$\int u_{ij}^\sigma(x)u_{kl}^\tau(x^{-1})\,dx = d_\sigma^{-1}\delta_{\sigma\tau}\delta_{ik}\delta_{jl};\ |d_\sigma| = 1. \tag{1}$$

(2) *the set $\{u_{ij}^\sigma : \sigma \in \Sigma,\ 1 \leqq i, j \leqq d_\sigma\}$ is an orthonormal basis for $L(G)$.*

Proof. To show (1) we may assume $\sigma = \tau$ (Lemma 4.4). There is a normal subgroup $H \in \Gamma_G$ such that the u_{ij}^σ are constant on the cosets of H. The left hand expression of (1) is then equal to

$$m(H) \sum_{g\in G/H} v_{ij}(g)v_{kl}(g^{-1}) \tag{3}$$

where $v_{ij}(x \bmod H) = u_{ij}(x)(x \in G)$. By [1], (32.5), (3) is equal to

$$\frac{m(H)(G:H)}{d_\sigma}\delta_{ik}\delta_{jl} = d_\sigma^{-1}\delta_{ik}\delta_{jl}.$$

Since $d_\sigma | (G:H)$ we have $|d_\sigma| = 1$. (2) follows from the fact that the

u_{ij}^{σ} span $\mathscr{R}(G)$ and from Lemma 0.1, by taking the u_{ij}^{σ} for e_{α}'s, and the maps $f \mapsto \int f(x)u_{kl}^{\tau}(x^{-1})dx$ for f_{α}'s.

4.8. Corollary. *Let K be a splitting field for G where G is abelian. Then the continuous characters: $G \to K$ form an orthonormal basis of $L(G)$.*

Proof. Let E be a simple G-module. By [1], (27.3), $\phi : E \to E$ is a G-module map implies $\phi = \lambda I(\lambda \in K)$. Hence $\dim E = 1$. Now apply Theorem 4.7 (2).

4.9. Corollary *Let K be a splitting field for G and let E and E' be finite dimensional G-modules. Then the following statements are equivalent.*

(1) $$E \sim E'$$

(2) $$\chi_E = \chi_{E'}$$

(3) $$\Phi_E = \Phi_{E'}.$$

Proof. By Theorem 3.2 and Theorem 3.3 we only have to show (2) $\to$ (1). Let first E, E' be simple. By Corollary 4.5 we may assume that E, E' are Hilbert G-modules. The orthogonality relations yield

$$\int \chi_E(x)\chi_{E'}(x^{-1})\,\mathrm{d}x = 0 \qquad \text{if } E \nsim E'$$

$$\int \chi_E(x)\chi_E(x^{-1})\,\mathrm{d}x = 1.$$

Hence the assumptions $\chi_E = \chi_{E'}$ and $E \nsim E'$ are contradictory. The general case can easily be proved by decomposing E and E' into a direct sum of simple modules.

4.10. Corollary. *Let K be a splitting field for G and let $(E_{\sigma})_{\sigma \in \Sigma}$ be as in Theorem 4.7. For $f \in L(G)$ define*

$$\|f\|^{\wedge} = \sup_{\sigma \in \Sigma} \|f\|_{\sigma},$$

where $\|f\|_{\sigma} = \sup \{\|f \cdot \xi\|/\|\xi\| : \xi \in E_{\sigma}, \xi \neq 0\}$. Then $\|f\|^{\wedge} = \|f\|$.

Proof. Write $f = \Sigma\, \lambda_{ij}^{\sigma} u_{ij}^{\sigma}$(Theorem 4.7 (2)). Then it follows that

$$\|f\|_{\sigma} = \sup \{|\lambda_{ij}^{\sigma}| : 1 \leqq i, j \leqq d_{\sigma}\}, \text{ whence } \|f\|^{\wedge} = \sup \{|\lambda_{ij}^{\sigma}| : 1 \leqq i, j \leqq d, \sigma \in \Sigma\} = \|f\|.$$

5. Hilbert G-modules for compact G.

The left regular representation of $L(G)$ can (in case K is a splitting field for G) be decomposed into a direct sum (in the sense of 1.3) of irreducible representations. This follows from Theorem 4.7. Note that $L(G)$ need not be a n.a. Hilbert space. We now try to decompose arbitrary representations. We can only get some results in case the representation space is a n.a. Hilbert space. (Theorem 5.3).

Throughout this section we assume that E is a Hilbert G-module. With the definition $x \cdot \bar{\xi} = \overline{x \cdot \xi}(x \in G, \xi \in E, \|\xi\| \leqq 1)$, $\bar{E}$ (see 0.2) becomes a G-module (in the algebraic sense). $E \mapsto \bar{E}$ is functorial. The crucial property is the following.

5.1. Theorem. *Let E, F be Hilbert G-modules and let $\alpha : \bar{E} \to \bar{F}$ be a G-module map. Then there is a G-module map $\phi : E \to F$ with $\bar{\phi} = \alpha$.*

Proof. There exists a continuous linear map $\psi : E \to F$ with $\bar{\psi} = \alpha$. Write $U_x(\xi) = x\xi$, $V_x(\eta) = x\eta (x \in G, \xi \in E, \eta \in F)$. Define

$$\phi(\xi) = \int V_{x^{-1}} \psi U_x(\xi)\, dx.$$

By Lemma 4.2, ϕ is a G-module map. Write $H_x = \psi U_x - V_x \psi$. Then $\overline{H_x} = \alpha \bar{U}_x - \bar{V}_x \alpha = 0$, so by 0.3, (i) we have $\|H_x\| < 1$ for all $x \in G$.

$$\begin{aligned}\phi(\xi) &= \int \psi(\xi)\, dx + \int V_{x^{-1}} H_x(\xi)\, dx \\ &= \psi(\xi) + H(\xi), \text{ where } \|H\| < 1.\end{aligned}$$

Hence $\bar{\phi} = \bar{\psi} = \alpha$.

5.2. Corollary. *Let E, F be Hilbert G-modules. Then $\bar{E} \sim \bar{F}$ implies $E \simeq F$. In case K is a splitting field we have, in addition, the following.*

(i) *E is simple if and only if $\bar{E}$ is simple.*

(ii) *Let E, F be simple and let $\phi : E \to F$ be a topological equivalence of G-modules. Then there is $\lambda \in K$ with $\phi = \lambda\psi$, where ψ an isometry. In particular, $E \sim F$ implies $E \simeq F$.*

Proof. Let $\alpha : \bar{E} \to \bar{F}$ be an isomorphism of G-modules. By Theorem 5.1 there is a G-module map $\phi : E \to F$ with $\bar{\phi} = \alpha$. By 0.3, (ii) ϕ is an isomorphism of Banach spaces. To prove (i), let E be be simple. Let $\alpha : \bar{E} \to \bar{E}$ be a G-module map. There is a G-module map $\phi : E \to E$ with $\bar{\phi} = \alpha$. By [1], (27.3), $\phi = \lambda I$, $|\lambda| \leqq 1$. Hence $\alpha = \bar{\lambda} I$, so $\bar{E}$ is simple.

Now let $\bar{E}$ be simple. Let $\phi : E \to E$ be a projection and a G-module map. Then $\bar{\phi}$ is a projection, so $\bar{\phi} = 0$ or I, whence $\phi = 0$ or I. Thus E is simple. To prove (ii), choose $\lambda^{-1} \in K$ such that $\|\lambda^{-1}\phi\| = 1$. Then $\overline{\lambda^{-1}\phi} \neq 0$, hence a bijection by Schur's lemma. It follows that $\lambda^{-1}\phi$ is an isometry.

5.3. THEOREM. *Let E be a Hilbert G-module. Then E is an orthogonal direct sum of simple G-modules. In case K is a splitting field this decomposition is unique in the following sense. Let $E = \bigoplus_{\alpha \in I} E_\alpha^{n_\alpha} = \bigoplus_{\alpha \in J} F_\alpha^{m_\alpha}$ be two decompositions in the above sense ($E_\alpha \not\simeq E_\beta$ if $\alpha \neq \beta$; n_α is the multiplicity; same for F_α's). Then there is a bijection $\sigma : I \to J$ such that*

$$m_{\sigma(\alpha)} = n_\alpha;\ E_\alpha \simeq F_{\sigma(\alpha)} \qquad (\alpha \in I).$$

PROOF. We first show that E has simple submodules. There exist $H \in \Gamma_G$ and $\xi \in E$ such that $u_H\, \xi \neq 0$. The G-submodule generated by this vector is finite dimensional, hence it contains simple G-submodules. The existence of a decomposition of E now can easily be shown, using Lemma 4.2 and Zorn's Lemma. To show the uniqueness property, let $\alpha \in I$, $\beta \in J$ and consider the G-module map

$$E_\alpha \to E \to F_\beta.$$

If $E_\alpha \nsim F_\beta$, this map is the zero map. Define $\sigma(\alpha)$ to be the only β for which the above maps are non-zero. The identity on E sends $E_\alpha^{n_\alpha}$ into $F_{\sigma(\alpha)}^{m_{\sigma(\alpha)}}$; it must be onto. Since $\dim E_\alpha < \infty$, a cardinality argument can show that $n_\alpha = m_{\sigma(\alpha)}$. By Corollary 5.2 (ii) $E_\alpha \sim F_{\sigma(\alpha)}$ implies $E_\alpha \simeq F_{\sigma(\alpha)}$.

6. The Tannaka duality theorem

We formulate the Tannaka Duality Theorem in the language of [3], II.3. A *Hopf algebra* is a K-algebra H with identity together with algebra homomorphisms $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ such that the diagrams

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "1 \otimes \Delta"] \\ H \otimes H \arrow[r, "\Delta \otimes 1"'] & H \otimes H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[dr, "\sim"'] & H \otimes H \arrow[d, "1 \otimes \varepsilon"] \\ & H \otimes K \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r] \arrow[dr, "\sim"'] & H \otimes H \arrow[d, "\varepsilon \otimes 1"] \\ & K \otimes H \end{tikzcd}$$

commute. It is useful to consider the multiplication as a map $\phi : H \otimes H \to H$ and the identity as a map $\eta : K \to H$. The Hopf algebra then is completely described by the 5-tuple $(H, \phi, \eta, \Delta, \varepsilon)$.

Let X be a compact 0-dimensional Hausdorff space and let K be a field. Let $\mathscr{R}(X)$ denote the space of the locally constant functions: $X \to K$. Then $\mathscr{R}(X)$ is a commutative K-algebra with identity, generated by its idempotents. Conversely, if A is such an algebra then the set of all algebra homomorphisms: $A \to K$ with the finite-open topology is a 0-dimensional compact Hausdorff space. In fact the above functors yield a duality between the category of 0-dimensional compact Hausdorff spaces and the

category of the commutative K-algebras with identity, generated by its idempotents. (See [9], Chapter III).
Now if S is a compact 0-dimensional semigroup with identity, let $\mathscr{R}(S)$ be the K-algebra of the locally constant functions on S. It is a commutative Hopf algebra under

$$\Delta : \mathscr{R}(S) \xrightarrow{\sigma} \mathscr{R}(S \times S) \xrightarrow{\pi^{-1}} \mathscr{R}(S) \otimes \mathscr{R}(S)$$
$$\varepsilon : \mathscr{R}(S) \to K,$$

where σ, π, ε are defined by

$$\sigma(f)(x, y) = f(xy) \qquad (x, y \in S; f \in \mathscr{R}(S))$$
$$\pi(f \otimes g)(x, y) = f(x)g(y) \qquad (f, g \in \mathscr{R}(S); x, y \in S)$$
$$\varepsilon(f) = f(e) \qquad (f \in \mathscr{R}(S),\ e \in S \text{ is the identity}).$$

If H is a commutative Hopf algebra, generated by idempotents, let $G(H)$ be the collection of all algebra homomorphisms $: H \to K$. Under the finite-open topology and with the multiplication

$$f * g = H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} K \otimes K \cong K,$$

$G(H)$ becomes a 0-dimensional compact Hausdorff semigroup with identity ε. We now have the following theorem the proof of which is left to the reader.

6.1. THEOREM. *Let $\mathscr{S}$ be the category of 0-dimensional Hausdorff semigroups with identity, and let $\mathscr{H}$ be the category of the commutative K-Hopf algebras generated by idempotents. Then the functors $\mathscr{R} : \mathscr{S} \rightleftarrows \mathscr{H} : G$ have the properties $G\mathscr{R} \sim \mathrm{id}_{\mathscr{S}}$ and $\mathscr{R}G \sim \mathrm{id}_{\mathscr{H}}$. These natural equivalences $j_S : S \to G\mathscr{R}(S)$ and $j_H : H \to \mathscr{R}G(H)$ are given by*

$$j_S(x)(f) = f(x) \qquad (f \in \mathscr{R}(S), x \in S)$$

and

$$j_H(f)(\alpha) = \alpha(f) \qquad (f \in H, \alpha \in G(H)).$$

We now can give a very short proof of a well-known result (See [4], (8.19)).

6.2. COROLLARY. *Every 0-dimensional Hausdorff semigroup with identity is projective limit of finite semigroups.*

PROOF. We show that every element $f \in H \in \mathscr{H}$ lies in a finite dimensional sub-Hopf algebra. By [6], (2.5) f is contained in a finite dimensional sub-coalgebra C. The subalgebra generated by C is easily seen to be a sub-Hopf algebra, and it is also finite dimensional since we can interpret H as an $\mathscr{R}(S)$ for some S.

An *antipode* in a Hopf algebra H is a K-linear map $\omega : H \to H$ such that the diagram

$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & H \otimes H \arrow[dd, shift right=2, "\omega \otimes 1"'] \arrow[dd, shift left=2, "1 \otimes \omega"] \\
K \arrow[d, "\eta"'] & \\
H & H \otimes H \arrow[l, "\phi"]
\end{tikzcd}$$

commutes. If $G \in \mathscr{S}$ is a topological group, then $\mathscr{R}(G)$ has an antipode defined by

$$\omega(f)(x) = f(x^{-1}) \qquad (x \in G, f \in \mathscr{R}(G))$$

It is not very hard to show that $\mathscr{R}(S)$ has an antipode if and only if S is a topological group. This remark, together with Theorem 6.1 and Lemma 4.3 form the non-archimedean analogue of the Tannaka Duality Theorem.

REFERENCES

C. Curtis, I. Reiner
[1] Representation theory of finite groups and associative algebras. New York, 1962.

E. Hewitt, K. Ross
[2] Abstract harmonic analysis. New York, I, 1963; II, 1970.

G. Hochschild
[3] The structure of Lie groups. San Francisco, 1965.

K. Hofmann, P. Mostert
[4] Elements of compact semigroups, Columbus Ohio, 1966.

G. Kalisch
[5] On p-adic Hilbert spaces. Ann. Math. 48 (1947). 180–192.

R. Larson
[6] Cocommutative Hopf algebras. Can. J. Math. 19 (1967), 350–360.

A. Monna, T. Springer
[7] Intégration non-archimédienne I, II. Indag. Math. 25 (1963), 634–654.

A. Monna, T. Springer
[8] Sur la structure des espaces de Banach non-archimédiens. Indag. Math. 27 (1965), 602–614.

M. van der Put
[9] Algèbres de fonctions continues p-adiques (Thesis). Utrecht, 1967.

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[10] Non-archimedean harmonic analysis (Thesis). Nijmegen, 1967.

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