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PARACOMPACTNESS AND THE LINDELÖF PROPERTY IN FINITE AND COUNTABLE CARTESIAN PRODUCTS

by

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1. Introduction

It is well known [16] that there are paracompact (in fact, hereditarily Lindelöf) spaces X for which X^2 is not even normal [2]. The purpose of this paper is to refute some plausible conjectures by showing that, in these respects, the higher powers X^n and X^ω of X can also behave quite unpredictably. (Here X^ω denotes the product of countably many copies of X.) This emphasizes the significance of certain special classes of paracompact spaces which have been studied in recent years, and whose relevant properties are summarized in section 2.

Most of our examples assume the Continuum Hypothesis, which is indicated by the symbol (CH).

EXAMPLE 1.1. There exists a space Y such that Y^n is paracompact for all $n \in N$, but Y^ω is not normal.

EXAMPLE 1.2. (CH). There exists a regular space Y such that Y^n is Lindelöf for all $n \in N$, but Y^ω is not normal.

EXAMPLE 1.3. (CH). There exist semi-metrizable, regular spaces Y_1 and Y_2 such that Y_1^ω and Y_2^ω are both hereditarily Lindelöf, but $Y_1 \times Y_2$ is not normal (and hence neither Y_1 nor Y_2 is cosmic) [3].

EXAMPLE 1.4. (CH). For all $n \in N$, there exists a regular space Y such that Y^n is Lindelöf and Y^{n+1} is paracompact, but Y^{n+1} is not Lindelöf [4].

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[2] The reader should recall that paracompact spaces are normal [4; p. 163], and that regular Lindelöf spaces are paracompact [4; p. 174]. A Lindelöf space X with all open subsets F_σ is hereditarily Lindelöf, and conversely if X is regular.

[3] Semi-metrizable spaces are defined, for instance, in [5]. Cosmic spaces are defined in section 2. All first-countable cosmic spaces are semi-metrizable.

[4] This example is new for $n = 1$. In contrast to this example, S. Willard has shown that, if $X \times Y$ is paracompact with X Lindelöf and Y *separable*, then $X \times Y$ must be Lindelöf.

EXAMPLE 1.5. (CH). For all $n \in N$, there exists a regular space Y such that Y^n is hereditarily Lindelöf but Y^{n+1} is not normal [5].

Example 1.1 is the space obtained from the reals by making the irrationals discrete (for a precise description, see Example 3.2). This space, which was studied in [10], is hereditarily paracompact and its product with the space $\boldsymbol{P}$ of irrationals is not normal. Examples 1.2 and 1.4 are subspaces of this space. The spaces of Example 1.3 are both subspaces of the plane with the 'bow-tie neighborhood' topology which was defined by R. W. Heath in [5]; for a precise description, see Example 3.6. Example 1.5, finally, is a subspace of the real line with the half-open interval topology which was introduced by R. Sorgenfrey in [16]; for a precise description, see Example 3.4. It follows that all five examples are first-countable and hereditarily paracompact, that Examples 1.1, 1.2, and 1.4 have a point-countable base, and that Examples 1.3 and 1.5 are hereditarily separable.

The paper is arranged as follows. Section 2 contains a summary of some known positive results, which may help to place our examples in proper perspective. Section 3 is devoted to some preliminary examples which form the building blocks for Examples 1.1–1.5, as well as some related lemmas and corollaries. Corollary 3.3 is an easy analogue of Examples 1.1 and 1.2 in which the factors are not all the same, while Corollary 3.7 provides a new proof of a recent result of E. S. Berney [2]. Example 1.1 is constructed in section 4, Examples 1.2 and 1.3 in section 5, and Examples 1.4 and 1.5 in section 6. The proofs become progressively more complicated. The paper concludes in section 7 with some open questions.

The reader may find that the proof of Lemma 3.1 serves as a helpful introduction to the proof of the more general Theorem 5.3, while the proof of Theorem 5.3 provides an introduction to the more complicated proof of Theorem 6.2.

The symbols $\boldsymbol{R}$, $\boldsymbol{Q}$, $\boldsymbol{P}$, and $\boldsymbol{N}$ will always denote, respectively, the reals, the rationals, the irrationals, and the positive integers, all with their usual topologies; the symbols R, Q, P, and N will always denote the underlying sets of these spaces. The symbol Ω denotes the first uncountable ordinal. Finally, the *weight* of a topological space (or a topology) is the smallest cardinality of a base.

[5] This example is new for $n = 2$. For $n = 1$, it is the classical example of Sorgenfrey [16].

2. A summary of positive results

In order to place our examples in proper perspective, let us summarize here some relevant positive results.

The following diagram lists some important classes of paracompact spaces which are preserved by countable products. Those classes whose elements are all Lindelöf (resp. hereditarily Lindelöf) are indicated by a star (resp. two stars). All these classes are preserved by closed subsets; the four classes in the upper right diamond are preserved by arbitrary subsets, and all their elements are perfectly normal.

Let us briefly explain the terms used. Call a collection $\mathscr{S}$ of subsets of X a *network* for X [1] if, whenever U is a neighborhood of x in X, then $x \in S \subset U$ for some $S \in \mathscr{S}$. A regular space is called *cosmic* [11] (resp. a *σ-space* [15]) if it has a countable (resp. σ-locally finite) network. For the definition of *Σ-spaces*, see [12]. The class of regular Lindelöf Σ-spaces coincides with the class of spaces defined in [7; Lemma 2.2].

All regular Lindelöf σ-spaces are cosmic, and so are all regular Σ-spaces X for which X^2 is hereditarily Lindelöf [12; Theorem 3.15].

$$
\begin{tikzcd}
 & & & (**)\text{ separable metrizable} \arrow[dl] \arrow[dr] & \\
(*)\text{ compact } T_2 \arrow[dr] & & (**)\text{ cosmic} \arrow[dl] \arrow[dr] & & \text{metrizable} \arrow[dl] \\
 & (*)\text{ regular Lindelöf } \Sigma \arrow[dr] & & \text{paracompact } \sigma \arrow[dl] & \\
 & & \text{paracompact } \Sigma & &
\end{tikzcd}
$$

In a somewhat different direction, N. Noble recently proved [13; Corollary 4.2] that a countable product of Lindelöf spaces, in each of which every G_δ-subset is open is again a Lindelöf space.

We conclude this section with the following result. Parts (a) and (b) are known (see below). Part (c) appears to be new, and is used to establish Example 1.3.

PROPOSITION 2.1. Let $X_1, X_2, \cdots$ be topological spaces.

(a) If, for all $n \in N$, $\prod_{i=1}^n X_i$ is a normal space in which every open subset is an F_σ, then so is $\prod_{i=1}^\infty X_i$.

(b) If, for all $n \in N$, $\prod_{i=1}^n X_i$ is a paracompact space in which every open subset is an F_σ, then so is $\prod_{i=1}^\infty X_i$.

(c) If, for all $n \in N$, $\prod_{i=1}^n X_i$ is a Lindelöf space in which every open subset is an F_σ, then so is $\prod_{i=1}^\infty X_i$.

PROOF. Part (a) was proved by M. Katětov [9; Theorem 2], and (b) by A. Okuyama in [14; Theorem 4.9] (where the result is credited to

K. Morita). The following proof of (c) is a simple adaptation of the proof of (b) in [14].

Suppose that, for all $n \in N$, $\prod_{i=1}^{n} X_i$ is Lindelöf with every open subset an F_σ. Then every open subset of $\prod_{i=1}^{\infty} X_i$ is an F_σ by [6; Proposition 2.1]. It remains to show that $\prod_{i=1}^{\infty} X_i$ is Lindelöf. So let $\mathscr{U}$ be an open cover of $\prod_{i=1}^{\infty} X_i$. For each $U \in \mathscr{U}$,

$$U = \bigcup_{n=1}^{\infty} (U(n) \times \prod_{i=n+1}^{\infty} X_i),$$

where $U(n)$ is open in $\prod_{i=1}^{n} X_i$. Let $V(n) = \bigcup \{U(n) : U \in \mathscr{U}\}$. Then $\{U(n) : U \in \mathscr{U}\}$ is an open cover of $V(n)$, and hence, since $\prod_{i=1}^{n} X_i$ is hereditarily Lindelöf (see footnote 2), there exists a countable subcollection $\mathscr{U}_n$ of $\mathscr{U}$ such that $V(n) = \bigcup \{U(n) : U \in \mathscr{U}_n\}$. But then

$$\{U(n) \times \prod_{i=n+1}^{\infty} X_i : U \in \mathscr{U}_n, n \in N\}$$

is a countable subcover of $\mathscr{U}$, and that completes the proof.

3. Some preliminary lemmas and examples

Examples 3.2, 3.4, and 3.6 in this section provide the foundation for the construction of Examples 1.1–1.5. Lemma 3.1 and the more general Lemma 3.5 are needed for Examples 3.2 and 3.6, respectively; they are special cases of Theorems 5.3 and 5.4, respectively.

LEMMA 3.1. (CH). *Let X be a T_1-space of weight $\leqq 2^{\aleph_0}$, and A a countable non-G_δ subset of X. Then there exists an uncountable Lindelöf subset Y of X containing A.*

PROOF. Let $\mathscr{U}$ be a base for X with card $\mathscr{U} \leqq 2^{\aleph_0}$; we may suppose that $\mathscr{U}$ is closed under countable unions. Let $\mathscr{U}^* = \{U \in \mathscr{U} : U \supset A\}$. Then (CH) implies that card $\mathscr{U}^* \leqq \aleph_1$, so we may write $\mathscr{U}^* = \{U_\alpha : \alpha < \Omega\}$. By transfinite induction, pick points $x_\alpha \in X - A$ for all $\alpha < \Omega$ such that

$$x_\alpha \in (\bigcap_{\beta < \alpha} U_\beta - \{x_\beta : \beta < \alpha\}) - A.$$

This can always be done, because A is not a G_δ in X. Let

$$Y = A \cup \{x_\alpha : \alpha < \Omega\}.$$

Clearly $Y - U_\alpha$ is countable for all $\alpha < \Omega$.

To see that Y is Lindelöf, let $\mathscr{V}$ be covering of Y by open subsets of X, and let us find a countable subcovering. We may suppose that $\mathscr{V} \subset \mathscr{U}$. Let $\mathscr{W}$ be a countable subcollection of $\mathscr{V}$ covering A, and let $W = \bigcup \mathscr{W}$. Then $W \in \mathscr{U}^*$, so $W = U_\alpha$ for some $\alpha < \Omega$. But then $Y - W$ is countable, and can therefore be covered by a countable subcollection $\mathscr{W}'$ of $\mathscr{V}$.

Hence $\mathscr{W} \cup \mathscr{W}'$ is a countable subcollection of $\mathscr{V}$ which covers Y, and that completes the proof.

The following example, which is used to establish Examples 1.1, 1.2, and 1.4, deals with a space which was studied in [10]. Part (c) was asserted without proof in [10; footnote 4], and is proved here with the aid of Lemma 3.1.

EXAMPLE 3.2. Let R^* be the space obtained from $\boldsymbol{R}$ by making the subset P discrete. In other words, R^* is the set R, topologized by taking as open sets all subsets of the form $U \cup T$, with U open in $\boldsymbol{R}$ and $T \subset P$.

(a) R^* is hereditarily paracompact and has a point-countable base.

(b) $R^* \times \boldsymbol{P}$ is not normal. More generally, $X \times \boldsymbol{P}$ is not normal if X is any subspace of R^* which contains Q as a non-G_δ subset.

(c) $X \times \boldsymbol{P}$ is not normal if X is any uncountable Lindelöf subspace of R^* containing Q.

(d) (CH). There exists an uncountable Lindelöf subspace of R^* containing Q.

PROOF. (a) It is known, and easy to verify, that any space obtained from a metric (or merely hereditarily paracompact) space by making a subset discrete is hereditarily paracompact. That R^* has a point-countable base is clear.

(b) That $R^* \times \boldsymbol{P}$ is not normal is proved in [10]. The same proof shows that $X \times \boldsymbol{P}$ is not normal for any $X \subset R^*$ which contains Q as a non-G_δ subset.

(c) If X is an uncountable Lindelöf subspace of R^* containing Q, then every neighborhood of Q in X has a countable complement in X, so that Q is not a G_δ in X, and hence $X \times \boldsymbol{P}$ is not normal by (b).

(d) This was asserted without proof in [10; footnote 4]. To prove it, note that Q is not a G_δ in $\boldsymbol{R}$, so it cannot be a G_δ in R^*. Hence our assertion follows from Lemma 3.1, with $X = R^*$ and $A = Q$.

That completes the proof.

We now apply Example 3.2 to give an elementary analogue of Examples 1.1 and 1.2 in which the factors are not all the same, but all except one are metrizable. The part dealing with Lindelöf spaces answers a question first raised by W. W. Comfort.

COROLLARY 3.3. *There exists a sequence of spaces* $X_1, X_2, \cdots$ (*with* $X_i = \boldsymbol{N}$ *for* $i > 1$) *such that* $\prod_{i=1}^{n} X_i$ *is hereditarily paracompact for all* $n \in N$, but $\prod_{i=1}^{\infty} X_i$ *is not normal. If* (CH) *is assumed, then we can also make* $\prod_{i=1}^{n} X_i$ *Lindelöf for all* $n \in N$.

PROOF. It was shown in Example 3.2 that there exists a hereditarily paracompact space X such that $X \times \boldsymbol{P}$ is not normal, and that X can be chosen Lindelöf if (CH) is assumed. Let $X_1 = X$ and $X_i = N$ for $i > 1$. This works because N^ω is homeomorphic to $\boldsymbol{P}$. That completes the proof.

Our next example deals with a space which was studied by R. Sorgenfrey in [16].

EXAMPLE 3.4. Let S be the Sorgenfrey line; that is, S is the set R, topologized by a base consisting of all half-open intervals $[a, b)$ with $a < b$.

(a) S is regular, first countable, hereditarily separable, and hereditarily Lindelöf.

(b) $S \times S$ is not normal.

(c) In S^ω, every open subset is an F_σ.

(d) If $n \in N$ and $r \in R$, then $D = \{x \in S^n : \sum_{i=1}^n x_i = r\}$ is closed and discrete in S^n.

(e) If $Y \subset S$ and Y is uncoutable, then Y is not cosmic.

PROOF. (a) That S is regular, first countable, and hereditarily separable is clear. That S is hereditarily Lindelöf is proved in [4; p. 164, Ex. 6].

(b) This is proved in [16] and, more simply, in [4; p. 144, Ex. 3].

(c) This is proved in [6].

(d) Clearly D is closed in $\boldsymbol{R}^n$ and hence in S^n. To see that D is discrete, note that, if $x \in D$, then $\{y \in D : y_i \geqq x_i \text{ for } i \leqq n\}$ is a neighborhood of x in D whose only element is x.

(e) To show that Y is not cosmic, we must show that every network $\mathscr{A}$ for S is uncountable: For each $y \in Y$, pick $A_y \in \mathscr{A}$ such that $y \in A_y$ and $A_y \subset [y, y+1)$. Then $A_x \neq A_y$ if $x \neq y$, and hence $\mathscr{A}$ is uncountable.

Before giving the final example of this section, we need the following generalization of Lemma 3.1. (Actually, we need the lemma only with card $\Lambda = 2$, but the general case is no harder to prove).

LEMMA 3.5. (CH). *Let $\{\mathscr{T}_\lambda : \lambda \in \Lambda\}$ be a family of T_1-topologies of weight $\leqq 2^{\aleph_0}$ on a set X, with card $\Lambda \leqq 2^{\aleph_0}$. Let A be a countable subset of X which is not the intersection of a countable subcollection of $\bigcup_{\lambda \in \Lambda} \mathscr{T}_\lambda$. Then there exists an uncountable subset Y of X containing A which is Lindelöf with respect to every $\mathscr{T}_\lambda$.*

PROOF. For each $\lambda \in \Lambda$, let $\mathscr{U}_\lambda$ be a base for $(X, \mathscr{T}_\lambda)$ of cardinality $\leqq 2^{\aleph_0}$; we may suppose that each $\mathscr{U}_\lambda$ is closed under countable unions. Let $\mathscr{U} = U_{\lambda \in \Lambda} \mathscr{U}_\lambda$, so that card $\mathscr{U} \leqq 2^{\aleph_0}$. Starting with this $\mathscr{U}$, we can now construct the required Y precisely as in the proof of Lemma 3.1. That completes the proof.

The following example was introduced by R. W. Heath [5; Remark 1, p. 105].

EXAMPLE 3.6. Let R^2 be the plane, and let $\mathscr{T}_1$ be the topology on R^2 generated by the base consisting of all 'horizontal bow-tie neighborhoods'

$$U_\varepsilon(x) = \{x\} \cup \{y \in R^2 : 0 < |y_1 - x_1| < \varepsilon, |m(x, y)| < \varepsilon\},$$

with $x \in R^2$ and $\varepsilon > 0$, where $m(x, y)$ denotes the slope of the line through x and y. Let $\mathscr{T}_2$ be the topology generated by the analogously defined 'vertical bow-tie neighborhoods'.

(a) $(R^2, \mathscr{T}_1)$ and $(R^2, \mathscr{T}_2)$ are homeomorphic, and they are completely regular and semi-metrizable.

(b) If $Q^2 \subset Z \subset R^2$ and card $Z = 2^{\aleph_0}$, then $(Z, \mathscr{T}_1) \times (Z, \mathscr{T}_2)$ is not normal.

(c) Q^2 is not the intersection of a countable subcollection of $\mathscr{T}_1 \cup \mathscr{T}_2$.

(d) (CH). There exists an uncountable subset Z of R^2 containing Q^2 such that $(Z, \mathscr{T}_1)$ and $(Z, \mathscr{T}_2)$ are homeomorphic and hereditarily Lindelöf.

PROOF. (a) The map $(x_1, x_2) \to (x_2, x_1)$ is a homeomorphism from $(X, \mathscr{T}_1)$ into $(X, \mathscr{T}_2)$. That $(X, \mathscr{T}_1)$ is completely regular is easily verified, and it is semi-metrizable by [5; Remark 1, p. 105].

(b) Let $E = (Z, \mathscr{T}_1) \times (Z, \mathscr{T}_2)$. Then $Q^2 \times Q^2$ is a countable dense subset of E, while $\{(x, x) : x \in Z\}$ is a closed, discrete subset of E of cardinality $2^{\aleph_0}$. Hence E is not normal by a theorem of F. B. Jones ([8] or [4; p. 144, Ex. 3]).

(c) If U is a $\mathscr{T}_1$-neighborhood of Q in R^2, then the $\boldsymbol{R}^2$-interior of U is dense in $\boldsymbol{R}^2$. Our assertion therefore follows from the Baire category theorem for $\boldsymbol{R}^2$.

(d) By (c) and Lemma 3.5, there exists an uncountable subset Y of R^2 containing Q^2 which is Lindelöf with respect to both $\mathscr{T}_1$ and $\mathscr{T}_2$. Let $Z = Y \cup Y^*$, where Y^* denotes $\{(x_2, x_1) : (x_1, x_2) \in Y\}$. Then $Z = Z^*$, so $(Z, \mathscr{T}_1)$ is homeomorphic to $(Z, \mathscr{T}_2)$. Now Y and Y^* are both $\mathscr{T}_1$-Lindelöf (the latter because Y is $\mathscr{T}_2$-Lindelöf), and hence so is Z. Since every open subset of a semi-metrizable space is an F_σ-subset, Z is hereditarily Lindelöf. That completes the proof.

The following result, which was recently obtained by E. S. Berney [2] by a somewhat different approach, follows immediately from Example 3.6 (b) and (d). (It also follows from Example 1.3.)

COROLLARY 3.7. (CH). *There exists a regular, hereditarily Lindelöf, semi-metrizable space Z such that Z^2 is not normal (and hence Z is not cosmic).*

4. Construction of Example 1.1

Let Y be the space R^* constructed in Example 3.2. Then Y is regular, $Y \times \boldsymbol{P}$ is not normal, and the set of non-isolated points of Y is Q and therefore is countable. That Y satisfies the requirements of Example 1.1 now follows immediately from Theorem 4.2 and Proposition 4.3 below. Before proving Theorem 4.2, we need a lemma which will also be used in the proof of Lemma 5.1.

LEMMA 4.1. *Let X be any topological space, and let $B \subset X$ with $X - B$ countable. If $n \in N$, then $X^n - B^n$ is the union of countably many subsets, each of which is a retract of X^n and homeomorphic to X^{n-1}.*

PROOF. For each $i \leqq n$ and each $a \in X - B$, let

$$Z_{i,a} = \{x \in X^n : x_i = a\}.$$

There are countably many such $Z_{i,a}$, and they clearly have all the required properties. That completes the proof.

THEOREM 4.2. *If X is a regular space with at most countably many non-isolated points, then X^n is paracompact for all $n \in N$.*

PROOF. By induction. For convenience, we let X^0 be a one-point set, so that the assertion is clear for $n = 0$. Now assume that X^{n-1} is paracompact and let us prove that X^n is paracompact. We denote the set of isolated points of X by B.

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X^n. By Lemma 4.1, there are retracts Z_j $(j \in N)$ of X^n which are homeomorphic to X^{n-1} such that

$$X^n - B^n = \bigcup_{j=1}^{\infty} Z_j.$$

By our inductive hypothesis, each Z_j is paracompact, so for each j there exists a locally finite relatively open (with respect to Z_j) refinement $\{V_{\lambda,j} : \lambda \in \Lambda\}$ of $\{U_\lambda \cap Z_j : \lambda \in \Lambda\}$. Let $r_j : X^n \to Z_j$ be a retraction for each $j \in N$, and let

$$W_{\lambda,j} = r_{\lambda,j}^{-1}(V_{\lambda,j}) \cap U_\lambda, \qquad \lambda \in \Lambda,\ j \in N.$$

Then $\mathscr{W} = \{W_{\lambda,j} : \lambda \in \Lambda, j \in N\}$ is a σ-locally finite collection of open subsets of X^n which covers $X^n - B^n$. But then $\mathscr{W}$, together with all $\{x\}$ with $x \in (X^n - \bigcup \mathscr{W})$, is a σ-locally finite open refinement of $\{U_\lambda : \lambda \in \Lambda\}$. Since X is regular, this implies that X is paracompact [4; p. 163, Theorem 2.3], and that completes the proof.

PROPOSITION 4.3. *If X is a space for which X^ω is normal, then $X \times \boldsymbol{P}$ is normal.*

PROOF. If X is countably compact, then $X \times \boldsymbol{P}$ is normal by a result, announced by A. H. Stone in [17; Footnote 2] and proved by J. Dieudonné in [3], which asserts that the product of a countably compact normal space and a metric space is normal [6].

If X is not countably compact, then X has a closed subset homeomorphic to $\boldsymbol{N}$. Hence X^{ω} has a closed subset which is homeomorphic to $\boldsymbol{N}^{\omega}$ and hence to $\boldsymbol{P}$. But X^{ω} is homeomorphic to $X \times X^{\omega}$, so X^{ω} has a closed subset homeomorphic to $X \times \boldsymbol{P}$, which implies that $X \times \boldsymbol{P}$ is normal. That completes the proof.

5. Construction of Examples 1.2 and 1.3

We begin with some preliminary results, the first of which will also be used in the next section. Example 1.2 is constructed after Theorem 5.3, and Example 1.3 after Theorem 5.4.

LEMMA 5.1. *Let Y be a topological space, and let $B \subset Y$ with $Y-B$ countable. Suppose that $n \in N$ and that, for each $m \leqq n$, the space Y^m has a base $\mathscr{W}_m$ which is closed under countable unions and has the property that $Y^m - W$ is countable whenever $W \in \mathscr{W}_m$ and $W \supset Y^m - B^m$. Then Y^n is Lindelöf.*

PROOF. We will prove by induction that Y^m is Lindelöf for all $m \leqq n$. This is clear for $m = 0$, where we define Y^0 to be a one-point space. We therefore assume that Y^{m-1} is Lindelöf, and will prove that Y^m is Lindelöf.

Let $\mathscr{U}$ be an open covering of Y^m, and let us find a countable subcovering. Without loss of generality, we may suppose that $\mathscr{U} \subset \mathscr{W}_m$. Now $Y^m - B^m$ is Lindelöf by our inductive hypothesis and Lemma 4.1, so $\mathscr{U}$ has a countable subcollection $\mathscr{V}$ which covers $Y^m - B^m$. Let $W = \bigcup \mathscr{V}$. Then $W \in \mathscr{W}_m$ and $W \supset Y^m - B^m$, so $Y^m - W$ is countable. Let $\mathscr{V}'$ be a countable subcollection of $\mathscr{U}$ which covers $Y^m - W$. Then $\mathscr{V} \cup \mathscr{V}'$ is a countable subcover of $\mathscr{U}$, and that completes the proof.

LEMMA 5.2. *Let X be a T_1-space, let A be a non-G_δ subset of X, and for each $m \in N$ let $\mathscr{U}_m$ be a collection of open subsets of X^m with card $\mathscr{U}_m \leqq \aleph_1$. Then there exists an uncountable subset B of $X - A$ such that, if $Y = A \cup B$, then $Y^m - U$ is countable whenever $m \in N$ and $U \in \mathscr{U}_m$ with $U \supset Y^m - B^m$.*

PROOF. Let $\mathscr{U} = \bigcup_{m=1}^{\infty} \mathscr{U}_m$. Then card $\mathscr{U} \leqq \aleph_1$, so we can write $\mathscr{U} = \{U_\alpha : \alpha < \Omega\}$. By transfinite induction, we will construct the

[6] This argument, which is needed to prove the proposition without superfluous hypotheses, was suggested by N. Noble. If X is assumed paracompact, which would suffice for our applications, the use of Stone's theorem in the proof can be avoided.

subset $B = \{y(\alpha) : \alpha < \Omega\}$ of $X - A$, with all $y(\alpha)$ distinct, so that, if $m \in N$, $U_\beta \in \mathscr{U}_m$, $U_\beta \supset Y^m - B^m$, and if $\alpha_1, \cdots, \alpha_m < \Omega$ with $\beta \leqq \max\{\alpha_1, \cdots, \alpha_m\}$, then $(y(\alpha_1), \cdots, y(\alpha_m)) \in U_\beta$. It is easily checked that this will suffice.

Let us suppose, as we may, that $U_0 = X$, and start the induction by letting $y(0)$ be any element of $X - A$. Now let $\alpha > 0$, and suppose that distinct y_β with $\beta < \alpha$ have been chosen so that our requirement is satisfied whenever $\max\{\alpha_1, \cdots, \alpha_m\} < \alpha$. We must choose $y(\alpha)$ so that it is also satisfied whenever $\max\{\alpha_1, \cdots, \alpha_m\} = \alpha$, and so that $y(\alpha) \neq y(\beta)$ for all $\beta < \alpha$. Clearly $g(\alpha) \neq y(\beta)$ if $\beta < \alpha$.

Let $B_\alpha = \{y(\beta) : \beta < \alpha\}$ and $Y_\alpha = A \cup B_\alpha$. For each $m \in N$, let

$$\Gamma_m = \{\beta \leqq \alpha : U_\beta \in \mathscr{U}_m,\ U_\beta \supset Y_\alpha^m - B_\alpha^m\}.$$

Then Γ_m is surely countable.

For each $m \in N$, let Φ_m be the set of all functions ϕ from $\{1, \cdots, m\}$ to $\{\beta : \beta \leqq \alpha\}$ such that $\phi(i) = \alpha$ for at least one $i \leqq m$. For each $\phi \in \Phi_m$, define $g_\phi : X \to X^m$ by

$$(g_\phi(x))_i = x \qquad \text{if } \phi(i) = \alpha,$$

$$(g_\phi(x))_i = y(\phi(i)) \qquad \text{if } \phi(i) < \alpha.$$

Let

$$W_m = \bigcap \{g_\phi^{-1}(U_\beta) : \phi \in \Phi_m, \beta \in \Gamma_m\},$$

$$W = \bigcap_{m=1}^{\infty} W_m - \{y(\beta) : \beta < \alpha\}.$$

Then W is a G_δ-subset of X containing A, and hence contains an element of $X - A$ which we take to be $y(\alpha)$.

It remains to show that our requirements are now satisfied whenever $\max\{\alpha_1, \cdots, \alpha_m\} = \alpha$. So suppose that $\beta \leqq \alpha$ and that $U_\beta \in \mathscr{U}_m$ and $U_\beta \supset Y^m - B^m$. Then $U_\beta \supset (Y_\alpha^m - B_\alpha^m)$, so $\beta \in \Gamma_m$. To show that $(y(\alpha_1), \cdots, y(\alpha_m)) \in U_\beta$, define $\phi \in \Phi_m$ by $\phi(i) = \alpha_i$ for $1 \leqq i \leqq m$. Then

$$g_\phi(y(\alpha)) = (y(\alpha_1), \cdots, y(\alpha_m)).$$

But $y_\alpha \in W \subset W_m \subset g_\phi^{-1}(U_\beta)$, so

$$g_\phi(y(\alpha)) \in U_\beta.$$

Hence $(y(\alpha_1), \cdots, y(\alpha_m)) \in U_\beta$, and that completes the proof.

The following theorem strengthens Lemma 3.1.

THEOREM 5.3. (CH). *Let X be a T_1-space of weight* $\leqq 2^{\aleph_0}$, and *A a countable, non-G_δ subset of X. Then X has an uncountable subset $Y \supset A$ such that Y^n is Lindelöf for all $n \in N$.*

PROOF. For each $m \in N$, the space X^m also has a base $\mathscr{U}_m$ of cardinality $\leqq 2^{\aleph_0}$, and we may suppose that $\mathscr{U}_m$ is closed under countable unions. By (CH), card $\mathscr{U}_m \leqq \aleph_1$. Now apply Lemma 5.2 to pick an uncountable subset B of $X-A$ such that, if $Y = A \cup B$, then $Y^m - U$ is countable whenever $m \in N$, $U \in \mathscr{U}_m$, and $U \supset Y^m - B^m$. Let $\mathscr{W}_m = \{U \cap Y^m : U \in \mathscr{U}_m\}$. Then Y, B, and the $\mathscr{W}_m$ satisfy the assumptions of Lemma 5.1 for all $n \in N$, so Y^n is Lindelöf for all $n \in N$. That completes the proof.

PROOF OF EXAMPLE 1.2. Let X be the space R^* of Example 3.2, and let $A = Q$. Then X and A satisfy the hypotheses of Theorem 5.3, so X has an uncountable subset $Y \supset A$ such that Y^n is Lindelöf for all $n \in N$. By Example 3.2, $Y \times \boldsymbol{P}$ is not normal, and hence Y^ω is not normal by Proposition 4.3. That completes the proof.

Before establishing Example 1.3, we need the following generalization of Theorem 5.3, which also strengthens Lemma 3.5. (As in Lemma 3.5, all we really need is the case where card $\Lambda = 2$).

THEOREM 5.4. (CH). *Let $\{\mathscr{T}_\lambda : \lambda \in \Lambda\}$ be a family of T_1-topologies of weight $\leqq 2^{\aleph_0}$ on a set X, with card $\Lambda \leqq 2^{\aleph_0}$. Let A be a countable subset of X which is not the intersection of a countable subcollection of $\bigcup_{\lambda \in \Lambda} \mathscr{T}_\lambda$. Then X has an uncountable subset Y containing A such that Y^n is Lindelöf with respect to every $\mathscr{T}_\lambda$ for all $n \in N$.*

PROOF. The proof is the same as for Theorem 5.3, with only minor verbal changes. In fact, Lemma 5.2 remains true (with unchanged proof) under the more general hypotheses of our theorem, provided $\mathscr{U}_m$ is a collection of subsets of X^m, each of which is $\mathscr{T}_\lambda$-open in X^m for some $\lambda \in \Lambda$. This generalization of Lemma 5.2 is then used to prove Theorem 5.4 in precisely the same way that Lemma 5.2 is used to prove Theorem 5.3. That completes the proof.

PROOF OF EXAMPLE 1.3. Let $X = R^2$, and let $\mathscr{T}_1$ and $\mathscr{T}_2$ be the two completely regular, semi-metrizable topologies on X defined in Example 3.6. Let $A = Q^2$. By Example 3.6(c), the assumptions of Theorem 5.4 are then satisfied, so X has an uncountable subset $Y \supset Q^2$ such that Y^n is Lindelöf with respect to both $\mathscr{T}_1$ and $\mathscr{T}_2$ for all $n \in N$.

Let $Y_1 = (Y, \mathscr{T}_1)$ and $Y_2 = (Y, \mathscr{T}_2)$. Then Y_1 and Y_2 are both semi-metrizable, and hence so are Y_1^n and Y_2^n for all $n \in N$. But every open subset of a semi-metrizable space is an F_σ, so that, by footnote 2, both Y_1^n and Y_2^n are hereditarily Lindelöf for all $n \in N$. Hence both Y_1^ω and Y_2^ω are hereditarily Lindelöf by Proposition 2.1(c).

That $Y_1 \times Y_2$ is not normal follows from Example 3.6(b). Hence neither Y_1 nor Y_2 is cosmic because, by [14; Theorem 4.6], the product of a cosmic space and an hereditarily Lindelöf space must be normal.

REMARK. In a similar way, the following special case of Example 1.3 can be obtained by applying Theorem 5.3 (which is simpler than Theorem 5.4) to Example 3.4: *If (CH) is assumed, there exists a regular, non-cosmic space Y such that Y^{ω} is hereditarily Lindelöf.* Unlike the space in Example 1.3, the space Y thus obtained is not semi-metrizable.

6. Construction of Examples 1.4 and 1.5

Throughout this section, X will denote the set R, equipped with a T_1-topology of weight $\leqq 2^{\aleph_0}$ such that, if $x \in Q$ and U is a neighborhood of x in X, then x is in the $\boldsymbol{R}$-closure of the $\boldsymbol{R}$-interior of U. Examples of such X are the space R^* of Example 3.2 and the space S of Example 3.4. We define

$$D_n = \{x \in X^n : \sum_{i=1}^{n} x_i = \sqrt{2}\},$$

and we call a subset of E of X^n *simple* if $y, y' \in E$ with $y \neq y'$ implies that $y_i \neq y'_i$ for all $i \leqq n$.

LEMMA 6.1. *Let $n \in N$, and for each $m \leqq n$ let $\mathscr{U}_m$ be a collection of open subsets of X^m with card $\mathscr{U}_m \leqq \aleph_1$. Then there exists a simple, uncountable subset E of $D_{n+1} \cap P^{n+1}$ satisfying the following condition, with $B = \{y_i : y \in E, i \leqq n+1\}$ and $Y = Q \cup B$.*

(a) If $m \leqq n$, and if $U \in \mathscr{U}_m$ and $U \supset Y^m - B^m$, then $Y^m - U$ is countable.

PROOF. Let $\mathscr{U} = \bigcup_{m=1}^{n} \mathscr{U}_m$. Then card $\mathscr{U} \leqq \aleph_1$, so we can write $\mathscr{U} = \{U_\alpha : \alpha < \Omega\}$. By transfinite induction, we will construct a simple subset $E = \{y(\alpha) : \alpha < \Omega\}$ of $D_{n+1} \cap P^{n+1}$, with $y(\alpha) \neq y(\alpha')$ whenever $\alpha \neq \alpha'$, to satisfy (with B and Y as above) the following condition (b) which is easily seen to imply (a):

(b) Suppose that $m \leqq n$, that $\alpha_1, \cdots, \alpha_m < \Omega$, that $\beta \leqq \max\{\alpha_1, \cdots, \alpha_m\}$, and that $U_\beta \in \mathscr{U}_m$ and $U_\beta \supset Y^m - B^m$. Then

$$(y_{i_1}(\alpha_1), \cdots, y_{i_m}(\alpha_m)) \in U_\beta$$

for all $i_1, \cdots, i_m \leqq n+1$.

To start our induction, we may suppose that $U_0 = X^n$, and we pick $y(0)$ to be any element of $D_{n+1} \cap P^{n+1}$. Now suppose that $\alpha > 0$, and that the $y(\beta)$ have been chosen for $\beta < \alpha$ to satisfy (b) whenever max $\{\alpha_1, \cdots, \alpha_m\} < \alpha$. We must choose $y(\alpha)$ so that $y_i(\alpha) \neq y_i(\beta)$ whenever $\beta < \alpha$ and $i \leqq n+1$, and such that (b) is satisfied for max $\{\alpha_1, \cdots, \alpha_n\} = \alpha$.

Let $B_\alpha = \{y_i(\beta) : \beta < \alpha, i \leqq n+1\}$ and $Y_\alpha = Q \cup B_\alpha$. For all $m \leqq n$, let

$$\Gamma_m = \{\beta \leqq \alpha : U_\beta \in \mathscr{U}_m, U_\beta \supset Y_\alpha^m - B_\alpha^m\}.$$

Then Γ_m is surely countable.

For each $m \leqq n$, let Φ_m be the family of all functions ϕ from $\{1, \cdots, m\}$ to $\{\beta : \beta \leqq \alpha\}$ such that the set $S(\phi) = \phi^{-1}(\alpha)$ is non-empty, and let Ψ_m be the family of all functions ψ from $\{1, \cdots, m\}$ to $\{1, \cdots, n+1\}$.

For each $\phi \in \Phi_m$ and each $\psi \in \Psi_m$, define [7]

$$g_{\phi\psi} : X^{S(\phi)} \to X^m$$

by

$$(g_{\phi\psi}(x))_j = x_j \qquad \text{if } j \in S(\phi),$$

$$(g_{\phi\psi}(x))_j = y_{\psi(j)}(\phi(j)) \quad \text{if } 1 \leqq j \leqq m \text{ and } j \notin S(\phi).$$

Clearly each $g_{\phi\psi}$ is continuous.

For each $m \leqq n$, $\beta \in \Gamma_m$, $\phi \in \Phi_m$ and $\psi \in \Psi_m$, let

$$U_{\beta\phi\psi} = g_{\phi\psi}^{-1}(U_\beta).$$

Then $U_{\beta\phi\psi}$ is open in $X^{S(\phi)}$, and our definitions imply that $U_{\beta\phi\psi} \supset Q^{S(\phi)}$. Let $V_{\beta\phi\psi}$ be the interior of $U_{\beta\phi\psi}$ with respect to $\boldsymbol{R}^{S(\phi)}$; our assumption about X at the beginning of this section implies that $V_{\beta\phi\psi}$ is dense in $\boldsymbol{R}^{S(\phi)}$.

Let $T = P - B_\alpha$. Then T is a dense G_δ in $\boldsymbol{R}$, so $T^{S(\phi)}$ is a dense G_δ in $\boldsymbol{R}^{S(\phi)}$.

Now consider D_{n+1} as a subspace of $\boldsymbol{R}^{n+1}$. Since D_{n+1} is closed in $\boldsymbol{R}^{n+1}$, it is a Baire space. For each $m \leqq n$ and $\phi \in \Phi_m$, let π_ϕ be the projection from D_{n+1} onto $\boldsymbol{R}^{S(\phi)}$. Then each π_ϕ is an open map, for if S is chosen so that $S(\phi) \subset S \subset \{1, \cdots, n+1\}$ and card $S = n$, then $\pi_\phi = p \circ q$, where q is a homeomorphism from D_{n+1} onto $\boldsymbol{R}^S$ and p is the open projection from $\boldsymbol{R}^S$ onto $\boldsymbol{R}^{S(\phi)}$ [8]. Hence if

$$W_{\beta\phi\psi} = \pi_\phi^{-1}(V_{\beta\phi\psi} \cap T^{S(\phi)}),$$

then each $W_{\beta\phi\psi}$ is a dense G_δ in D_{n+1}, and hence so is

$$W = \bigcap \{W_{\beta\phi\psi} : \beta \in \Gamma_m, \phi \in \Phi_m, \psi \in \Psi_m, m \leqq n\}.$$

In particular, W is non-empty, and we take $y(\alpha)$ to be any element of W.

It is clear that $y(\alpha) \in D_{n+1} \cap P^{n+1}$ and that $y_i(\alpha) \neq y_i(\beta)$ for all $\beta < \alpha$ and $i \leqq n+1$. We therefore only have to verify that (b) is now satisfied whenever $\max\{\alpha_1, \cdots, \alpha_m\} = \alpha$. So suppose that $\beta \leqq \alpha$, that $m \leqq n$, and that $U_\beta \in \mathscr{U}_m$ and $U_\beta \supset Y^m - B^m$. Then surely $U_\beta \supset Y^m_\alpha - B^m_\alpha$, so $\beta \in \Gamma_m$. Now let $i_1, \cdots, i_m \leqq n+1$ be given. Define $\phi \in \Phi_m$ by $\phi(j) = \alpha_j$

[7] Following the usual convention, X^A denotes $\prod_{j \in A} X_j$ with $X_j = X$ for all $j \in A$. Thus X^n is shorthand for $X^{\{1, \cdots, n\}}$.

[8] I am grateful to V. L. Klee for this observation.

for all $j \leqq m$, and define $\psi \in \Psi_m$ by $\psi(j) = i_j$ for all $j \leqq m$. Let $x = \pi_\phi(y(\alpha))$. Then $x \in X^{S(\phi)}$, and

$$g_{\phi\psi}(x) = (y_{i_1}(\alpha_1), \cdots, y_{i_m}(\alpha_m)). \tag{1}$$

On the other hand,

$$y(\alpha) \in W \subset W_{\beta\phi\psi} \subset \pi_\phi^{-1}(V_{\beta\phi\psi}) \subset \pi_\phi^{-1}(U_{\beta\phi\psi}),$$

so

$$x = \pi_\phi(y(\alpha)) \in U_{\beta\phi\psi} = g_{\phi\psi}^{-1}(U_\beta),$$

and hence

$$g_{\phi\psi}(x) \in U_\beta. \tag{2}$$

By (1) and (2) it follows that

$$(y_{i_1}(\alpha_1), \cdots, y_{i_m}(\alpha_m)) \in U_\beta,$$

and that completes the proof.

Retaining the hypotheses at the beginning of this section, we now have the following theorem.

THEOREM 6.2. (CH). *Let $n \in N$. Then X has a subspace Y such that Y^n is Lindelöf and such that $Y^{n+1} \cap D_{n+1}$ contains an uncountable, simple subset E.*

PROOF. For each $m \leqq n$, the space X^m has a base $\mathscr{U}_m$ of cardinality $\leqq 2^{\aleph_0}$, and we may suppose that each $\mathscr{U}_m$ is closed under countable unions. By (CH), card $\mathscr{U}_m \leqq \aleph_1$ for all $m \leqq n$. Now apply Lemma 6.1 to pick Y, $B \subset Y$, and $E \subset Y^{n+1}$. We need only show that Y^n is Lindelöf. To do that, let $\mathscr{W}_m = \{U \cap Y : U \in \mathscr{U}_m\}$ for all $m \leqq n$. Then Y, B, and the $\mathscr{W}_m$ satisfy the hypothesis of Lemma 5.1, so Y^n is Lindelöf. That completes the proof.

PROOF OF EXAMPLE 1.4. Let X be the space R^* of Example 3.2. Then all the hypotheses of Theorem 6.2 apply, so we can pick Y and E as in that theorem. That Y^n is Lindelöf is asserted in the theorem.

To show that Y^{n+1} is not Lindelöf, we will show that the uncountable subset E of Y^{n+1} is closed and discrete. Now D_{n+1} is closed in $\boldsymbol{R}^{n+1}$, and hence surely in X^{n+1}. We therefore need only show that, if $x \in D_{n+1}$, then x has a neighborhood in X^{n+1} containing at most one element of E. Now clearly $D_{n+1} \cap Q^{n+1} = \emptyset$, so $x_i \in P$ for some $i \leqq n+1$, and hence, remembering that E is simple, $\{x' \in X^{n+1} : x'_i = x_i\}$ is precisely such a neighborhood of x.

The paracompactness of X^{n+1} (in fact, of X^i for all $i \in N$) follows from Theorem 4.2, and that completes the proof.

PROOF OF EXAMPLE 1.5. Let X be the space S of Example 3.4. Then all the hypotheses of Theorem 6.2 apply, so we can pick Y and E as in that theorem. That Y^n is Lindelöf is asserted in the theorem. Since open subsets of X^n – and hence of Y^n – are F_σ by Example 3.4(c), it follows that Y^n is hereditarily Lindelöf (see footnote 2).

To see that Y^{n+1} is not normal, observe first that Y is separable by 3.4(a), so Y^{n+1} is also separable. Since we are assuming (CH), card $E = 2^{\aleph_0}$. By a result of F. B. Jones (see [8] or [4, p. 144, Ex. 3]), it therefore suffices to show that E is discrete and closed in Y^{n+1}. But D_{n+1} is discrete and closed in X^{n+1} by Example 3.4(d), and hence surely its subset E is discrete and closed in Y^{n+1}. That completes the proof.

7. Some open questions

7.1. Is (CH) essential in Examples 1.2–1.5 and elsewhere in this paper where it is assumed?

7.2. (M. Maurice). If X and Y are paracompact, and if $X \times Y$ is normal, must $X \times Y$ be paracompact? What if X or Y is metrizable?

7.3. The spaces Y which we chose for Examples 1.1, 1.2, and 1.4 are hereditarily paracompact, but Y^2 is not hereditarily normal (= completely normal) in any of those examples by [9; Corollary 1]. Can the spaces Y in these examples be chosen so that Y^n is hereditarily paracompact for all $n \in N$? (Note that, by Proposition 2.1(c) and footnote 2, the spaces Y^n can *not* all be chosen hereditarily Lindelöf in Example 1.2.)

7.4. Can the space Y in Example 1.5 be chosen semi-metrizable?

Added in proof: O. T. Alas has shown that Lemma 3.1 is equivalent to *(CH)*. It remains unknown, however, whether *(CH)* is needed in Examples 1.2–1.5 or in the modification of Lemma 3.1 obtained when $2^{\aleph_0}$ is replaced by $\aleph_1$. I am also grateful to O. T. Alas for pointing out that the second question in 7.2 was asked by H. Tamano on p. 351 of [Normality and product spaces, General topology and its relations to modern analysis and algebra. II. Academic Press (1967)].

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