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D. VAN DULST

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ON STRICTLY SINGULAR OPERATORS

by

D. van Dulst [1]

Introduction

In this paper we continue our study of strict singularity begun in the final chapter of [2]. In Section 1 we prove some theorems on linear operators in normed linear spaces. These are of some interest in themselves and will be used in Section 2 where we consider operators in locally convex spaces. In [2] the author introduced the notion of super strict singularity for such operators. Here we study super strictly singular operators in terms of operators they induce in associated normed spaces. T. Kato [4] showed that every strictly singular operator mapping a Hilbert space X into a Hilbert space Y is compact. We prove a close analogue of this: if E and F are locally convex spaces which are generalized Hilbert spaces, then every bounded super strictly singular operator mapping E into F is precompact. Here, a generalized Hilbert space is a locally convex space whose topology is generated by a system of seminorms arising from inner products. It is well-known that every nuclear space is such a generalized Hilbert space.

If we drop the assumption that E and F are generalized Hilbert spaces, the above result no longer holds. Nevertheless, if $E = F$, and with appropriate restrictions on E, a bounded super strictly singular operator B mapping E into itself closely resembles a compact operator in its spectral properties: it has at most countably many eigenvalues with 0 as the only possible accumulation point. Furthermore, for every complex $\lambda \neq 0$, $\lambda I - B$ is a homomorphism with finite ascent and finite descent.

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Throughout this section X and Y will be normed linear spaces, X' and Y' their duals and $\tilde{X}$ and $\tilde{Y}$ their completions. A linear operator B mapping X into Y is denoted by $B : X \to Y$. By this notation we imply that the domain of definition of B is all of X, unless otherwise stated. Further-

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more, 'subspace' always means 'linear subspace'. If M is a subspace of X, the restriction of B to M is denoted by B_M.

DEFINITION 1.1. A continuous linear operator $B : X \to Y$ is called *strictly singular* (*s.s.*) if it is not a topological isomorphism when restricted to any infinite-dimensional subspace of X.

The s.s. operators have been characterized as follows (cf. [3]).

THEOREM 1.2. *For a continuous linear operator $B : X \to Y$ the following statements are equivalent.*

(i) *B is s.s..*

(ii) *Given $\varepsilon > 0$ and given an infinite-dimensional subspace $M \subset X$, there exists an infinite-dimensional subspace $N \subset M$ such that B_N has norm not exceeding ε.*

In the next theorems we settle the following two questions.

1. If a linear operator $B : X \to Y$ satisfies (ii) above, does this imply that B is continuous? The most we can say, in general, is that for some subspace $L \subset X$ with $\dim X/L < \infty$, B_L is continuous. If Y is complete and B is closed, then B itself must be continuous.

2. If $B : X \to Y$ is s.s. and if $\tilde{B} : \tilde{X} \to \tilde{Y}$ is the unique continuous extension of B to the completion $\tilde{X}$, is $\tilde{B}$ s.s.? The answer turns out to be affirmative.

LEMMA 1.3. *Let X be an infinite-dimensional normed linear space. Given an arbitrary $\varepsilon \in (0, 1)$ and an arbitrary subspace $N \subset X$ with* $\dim N < \infty$, *there exists a subspace $M \subset X$ with* $\dim X/M < \infty$ *such that*

$$\inf_{\substack{x \in N \\ ||x||=1}} \inf_{y \in M} ||x+y|| > 1-\varepsilon \tag{1}$$

REMARK. The expression in the left member of (1) is the distance from the unit sphere of N to M. This distance is positive, so in particular $N \cap M = \{0\}$. Therefore we can define a projection P on $N+M$ by

$$P(x+y) = x \qquad (x \in N, y \in M)$$

It is easily seen that (1) is equivalent to $||P|| < (1-\varepsilon)^{-1}$.

PROOF. We choose finitely many unit vectors $x_1, \cdots, x_k$ in N such that for every $x \in N$, $||x|| = 1$ we have $||x-x_i|| < \varepsilon$ for some i, $1 \leqq i \leqq k$. By the Hahn-Banach theorem there exists for every x_i an $x'_i \in X'$ with

$$||x'_i|| = 1 \quad \text{and} \quad x'_i(x_i) = 1 \qquad (i = 1, \cdots, k).$$

Let $M = \bigcap_{i=1}^k N(x'_i)$, where $N(x'_i)$ denotes the null space of x'_i, Then clearly $\dim X/M < \infty$. We show that (1) holds.

Let $x \in N$, $||x|| = 1$ be arbitrary. Choose x_i such that $||x-x_i|| < \varepsilon$. Then we have for any $y \in M$,

$$||x+y|| \geqq ||x_i+y||-||x-x_i|| > 1-\varepsilon$$

since

$$||x_i+y|| \geqq |x'_i(x_i+y)| = |x'_i(x_i)| = 1.$$

This proves (1).

COROLLARY 1.4. *Let X, M, N and ε be as in the Lemma. Then for every $x_0 \in M$ there exists an $x'_0 \in X'$ such that*

$$x'_0(x_0) = ||x_0||,\ x'_0(x) = 0 \qquad \text{for } x \in N, \quad \text{and} \quad ||x'_0|| < 1+(1-\varepsilon)^{-1}.$$

PROOF. We may assume that $||x_0|| = 1$. Let x'_0 be defined on $\text{sp}\{x_0, N\}$ by

$$x'_0(\alpha x_0+x) = \alpha \qquad (x \in N).$$

We must show that x'_0 has norm $< 1+(1-\varepsilon)^{-1}$ on $\text{sp}\{x_0, N\}$. The Hahn-Banach theorem then gives an extension of x'_0 to X with the same bound. But, P being defined as above, we have for all α and for all $x \in N$,

$$|x'_0(\alpha x_0+x)| = |\alpha| = ||\alpha x_0|| = ||(I-P)(\alpha x_0+x)|| \leqq ||I-P||\,||\alpha x_0+x||.$$

Hence $||x'_0|| \leqq ||I-P|| \leqq 1+||P|| < 1+(1-\varepsilon)^{-1}$.

THEOREM 1.5. *Let $B : X \to Y$ be a linear operator. Suppose that there exists a constant $c > 0$ with the property that every infinite-dimensional subspace of X contains a vector x such that $||x|| = 1$ and $||Bx|| < c$. Then there exists a subspace $L \subset X$ with* dim $X/L < \infty$ *such that B_L is continuous.*

PROOF. Clearly we may assume that $\dim Y = \infty$. Suppose that no such L exists. This will lead to a contradiction.

We begin by choosing $x_1 \in X$ such that $||x_1|| = 1$ and $||Bx_1|| > 6c$. By the Hahn-Banach theorem we can select $x'_1 \in X'$ and $y'_1 \in Y'$ such that $||x'_1|| = ||y'_1|| = 1$ and $x'_1(x_1) = 1$, $y'_1(Bx_1) = ||Bx_1||$. We denote the null spaces of x'_1 and y'_1 by $N(x'_1)$ and $N(y'_1)$, respectively.

By Lemma 1.3 and Corollary 1.4, applied with $N^{(1)} = \text{sp}\{Bx_1\}$ and $\varepsilon < \frac{1}{2}$, there exists a subspace $M^{(1)} \subset Y$, $\dim Y/M^{(1)} < \infty$ such that for every $y \in M^{(1)}$ there exists a $y' \in Y'$ with

$$||y'|| < 3, \quad y'(Bx_1) = 0, \quad \text{and} \quad y'(y) = ||y||.$$

Putting $N_1 = N(x'_1) \cap B^{-1}N(y'_1) \cap B^{-1}M^{(1)}$, N_1 is a subspace of finite codimension in X. By our supposition B_{N_1} is therefore unbounded. Hence an $x_2 \in N_1$ exists with

$$||x_2|| = 1 \text{ and } ||Bx_2|| > m_2, \text{ where } m_2 = 2^2(6c+||Bx_1||)$$

Next we choose $x'_2 \in X'$, $y'_2 \in Y'$ such that

$$||x'_2|| = x'_2(x_2) = 1 \quad \text{and}$$
$$||y'_2|| < 3, y'_2(Bx_1) = 0, \quad y'_2(Bx_2) = ||Bx_2||,$$

again using the Hahn-Banach theorem. The choice of y'_2 is possible by the preceding, since $x_2 \in N_1$ implies $Bx_2 \in M^{(1)}$.

For the next step we apply Lemma 1.3 again, this time with $N^{(2)} = \text{sp}\{Bx_1, Bx_2\}$ and $\varepsilon < \frac{1}{2}$. If $M^{(2)}$ satisfies the Lemma for this choice of $N^{(2)}$ and ε, we put $N_2 = N_1 \cap N(x'_2) \cap B^{-1}N(y'_2) \cap B^{-1}M^{(2)}$. Obviously codim $N_2 < \infty$. By assumption B_{N_2} is then unbounded and we can find an $x_3 \in N_2$ such that $||x_3|| = 1$ and $||Bx_3|| > m_3$, where

$$m_3 = 2^3(9c + \sum_{k=1}^{2} 2^{k-1}||Bx_k||)$$

Next we choose $x'_3 \in X'$, $y'_3 \in Y'$ such that

$$||x'_3|| = x'_3(x_3) = 1$$

and

$$||y'_3|| < 3, \quad y'_3(Bx_1) = y'_3(Bx_2) = 0, \quad y'_3(Bx_3) = ||Bx_3||.$$

y'_3 can be so chosen by Corollary 1.4, since $Bx_3 \in M^{(2)}$.

Inductively, we select sequences (x_n) in X, (x'_n) in X' and (y'_n) in Y' such that

(1) $$||x'_k|| = ||x_k|| = 1 = x'_k(x_k) \qquad (k = 1, 2, \cdots),$$

(2) $$x'_k(x_i) = 0 \qquad \text{for } i > k,$$

(3) $$||Bx_k|| > m_k = 2^k(3kc + \sum_{i=1}^{k-1} 2^{i-1}||Bx_i||) \qquad (k = 2, 3, \cdots),$$

(4) $$y'_k(Bx_k) = ||Bx_k|| \qquad (k = 1, 2, \cdots),$$

(5) $$y'_k(Bx_i) = 0 \qquad \text{for } i \neq k.$$

(6) $$||y'_k|| < 3 \qquad (k = 1, 2, \cdots).$$

The sequence (x_n) is easily seen to be linearly independent. Its linear span $M = \text{sp}\{x_1, x_2, \ldots\}$ is therefore infinite-dimensional. We shall eventually show that M cannot contain an element x such that $||x|| = 1$ and $||Bx|| < c$, thus arriving at the desired contradiction.

Let $x = \sum_{k=1}^{n} \alpha_k x_k \in M$ be arbitrary. Then

$$|\alpha_1| = |x'_1(x)| \leqq ||x'_1|| \cdot ||x|| = ||x||.$$

By induction, we prove that

(7) $$|\alpha_k| \leqq 2^{k-1}||x||, \qquad 1 \leqq k \leqq n.$$

Suppose that (7) is true for all k such that $1 \leqq k \leqq j$, for some $j < n$. Since, by (2)

$$x'_{j+1}(x) = \sum_{i=1}^{j} \alpha_i x'_{j+1}(x_i) + \alpha_{j+1},$$

we have, by the induction hypothesis,

$$\begin{aligned} |\alpha_{j+1}| &\leqq |x'_{j+1}(x)| + \sum_{i=1}^{j} |\alpha_i|\, |x'_{j+1}(x_i)| \\ &\leqq ||x|| + \sum_{i=1}^{j} 2^{i-1}||x|| = 2^j||x||. \end{aligned}$$

This proves (7).

Since $\dim M = \infty$, there must exist an $x = \sum_{k=1}^{n} \alpha_k x_k \in M$ such that $||x|| = 1$ and $||Bx|| < c$. By (4) and (5), we have for $k \leqq n$

$$y'_k(Bx) = \sum_{i=1}^{n} \alpha_i y'_k(Bx_i) = \alpha_k ||Bx_k||,$$

so

$$\alpha_k = \frac{y'_k(Bx)}{||Bx_k||}.$$

This implies that for any $j \leqq n$,

$$||\sum_{k=1}^{j} \alpha_k Bx_k|| \leqq \sum_{k=1}^{j} \frac{|y'_k(Bx)|}{||Bx_k||} ||Bx_k|| \leqq 3j||Bx|| < 3jc,$$

since $||y'_k|| < 3$ for all k. Taking $j = 1$ we find that $||\alpha_1 Bx_1|| < 3c$. This implies, since $||Bx_1|| > 6c$, that $|\alpha_1| < \frac{1}{2}$. Also, for every $2 \leqq j \leqq n$,

$$\begin{aligned} 3jc > ||\sum_{k=1}^{j} \alpha_k Bx_k|| &\geqq ||\alpha_j Bx_j|| - ||\sum_{k=1}^{j-1} \alpha_k Bx_k|| \\ &> |\alpha_j| 2^j (3jc + \sum_{k=1}^{j-1} 2^{k-1}||Bx_k||) - \sum_{k=1}^{j-1} 2^{k-1}||Bx_k||. \end{aligned}$$

This implies that $|\alpha_j| < 2^{-j}$ $(j = 2, \cdots, n)$. But then

$$||x|| = ||\sum_{k=1}^{n} \alpha_k x_k|| \leqq \sum_{k=1}^{n} |\alpha_k| < \sum_{k=1}^{n} 2^{-k} < 1,$$

contrary to the choice of x. This completes the proof.

REMARK 1.6. The conclusion of the theorem is the strongest possible. We cannot expect continuity of B on all of X. Indeed, let $B : X \to Y$ be continuous. Choose a dense hyperplane $H \subset X$ with $0 \in H$ and an $x_0 \in X \setminus H$. Then $X = H \oplus \text{sp}\{x_0\}$. Define $B_1 : X \to Y$ by putting

$$B_1 x = Bx \text{ for } x \in H \text{ and}$$

$$B_1 x_0 \text{ arbitrary, but } \neq Bx_0,$$

and extending linearly. Then B_1 is not continuous on all of X, but still satisfies the hypothesis of the theorem.

From Theorem 1.5 we derive the following sufficient condition for a closed linear operator to be bounded.

COROLLARY 1.7. *Let $B : X \to Y$ be a closed linear operator and let Y be complete. Suppose that there exists a constant $c > 0$ with the property that every infinite-dimensional subspace of X contains an x such that $||x|| = 1$ and $||Bx|| < c$. Then B is continuous.*

PROOF. By the previous theorem there exists a subspace $L \subset X$ with $\dim X/L < \infty$ such that B_L is continuous. Let $\bar{L}$ be the closure of L in X. Then $B_{\bar{L}}$ is continuous, since it is closed, and B_L is continuous. Let N be any complementary subspace of $\bar{L}$ in X. Then $\dim N < \infty$ and X is the topological direct sum of $\bar{L}$ and N. Hence

$$B = B_{\bar{L}}P + B_N(I-P),$$

where P is the continuous projection of X onto $\bar{L}$ with null space N. Since B_N is continuous, this implies that B is continuous.

COROLLARY 1.8. *If Y is complete, then for a closed linear operator $B : X \to Y$ the statements* (i) *and* (ii) *in Theorem* 1.2 *are equivalent.*

PROOF. (ii) implies that the hypothesis of Theorem 1.5 is satisfied. By Corollary 1.7, B is then continuous. Therefore (i) holds by Theorem 1.2.

COROLLARY 1.9. *For an arbitrary linear operator $B : X \to Y$,* (ii) *is equivalent to*

(i′) *B_L is s.s., for some subspace $L \subset X$ with* $\dim X/L < \infty$.

PROOF. If (ii) holds for B, the condition of Theorem 1.5 is satisfied so that B_L is continuous for some L with $\dim X/L < \infty$. Clearly (ii) also holds for B_L. Hence B_L is s.s. by Theorem 1.2.

Conversely, if (ii) holds for some B_L, L a subspace of X with $\dim X/L < \infty$, then (ii) holds for B.

REMARK 1.10. For an arbitrary, not necessarily continuous linear operator $B : X \to Y$ there are two possible ways to define strict singularity:

(iii) B does not have a bounded inverse when restricted to any infinite-dimensional subspace.

(iv) B is not a topological isomorphism when restricted to any infinite-dimensional subspace.

Clearly, (iii) $\Leftrightarrow$ (iv) when B is continuous. It is not difficult to show that in general (iii) and (iv) are not equivalent. Finally, it is known that (iii) $\Leftrightarrow$ (ii) (cf. [3]). So, by Corollary 1.9, (iii) implies the continuity of B_L, where L is some subspace of X with $\dim X/L < \infty$.

We now proceed to deal with question 2.

THEOREM 1.11. *Let X, Y be normed linear spaces with completions $\tilde{X}$, $\tilde{Y}$, respectively. If $B : X \to Y$ is strictly singular, then its unique continuous extension $\tilde{B} : \tilde{X} \to \tilde{Y}$ is also s.s.*

PROOF. Suppose that $\tilde{B} : \tilde{X} \to \tilde{Y}$ is not s.s.. Then there is a subspace $M \subset \tilde{X}$, $\dim M = \infty$ and such that $\tilde{B}_M$ is an isomorphism. We may assume that $M \cap X = \{0\}$, since at any rate $\dim M \cap X < \infty$. Hence, for some $c > 0$ we have

$$\|\tilde{B}x\| > c\|x\| \quad \text{for all } x \in M.$$

Choose $x_1 \in M$ with $\|x_1\| = 1$. By the Hahn-Banach Theorem we can select $x'_1 \in M'$ with $\|x'_1\| = \|x_1\| = x'_1(x_1) = 1$. Choose $x_2 \in N(x'_1)$ with $\|x_2\| = 1$. Let $x'_2 \in M'$ be such that $\|x'_2\| = \|x_2\| = x'_2(x_2) = 1$. Choose $x_3 \in N(x'_1) \cap N(x'_2)$ with $\|x_3\| = 1$, etc..

Inductively, we construct sequences (x_n) in M, (x'_n) in M' such that

(1) $\|x_n\| = \|x'_n\| = x'_n(x_n) = 1, \qquad 1 \leqq n < \infty,$

(2) $x_n \in \bigcup_{i=1}^{n-1} N(x'_i)$ or, equivalently, $x'_i(x_k) = 0, \quad 1 \leqq i < k.$

It is easily verified that the sequence (x_n) is linearly independent. Then $N = \mathrm{sp}\,\{x_1, x_2, \cdots\}$ is an infinite-dimensional subspace of M. As in Theorem 1.5, we have for an arbitrary element $x = \sum_{i=1}^n \alpha_i x_i \in N$ that

$$|\alpha_i| \leqq 2^{i-1}\|x\|, \qquad 1 \leqq i \leqq n. \tag{3}$$

Let

$$A = \{(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) : \textstyle\sum \alpha_i x_i \in N \text{ and } \|\sum \alpha_i x_i\| = 1\}.$$

Then, for every sequence $(\alpha_1, \alpha_2, \cdots) \in A$ we have $\alpha_i = 0$ for i sufficiently large and, by (3), $|\alpha_i| \leqq 2^{i-1}$, $1 \leqq i < \infty$.

Let $(x_{1,k})_{k=1}^{\infty}$ be a sequence in X such that $\lim_{k\to\infty} x_{1,k} = x_1$. Then, for a fixed $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ we have

$$\lim_{k\to\infty} \frac{\|\alpha_1 Bx_{1,k} + \alpha_2 \tilde{B}x_2 + \cdots + \alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k} + \alpha_2 x_2 + \cdots + \alpha_n x_n\|} = \frac{\|\alpha_1 \tilde{B}x_1 + \alpha_2 \tilde{B}x_2 + \cdots + \alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|} > c.$$

Therefore

$$\frac{\|\alpha_1 Bx_{1,k}+\alpha_2 \tilde{B}x_2+\cdots+\alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k}+\alpha_2 x_2+\cdots+\alpha_n x_n\|} > c\,\frac{1-\frac{1}{2}}{1+\frac{1}{2}} \tag{4}$$

for k sufficiently large. We shall show that there exists a k_1 such that (4) holds for $k \geqq k_1$ and for all $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$. (Note that n varies with the particular element of A under consideration.)

Indeed, for every $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ we have

$$|\alpha_1| \leqq 1, \|\alpha_1 x_1+\cdots+\alpha_n x_n\| = 1, \ \|\alpha_1 \tilde{B}x_1+\cdots+\alpha_n \tilde{B}x_n\| > c.$$

Hence, for every $k \in \mathbb{N}$ and for every $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ we have

$$\begin{aligned} 1-\|x_1-x_{1,k}\| &\leqq \|\alpha_1 x_1+\cdots+\alpha_n x_n\|-\|\alpha_1(x_1-x_{1,k})\| \\ &\leqq \|\alpha_1 x_{1,k}+\alpha_2 x_2+\cdots+\alpha_n x_n\| \\ &\leqq \|\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n\|+\|\alpha_1(x_1-x_{1,k})\| \\ &\leqq 1+\|x_1-x_{1,k}\| \end{aligned}$$

and

$$\begin{aligned} \|\alpha_1 Bx_{1,k}+\alpha_2 \tilde{B}x_2+\cdots+\alpha_n \tilde{B}x_n\| &\geqq \|\alpha_1 \tilde{B}x_1+\cdots+\alpha_n \tilde{B}x_n\| \\ &\quad -\|\alpha_1(\tilde{B}x_1-Bx_{1,k})\| > c-\|\tilde{B}\|\,\|x_1-x_{1,k}\|. \end{aligned}$$

Therefore there is a $k_1 \in \mathbb{N}$ such that for all $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ we have

$$1-\tfrac{1}{2} \leqq \|\alpha_1 x_{1,k_1}+\alpha_2 x_2+\cdots+\alpha_n x_n\| \leqq 1+\tfrac{1}{2},$$
$$\|\alpha_1 Bx_{1,k_1}+\alpha_2 \tilde{B}x_2+\cdots+\alpha_n \tilde{B}x_n\| > c(1-\tfrac{1}{2}),$$

hence

$$\frac{\|\alpha_1 Bx_{1,k_1}+\alpha_2 \tilde{B}x_2+\cdots+\alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k_1}+\alpha_2 x_2+\cdots+\alpha_n x_n\|} > c\,\frac{1-\frac{1}{2}}{1+\frac{1}{2}}.$$

Next we choose a sequence $(x_{2,k})_{k=1}^{\infty}$ in X such that $\lim_{k\to\infty} x_{2,k} = x_2$. Then

$$\begin{aligned} &\lim_{k\to\infty} \frac{\|\alpha_1 Bx_{1,k_1}+\alpha_2 Bx_{2,k}+\alpha_3 \tilde{B}x_3+\cdots+\alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k_1}+\alpha_2 x_{2,k}+\alpha_3 x_3+\cdots+\alpha_n x_n\|} \\ &\qquad = \frac{\|\alpha_1 Bx_{1,k_1}+\alpha_2 \tilde{B}x_2+\alpha_3 \tilde{B}x_3+\cdots+\alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k_1}+\alpha_2 x_2+\alpha_3 x_3+\cdots+\alpha_n x_n\|} > c\,\frac{1-\frac{1}{2}}{1+\frac{1}{2}} \end{aligned}$$

for every fixed $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$. Since for every $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ we have $|\alpha_2| \leqq 2$ and

$$\begin{aligned}1-\tfrac{1}{2}-2\|x_{2,k}-x_2\| &\leqq \|\alpha_1 x_{1,k_1}+\alpha_2 x_2+\alpha_3 x_3+\cdots+\alpha_n x_n\|\\ &\quad -\|\alpha_2(x_{2,k}-x_2)\|\\ &\leqq \|\alpha_1 x_{1,k_1}+\alpha_2 x_{2,k}+\alpha_3 x_3+\cdots+\alpha_n x_n\|\\ &\leqq \|\alpha_1 x_{1,k_1}+\alpha_2 x_2+\alpha_3 x_3+\cdots+\alpha_n x_n\|\\ &\quad +\|\alpha_2(x_{2,k}-x_2)\|\\ &\leqq 1+\tfrac{1}{2}+2\|x_{2,k}-x_2\|\end{aligned}$$

as well as

$$\begin{aligned}&\|\alpha_1 Bx_{1,k_1}+\alpha_2 Bx_{2,k}+\alpha_3 \tilde{B}x_3+\cdots+\alpha_n \tilde{B}x_n\|\\ &\quad \geqq \|\alpha_1 Bx_{1,k_1}+\alpha_2 \tilde{B}x_2+\alpha_3 \tilde{B}x_3+\cdots+\alpha_n \tilde{B}x_n\|-\|\alpha_2(Bx_{2,k}-\tilde{B}x_2)\|\\ &\quad \geqq c(1-\tfrac{1}{2})-2\|\tilde{B}\|\,\|x_{2,k}-x_2\|,\end{aligned}$$

there exists a $k_2 \in \mathbb{N}$ such that for all $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ we have

$$\begin{aligned}(1-\tfrac{1}{2})(1-1/2^2) &\leqq \|\alpha_1 x_{1,k_1}+\alpha_2 x_{2,k_2}+\alpha_3 x_3+\cdots+\alpha_n x_n\|\\ &\leqq (1+\tfrac{1}{2})(1+1/2^2)\end{aligned}$$

and

$$\|\alpha_1 Bx_{1,k_1}+\alpha_2 Bx_{2,k_2}+\alpha_3 \tilde{B}x_3+\cdots+\alpha_n \tilde{B}x_n\| > c(1-\tfrac{1}{2})(1-1/2^2)$$

and therefore

$$\frac{\|\alpha_1 Bx_{1,k_1}+\alpha_2 Bx_{2,k_2}+\alpha_3 \tilde{B}x_3+\cdots+\alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k_1}+\alpha_2 x_{2,k_2}+\alpha_3 x_3+\cdots+\alpha_n x_n\|} > c\,\frac{(1-\frac{1}{2})(1-1/2^2)}{(1+\frac{1}{2})(1+1/2^2)}.$$

We take care to choose x_{2,k_2} so that it is independent of x_{1,k_1}. This can be done by choosing k_2 sufficiently large. (Note that x_2 and x_{1,k_1} are linearly independent, since $M \cap X = \{0\}$.)

Inductively, using the fact that $|\alpha_k| \leqq 2^{k-1}$ for every $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$, we choose a linearly independent sequence $(x_{n,k_n})_{n=1}^{\infty}$ in X such that, for every $j \in \mathbb{N}$ and for every $(\alpha_1, \cdots, \alpha_n, 0, \cdots) \in A$ the following inequality holds.

$$\frac{\|\alpha_1 Bx_{1,k_1}+\alpha_2 Bx_{2,k_2}+\cdots+\alpha_j Bx_{j,k_j}+\alpha_{j+1}\tilde{B}x_{j+1}+\cdots+\alpha_n \tilde{B}x_n\|}{\|\alpha_1 x_{1,k_1}+\alpha_2 x_{2,k_2}+\cdots+\alpha_j x_{j,k_j}+\alpha_{j+1} x_{j+1}+\cdots+\alpha_n x_n\|} > c\prod_{i=1}^{j}\frac{1-2^{-i}}{1+2^{-i}}.$$

Let
$$\gamma = \prod_{i=1}^{\infty}\frac{1-2^{-i}}{1+2^{-i}} > 0.$$

In particular, we then have for every $j \in \mathbb{N}$ and for every $(\alpha_1, \alpha_2, \cdots, \alpha_j, 0, \cdots) \in A$,

$$(5) \qquad \frac{||\alpha_1 Bx_{1,k_1}+\alpha_2 Bx_{2,k_2}+\cdots+\alpha_j Bx_{j,k_j}||}{||\alpha_1 x_{1,k_1}+\alpha_2 x_{2,k_2}+\cdots+\alpha_j x_{j,k_j}||} > c\gamma > 0.$$

The linear subspace $L = \mathrm{sp}\{x_{1,k_1}, x_{2,k_2}, \cdots\}$ is infinite dimensional and contained in X. If $x = \sum_{i=1}^{n} \alpha_i x_{i,k_i} \neq 0$ is an arbitrary element of L, there exists a constant $\alpha \neq 0$ such that $(\alpha\alpha_1, \cdots, \alpha\alpha_n, 0, \cdots) \in A$. Hence, in virtue of (5) and the homogeneity of the norm, we have

$$\frac{||Bx||}{||x||} > c\gamma.$$

The last inequality implies that B_L is an isomorphism, which contradicts the fact that B is s.s.. This completes the proof.

COROLLARY 1.12. *Let $B : X \to Y$ be a linear operator which does not have a bounded inverse when restricted to any infinite-dimensional subspace. Then B has a linear extension $\bar{B} : \tilde{X} \to \tilde{Y}$ with the same property.*

PROOF. By Corollary 1.9 and Remark 1.10 there is a subspace $L \subset X$ with $\dim X/L < \infty$ such that B_L is s.s.. We may identify $\tilde{L}$ with the closure of L in $\tilde{X}$. Since $\dim X/L < \infty$, also $\dim \tilde{X}/\tilde{L} < \infty$ and

$$\tilde{X} = \tilde{L}+N_1, \text{ with } \tilde{L}\cap N_1 = \{0\}, N_1 \subset X, \dim N_1 < \infty.$$

Also $\tilde{L}$ can be written as

$$\tilde{L} = M+N_2, \text{ with } M \cap N_2 = \{0\},$$
$$M \cap X = L, N_2 \subset X \text{ and } \dim N_2 < \infty.$$

Hence $\tilde{X} = M+N_1+N_2$.

We define $\bar{B}$ on $\tilde{X}$ by putting

(1) $\qquad \bar{B}x = Bx \quad$ for $x \in N_1+N_2$,

(2) $\qquad \bar{B}x = \tilde{B}_L x \quad$ for $x \in M$,

where $\tilde{B}_L$ is the unique continuous extension of B_L to $\tilde{L}$,

and extending linearly to $\tilde{X} = M+N_1+N_2$. Then $\bar{B} = B$ on X. Also $\bar{B}$ is continuous on M and therefore $\bar{B}_M$ is s.s. by Theorem 1.11. Since $\dim \tilde{X}/M < \infty$, $\bar{B}$ does not have a bounded inverse on any infinite-dimensional subspace of $\tilde{X}$ (Cf. Corollary 1.9 and Remark 1.10).

2

In this section E and F will be locally convex spaces (l.c.s.). 0-neighborhoods U and V will always be assumed to be absolutely convex and closed. $L(E, F)$ is the set of all continuous linear operators mapping E

into F. The null space and the range of a linear operator $B : E \to F$ are denoted by $N(B)$ and $R(B)$, respectively.

DEFINITION 2.1. A linear operator $B : E \to F$ is said to be *bounded* (*precompact*, *compact*) if there exists a 0-neighborhood U in E such that BU is a bounded (precompact, relatively compact) subset of F.

THEOREM 2.2. [2]. *A continuous linear operator $B : E \to F$ is precompact if and only if there exists a 0-neighborhood U in E with the property that for every 0-neighborhood V in F there is a closed subspace $L \subset E$ of the form $L = \bigcap_{i=1}^{n} N(x_i')$, with $x_i' \in E'$ and x_i' bounded on U $(i = 1, \cdots, n)$ such that $B(U \cap L) \subset V$.*

In [2] the author introduced the following generalization of precompactness that proved useful in stability theory.

DEFINITION 2.3. *A continuous linear operator $B : E \to F$ is called super strictly singular* (*s.s.s.*) *if there exists a 0-neighborhood U in E with the property that for every infinite-dimensional subspace $M \subset E$ such that $M \cap N(B) = \{0\}$ and for every 0-neighborhood V in F there exists an infinite-dimensional subspace $N \subset M$ such that $N \not\subset U$ and $B(U \cap N) \subset V$.*

The s.s.s. operators mapping E into F form a two sided ideal in $L(E, F)$ which coincides with the s.s. operators when E and F are normed linear spaces. For the proofs of these statements and for a theorem concerning perturbations of Fredholm operators by s.s.s. operators we refer to [2].

Let U be a 0-neighborhood in E and let p_U be its gauge. Then E_U is the quotient space $E/p_U^{-1}(0)$ equipped with the norm

$$\|[x]\| = p_U(x) \qquad (x \in [x], [x] \in E_U).$$

The completion of E_U is denoted by $\tilde{E}_U$.

DEFINITION 2.4. *A l.c.s. E is called a generalized Hilbert space if there exists a 0-neighborhood base $\mathfrak{U}$ such that for all $U \in \mathfrak{U}$ the spaces $\tilde{E}_U$ are Hilbert spaces.*

The next result is due to T. Kato [4].

THEOREM 2.5. *If X and Y are Hilbert spaces, then every s.s. operator $B : X \to Y$ is compact.*

In the following we shall prove a more general statement (Corollary 2.7).

THEOREM 2.6. *Let E and F be generalized Hilbert spaces and let $B : E \to F$ be s.s.s.. Then there exists a 0-neighborhood U in E with the property that for every 0-neighborhood V in F there exists a subspace $L \subset E$ with* $\dim E/L < \infty$ *and such that $\phi_V B(U \cap L)$ is precompact in F_V, where $\phi_V : F \to F_V$ is the quotient map.*

PROOF. Let U be a 0-neighborhood in E satisfying the condition of Definition 2.3. If p_U is the gauge of U, then $p_U^{-1}(0) = \{x : x \in E \text{ and } p_U(x) = 0\}$ is a subspace of E and Definition 2.3 implies that $\dim p_U^{-1}(0)/N(B) \cap p_U^{-1}(0) < \infty$. Therefore, replacing U by a smaller 0-neighborhood if necessary, we may assume that $p_U^{-1}(0) \subset N(B)$. Let V be an arbitrary 0-neighborhood in F. Then, in virtue of $p_U^{-1}(0) \subset N(B)$, B induces a linear operator $B^{U,V} : E_U \to F_V$ defined by

$$B^{U,V}[x] = [Bx] \qquad (x \in [x], [x] \in E_U).$$

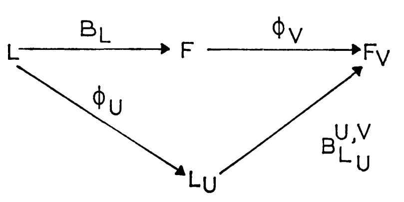
It follows from Definition 2.3 that $B^{U,V}$ satisfies (ii) of Theorem 1.2, with X and Y replaced by E_U and F_V, respectively. Although $B^{U,V}$ might not be continuous, by Theorem 1.5 there exists a subspace $L_U \subset E_U$ with $\dim E_U/L_U < \infty$ such that the restriction $B_{L_U}^{U,V}$ is continuous. Therefore $B_{L_U}^{U,V}$ is s.s. (cf. Corollary 1.9). Our assumption that E and F are generalized Hilbert spaces allows us to assume that the completions $\tilde{L}_U$ and $\tilde{F}_V$ are Hilbert spaces. By Theorem 1.11 the unique continuous extension $(B_{L_U}^{U,V})^{\sim} : \tilde{L}_U \to \tilde{F}_V$ is also s.s.. Hence, by Theorem 2.5, $(B_{L_U}^{U,V})^{\sim}$ is compact and therefore $B_{L_U}^{U,V} : L_U \to F_V$ is precompact. $\phi_U : E \to E_U$ being the quotient map, let $L = \phi_U^{-1}(L_U)$. Then clearly $\dim E/L < \infty$ and $\phi_V B(U \cap L)$ is precompact in F_V.

COROLLARY 2.7. *Let, in addition to the hypotheses of Theorem* 2.6, *be given that B is bounded. Then B is precompact.*

PROOF. We can choose U in the above proof so small that BU is bounded in F. Then it is obvious that $B^{U,V} : E_U \to F_V$ is continuous for every V. Therefore, in the preceding proof we can take $L_U = E_U$, hence $L = E$. Hence $B^{U,V} : E_U \to F_V$ is precompact for every V. This implies that $B : E \to F$ is precompact, since V is arbitrary.

Implicit in the proof of Theorem 2.6 is the following.

COROLLARY 2.8. *Let $B : E \to F$ be s.s.s.. Then there exists a* 0*-neighborhood U in E such that for every* 0*-neighborhood V in F there is a subspace $L \subset E$ with* $\dim E/L < \infty$ *for which the following diagram commutes*



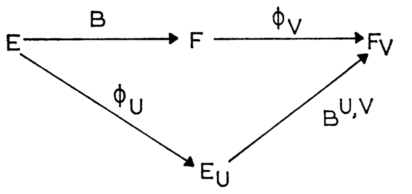
where ϕ_U, ϕ_V are the quotient maps, $L_U = \phi_U(L)$, $L = \phi_U^{-1}(L_U)$, and $B_{L_U}^{U,V}$ is s.s..

In the special case when E and F are generalized Hilbert spaces, U can be so chosen that all $B_{L_U}^{U,V}$ *are precompact.*

If B is bounded, L in the diagram can be replaced by E.

For bounded operators the converse also holds.

THEOREM 2.9. *Let* $B : E \to F$ *be bounded. Then B is s.s.s. if and only if there exists a* 0*-neighborhood U in E such that for every* 0*-neighborhood V in F the diagram*



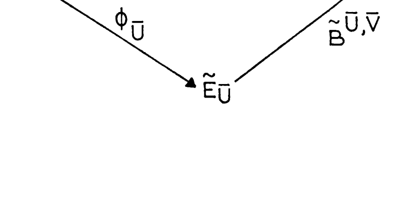
commutes and $B^{U,V}$ *is s.s.*

PROOF. Let U be as above. Then obviously $p_U^{-1}(0) \subset N(B)$. Indeed, for every $x \in E$, $x \notin N(B)$ we have $\phi_V Bx \neq 0$ for some V. Hence $\phi_U(x) \neq 0$, since the diagram commutes.

If M is a subspace of E with $\dim M = \infty$ and $M \cap N(B) = \{0\}$, then $\dim \phi_U M = \infty$. Let V be an arbitrary 0-neighborhood in F. Then, by the strict singularity of $B^{U,V}$, there exists a subspace $N \subset \phi_U M$ with $\dim N = \infty$ such that $B^{U,V} N \subset V$. Then $(\phi_U^{-1} N) \cap M$ is an infinite-dimensional subspace of M, not contained in U, with $B((\phi_U^{-1} N) \cap M \cap U) \subset V$. We have shown that U satisfies Definition 2.3. Hence B is s.s.s.

COROLLARY 2.10. *If* $B : E \to F$ *is bounded and s.s.s., then its unique continuous extension* $\tilde{B} : \tilde{E} \to \tilde{F}$, $\tilde{E}$ *and* $\tilde{F}$ *the completions of E and F respectively, is also bounded and s.s.s..*

PROOF. Let $\mathfrak{U}$ and $\mathfrak{V}$ be 0-neighborhood bases of E and F, respectively. If $\bar{U}$ $(U \in \mathfrak{U})$ and $\bar{V}$ $(V \in \mathfrak{V})$ are the closures of U and V in $\tilde{E}$ and $\tilde{F}$, respectively, then $\{\bar{U} : U \in \mathfrak{U}\}$ and $\{\bar{V} : V \in \mathfrak{V}\}$ are 0-neighborhood bases in the completed spaces. Clearly E_U is a dense subspace of $\tilde{E}_{\bar{U}}$, so $(E_U)^{\sim} \supset \tilde{E}_{\bar{U}}$ and likewise $(F_V)^{\sim} \supset \tilde{F}_{\bar{V}}$. Let U be as in Theorem 2.9. We only have to prove, by Theorem 2.9, that $\tilde{B}\bar{U}$ is bounded in $\tilde{F}$, which is trivial, and that for every $\bar{V}$ the diagram



commutes and $\tilde{B}^{\bar{U},\bar{V}}$ is s.s.. Obviously the diagram commutes since all the operators appearing in it are continuous extensions of the operators in the corresponding commuting diagram for B. Also, for every V, $\tilde{B}^{\bar{U},\bar{V}}$ coincides with the restriction to $\tilde{E}_{\bar{U}}$ of the operator $(B^{U,V})^{\sim} : (E_U)^{\sim} \to (F_V)^{\sim}$. The latter is s.s. by Theorem 1.11, therefore $\tilde{B}^{\bar{U},\bar{V}}$ is also s.s. This completes the proof.

Finally, we turn our attention to spectral properties of a s.s.s. operator B mapping a l.c.s. E into itself. G. F. C. de Bruyn [1] introduced the following:

DEFINITION 2.11. A linear operator $B : E \to E$ is called a *Riesz-transformation* if the following holds.

(a) For any complex $\lambda \neq 0$

(i) $\lambda I - B$ is a σ-transformation (i.e., $\lambda I - B$ is a homomorphism with $\dim N(\lambda I - B) < \infty$, $\dim E/R(\lambda I - B) < \infty$ and $R(\lambda I - B)$ closed),

(ii) the ascent and descent of $\lambda I - B$ are finite.

(b) The eigenvalues of B form a finite set or a sequence convergent to 0.

The next theorem is due to de Bruyn [1].

THEOREM 2.12. *If $B : E \to E$ is a bounded linear operator, then B is a Riesz-transformation if and only if $\lambda I - B$ is a σ-transformation for every non-zero λ.*

THEOREM 2.13. *Let E be a Fréchet space which is superprojective (cf.* [2]*). Then every bounded s.s.s. operator $B : E \to E$ is a Riesz-transformation.*

PROOF. If E is superprojective, it follows from [2, III 2.4] that for every $\lambda \neq 0$, $\lambda I - B$ is a σ-transformation. By Theorem 2.12, B is then a Riesz-transformation.

REMARK 2.14. The hypothesis that E is a Fréchet space may be weakened. It is sufficient that E is fully barreled and $E \times E$ is a Ptak space (cf. [2]).

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University of Maryland
Department of Mathematics
College Park, Maryland 20742
U.S.A.