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A MODEL FOR BARRECURSION OF HIGHER TYPES

by

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Introduction

Problems connected with the foundations of mathematics led C. Spector to consider a certain kind of functional equation. The solution of this functional equation is provided by a certain principle, the 'principle of barrecursion". One problem with respect to this functional equation has up to now remained open, namely to find a family $\boldsymbol{F}$ of functionals with the property: if the parameters of the equation belong to $\boldsymbol{F}$ then there is a solution which belongs to $\boldsymbol{F}$. There is so to speak a weak and a strong version of this problem: a) the weak version is that one given above, b) the strong version requires that the elements of $\boldsymbol{F}$ are constructive in one sense or the other. Here we propose a solution of the weak problem. More precisely, we construct two families $\boldsymbol{S}$ and $\boldsymbol{K}$. The first is in essence already described in [2] but it has the disadvantage that its construction leads beyond classical analysis. The second, $\boldsymbol{K}$, is a more elaborate version of $\boldsymbol{S}$ and its construction remains within the scope of classical analysis. A few applications of these models are given.

I. A family of functionals of higher types

1.1. *Syntax*

Our notation follows loosely the one used in [4]. Types are inductively given as follows: 1) 0 is a type, 2) if $\sigma_1, \cdots, \sigma_s, \tau$ are types then $(\sigma_1, \cdots, \sigma_s/\tau)$ is a type. For each type σ there is a denumerable list of free variables $X_1^\sigma, X_2^\sigma, \cdots, Y_1^\sigma, Y_2^\sigma, \cdots$; for simplicity we often omit the superscripts. For every type σ there is also a denumerable list of bound variables $B_1^\sigma, B_2^\sigma, \cdots, B$. In addition we have a symbol λ, called abstraction operator and two kind of brackets, (,) and [,].

1.2. *Topologies defined by convergence*

Let X be a set on which a notion of convergence, denoted by $\rightarrow$, is given. By $p_n \rightarrow p$, $n = 1, 2, \cdots$ we understand that the sequence $p_1, p_2, \cdots$ converges against p; we often simply write $p_n \rightarrow p$. We assume that $\rightarrow$

satisfies the following axioms: 1) if $p_n \to p$, if $k_1 < k_2 < \cdots$ then $p_{k_i} \to p$, $i = 1, 2, \cdots$, 2) if $p_n = p$ with the exception of finitely many n's then $p_n \to p$, 3) if p_n, $n = 1, 2, \cdots$ does not converge against p then there is a list $k_1 < k_2 < \cdots$ such that no subsequence of $p_{k_1}, p_{k_2}, \cdots$ converges against p, 4) if $p_n \to p$, $p_n \to q$ then $p = q$. The pair $(X, \to)$ will be called an L-space. Such spaces have already been investigated among others by Kuratowski [2, pg. 93]. We refer to axioms 1)–4) as to Kuratowski's axioms. Below we have to consider different L-spaces $(X_1, \to_1)$, $(X_2, \to_2)$, $\cdots$; then we often omit the superscripts in $\to_1$, $\to_2$, $\cdots$ and write simply $\to$ since it will be clear from the context on what space $\to$ is supposed to operate.

1.3. *Topologies on families of continuous functions*

Let $(X_i, \to)$ $i = 1, \cdots, s$ and $(Y, \to)$ be L-spaces. A mapping f from $X_1 \times \cdots \times X_s$ into Y is said to be continuous if $f(x_1^n, \cdots, x_s^n) \to f(x_1, \cdots, x_s)$ whenever $x_i^n \to x_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ (with x_i^n and x_i all in X_i for $n = 1, 2, \cdots$, $i = 1, 2, \cdots, s$). By $C(X_1, \cdots, X_s, Y)$ we denote the set of continuous mappings from $X_1 \times \cdots \times X_s$ into Y. On $C(X_1, \cdots, X_s, Y)$ we introduce a notion of convergence according to [2] as follows: if f and f_n, $n = 1, 2, \cdots$ belong to $C(X_1, \cdots, X_s, Y)$ then $f_n \to f$, $n = 1, 2, \cdots$ iff $f_n(x_1^n, \cdots, x_s^n) \to f(x_1, \cdots, x_s)$ for all x_i^n and x_i such that $x_i^n \to x_i$, $n = 1, 2, \cdots, i = 1, \cdots, s$. As shown in [2], $(C(X_1, \cdots, X_s, Y), \to)$ thus defined satisfies axioms 1), 2), 3) above and it is trivial to verify that axiom 4) is also satisfied. We call the convergence notion just defined the convergence notion induced by $(X_i, \to)$ $i = 1, \cdots, s$ and $(Y, \to)$ on $C(X_1, \cdots, X_s, Y)$.

1.4. *Continuous functionals for each type*

N is the family of natural numbers provided with the following notion of convergence: $x_n \to x$ iff $x_n = x$ with the exception of finitely many n's. We put $S(0) = N$. Let σ be $(\sigma_1, \cdots, \sigma_s/\tau)$ and assume that L-spaces $(S(\sigma_i), \to)$, $i = 1, \cdots, s$ and $(S(\tau), \to)$ have already been defined. Then we put $S(\sigma) = C(S(\sigma_1), \cdots, S(\sigma_s), S(\tau))$ and as notion of convergence on $S(\sigma)$ we take the convergence notion induced by $(S(\sigma_i), \to)$, $i = 1, \cdots, s$ and $(S(\tau), \to)$ on $S(\sigma)$. Our first goal is to prove

THEOREM 1. *The family* $\boldsymbol{S} = \bigcup_\sigma S(\sigma)$ *is closed under primitive recursion.*

THEOREM 2. $\boldsymbol{S}$ *is closed under barrecursion of higher type.*

The rather trivial proof of th. 1 will be sketched only, while the less trivial, but still simple proof of th. 2 will be worked out in detail.

1.5. *Some properties of* $\boldsymbol{S}$

a) In order to discuss some properties of S and also for later use we introduce below the notion of term. Prior to this we list a few functionals which belong to $\boldsymbol{S}$. b) Let $F \in \boldsymbol{S}$ be of type $(\sigma_1, \cdots, \sigma_s/\tau)$ and $\mu_1, \cdots, \mu_t$ a list of types. Then there is a $G \in \boldsymbol{S}$ with: if $F_i \in S(\sigma_i)$, $G_k \in S(\mu_k)$ then $G(F_1, \cdots, F_s, G_1, \cdots, G_t) = F(F_1, \cdots, F_s)$. If $\sigma_{t_1}, \cdots, \sigma_{t_s}$ is a permutation of $\sigma_1, \cdots, \sigma_s$ then there exists a $\tilde{F} \in \boldsymbol{S}$ with: $\tilde{F}(F_{t_1}, \cdots, F_{t_s}) = F(F_1, \cdots, F_s)$ for all F_i, $i = 1, \cdots, s$. The projections $P_i^\sigma \in \boldsymbol{S}$ are given by: $P_i^\sigma(F_1, \cdots, F_s) = F_i$ where $1 \leqq i \leqq s$. The successor function s is defined as usual by $s(x) = x+1$, where x runs over N. c) Let G be of type $(\sigma_1, \cdots, \sigma_s/\tau)$ and F_i of type $(\mu_1, \cdots, \mu_t/\sigma_i)$, $i = 1, \cdots, s$. Then there is an H in $\boldsymbol{S}$ of type $(\mu_1, \cdots, \mu_t/\tau)$ with the property: $H(T_1, \cdots, T_t) = G(F_1(T_1, \cdots, T_t), \cdots, F_s(T_1, \cdots, T_t))$ for all T_i, $i = 1, \cdots, t$. We say that H is obtained from $G, F_1, \cdots, F_s$ by substitution. d) Let E_μ be type $\mu = ((\sigma_1, \cdots, \sigma_s/\tau), \sigma_1, \cdots, \sigma_s/\tau)$ whose value $E(F, G_1, \cdots, G_s)$ is given by $F(G_1, \cdots, G_s)$. It is easy to verify that E_μ belongs to $\boldsymbol{S}$: if F_n converges against F and G_i^n against G_i then $F_n(G_1^n, \cdots, G_s^n)$ converges against $F(G_1, \cdots, G_s)$ in virtue of our definition of induced convergence notion (see also [2], pg. 94). e) Now to the notion of term. The definition is inductively: 1) if $F \in S(\sigma)$ then F is a term of type σ, 2) a free variable of type σ is a term of type σ, 3) if T is a term of type $(\sigma_1, \cdots, \sigma_s/\tau)$, if Q_i is a term of type σ_i, $i = 1, \cdots, s$ then $T[Q_1, \cdots, Q_s]$ is a term of type τ, 4) if T is a term of type τ, B a bound variable of type μ not occurring in T, if Y is a free variable of type μ then $(\lambda B S_Y^B T)$ is a term of type (μ/τ) (where $S_Y^B T$ denotes here and below the result of replacing every occurrence of Y in T by B). Next, let $Z_1, \cdots, Z_N$ be a list of free variables of types $\sigma_1, \cdots, \sigma_N$ respectively (with $N = 0$ admitted). We write $T/Z_1, \cdots, Z_N$ in order to indicate that the free variables of T occur among the Z_i's. With each expression $T/Z_1, \cdots, Z_N$ we associate an element $(T/Z_1, \cdots, Z_N)^*$ belonging to $S(\sigma_1, \cdots, \sigma_N/\tau)$ where τ is the type of T. The definition is by induction according to the clauses below; by definition $S(\sigma_1, \cdots, \sigma_N/\tau)$ denotes $S(\tau)$ whenever $N = 0$. α) If T is $F \in S(\sigma)$ then $(T/Z_1, \cdots, Z_N)^*(F_1, \cdots, F_N) = F$. β) If T is Z_i then $(T/Z_1, \cdots, Z_N)^*$ is P_i^σ with $\sigma = (\sigma_1, \cdots, \sigma_N/\sigma_i)$. γ) If T is $(\lambda B S_Y^B Q)$ with Y and B of type μ, Q of type τ and Y a free variable not occurring in $Z_1, \cdots, Z_N$ then $(T/Z_1, \cdots, Z_N)^*$ is the welldetermined element $G \in \boldsymbol{S}$ of type $(\sigma_1, \cdots, \sigma_N/(\mu/\tau))$ given by: $G(F_1, \cdots, F_N)(H) = (Q/Z_1, \cdots, Z_N, Y)^*(F_1, \cdots, F_N, H)$. δ) If T is $P[Q_1, \cdots, Q_s]$ and $(P/Z_1, \cdots, Z_N)^* = G$, $(Q_i/Z_1, \cdots, Z_N)^* = H_i$ then $(T/Z_1, \cdots, Z_N)^*$ is the functional $D \in \boldsymbol{S}$ given by: $D(F_1, \cdots, F_N) = G(F_1, \cdots, F_N)(H_1(F_1, \cdots, F_N), \cdots, H_s(F_1, \cdots, F_N))$ (that is, D is obtained from $G, H_1, \cdots, H_s$ by substitution). If in particular there are no free variables in T then we obtain for $N = 0 : (P[Q_1, \cdots, Q_s])^* = P^*(Q_1^*, \cdots, Q_s^*)$. Our assignment $*$ has a few properties, described by the following

LEMMA 1. *Let $Z_{\alpha_1}, \cdots, Z_{\alpha_s}$ be precisely the free variables occurring in T. Then $(T/Z_{\alpha_1}, \cdots, Z_{\alpha_s})^*(F_{\alpha_1}, \cdots, F_{\alpha_s}) = (T/Z_1, \cdots, Z_N)^*(F_1, \cdots, F_N)$.*

We omit the easy proof of the lemma which is by induction with respect to the complexity of T. If in particular T is a constant term then $(T/Z_1, \cdots, Z_N)^*(F_1, \cdots, F_N)$ does not depend on the F_i's. We denote this value by T^*. For $N = 0$ we then have $(T/Z_1, \cdots, Z_N)^* = T^*$. We note in particular: if $F \in \boldsymbol{S}$ then $F^* = F$. We also have

LEMMA 2. *Let T be a term whose free variables are among $Z_1, \cdots, Z_N$. Let $Q_1, \cdots, Q_s$ be terms without free variables and denote by T_0 the result of replacing each occurrence of Z_i in T by Q_i, $i = 1, \cdots, s$. Then*

$$(T_0/Z_{s+1}, \cdots, Z_N)^*(F_1, \cdots, F_{N-s}) = (T/Z_1, \cdots, Z_N)^*(Q_1^*, \cdots, Q_s^*, F_1, \cdots, F_{N-s}).$$

This proof too is by a straightforward induction with respect to the complexity of T and hence omitted. For $s = N$ the lemma reduces to: $T^* = (T/Z_1, \cdots, Z_N)^*(Q_1^*, \cdots, Q_N^*)$. Without danger of confusion we use the following notation at a few places: for $F \in \boldsymbol{S}$ of type $(\sigma_1, \cdots, \sigma_N/\tau)$ we write simply $(\lambda BF[Z_1, \cdots, Z_N, B])$ in place of $(\lambda BF[Z_1, \cdots, Z_N, B]/Z_1, \cdots, Z_N)^*$.

f) For every type σ there is a distinguished element $0_\sigma \in S(\sigma)$ whose inductive definition is as follows: 1) $0_0 = 0$, 2) if $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$ then $0_\sigma(F_1, \cdots, F_s) = 0_\tau$ for all $F_i \in S(\sigma_i)$. We often write 0 in place of 0_σ whenever it is clear from the context which type 0 is supposed to have. g) For every σ there is an element Δ_1 in $\boldsymbol{S}$ of type $((0/\sigma), 0/(0/\sigma))$ whose value for $\alpha \in S(0/\sigma)$ is given as follows: 1) $\Delta_1(\alpha, n)(i) = \alpha(i)$ for $i < n$, 2) $\Delta_1(\alpha, n)(i) = 0$ if $n \leqq i$. We write more suggestively: $\bar{\alpha}(n)$ in place of $\Delta_1(\alpha, n)$. h) For every type σ there is an element Δ_2 in $\boldsymbol{S}$ of type $((0/\sigma), 0, \sigma/(0/\sigma))$ whose value for $\alpha \in S(0/\sigma)$, $a \in S(\sigma)$ is given as follows: 1) $\Delta_2(\alpha, n, a)(i) = \alpha(i)$ if $i < n$, 2) $\Delta_2(\alpha, n, a)(n) = a$, 3) $\Delta_2(\alpha, n, a)(i) = 0$ if $i > n$. Without danger of confusion we often write $\bar{\alpha}(n) * a$ in place of $\Delta_2(\alpha, n, a)$. i) There is a $\Delta_3 \in \boldsymbol{S}$ of type $((0/\sigma), 0, (0/\sigma)/(0/\sigma))$ whose value for α, β in $S(0/\sigma)$ is given as follows: 1) $\Delta_3(\alpha, n, \beta)(i) = \alpha(i)$ for $i < n$, 2) $\Delta_3(\alpha, n, \beta)(i) = \beta(i-n)$ for $i \geqq n$. We often write $\bar{\alpha}(n) * \beta$ in place of $\Delta_3(\alpha, n, \beta)$. We write $\bar{\alpha}(n) * a * \beta$ in place of $\Delta_3(\bar{\alpha}(n) * a, n+1, \beta)$ and $a * \beta$ instead of $\bar{\alpha}(0) * a * \beta$. Note: $\bar{\alpha}(0)(i) = 0$ for all i. j) Of basic importance are the two lemmas listed below.

LEMMA 3. *Let Y of type $((0/\sigma)/0)$ be in $\boldsymbol{S}$. Then there is for every $\alpha \in S(0/\sigma)$ an n such that $Y(\alpha) = Y(\beta)$ whenever $\bar{\alpha}(n) = \bar{\beta}(n)$.*

PROOF. Assume the contrary. Then there is an α with the property: for every n there is a $\beta_n \in S(0/\sigma)$ such that $\bar{\beta}_n(n) = \bar{\alpha}(n)$ and such that

$Y(\beta_n) \neq Y(\alpha)$. However $\bar{\beta}_n(n) = \bar{\alpha}(n)$, $n = 1, 2, \cdots$ implies $\beta_n \to \alpha$ and hence $Y(\beta_n) = Y(\alpha)$ with the exception of finitely many n's. This contradicts $Y(\beta_n) \neq Y(\alpha)$, $n = 1, 2, \cdots$.

COROLLARY 1. *For every α there is an n such that $Y(\bar{\alpha}(n) * \beta) = Y(\alpha)$ for all β.*

COROLLARY 2. *For every k and every α there is an n such that $Y(\bar{\alpha}(n) * \beta) < n+k$ for all β.*

LEMMA 4. *Let F be an arbitrary mapping from $\mathbf{N}$ into $S(\sigma)$. Then $F \in S(0/\sigma)$.*

We omit the obvious proof.

REMARK. In this section we have mostly stated that some particular functionals belong to $\boldsymbol{S}$ or if some functionals belong to $\boldsymbol{S}$ then some others belong to $\boldsymbol{S}$. The proofs are completely trivial and therefore we have omitted them.

1.6. *Proof of theorem* 1

In order to prove theorem 1 we show that for every appropriate type there is a functional J which satisfies the following equations:

1) $J(0, F_1, \cdots, F_s, G, H) = G(F_1, \cdots, F_s)$,

2) $J(n+1, F_1, \cdots, F_s, G, H) = H(n, F_1, \cdots, F_s, J(n, F_1, \cdots, F_s, G, H))$,

(with type compatibility tacitly assumed). For simplicity we consider the case where just one parameter is present, that is where $s = 1$. Now it is evident that there exists a mapping J (of appropriate type) which satisfies equations 1), 2); what we have to do is to convince ourself that J is indeed an element of $\boldsymbol{S}$. This is achieved if we can show: $J(n, F_i, G_i, H_i) \to J(n, F, G, H)$ whenever $F_i \to F$, $G_i \to G$, $H_i \to H$ for $i = 1, 2, \cdots$. We prove this by induction with respect to $\dot{N}$. If $n = 0$ then the statement reduces to $G_i(F_i) \to G(F)$. But this is a consequence of our definition of convergence. Now assume that continuity of J (with respect to G, H, F) has been proved up to n. Now $J(n+1, F_i, G_i, H_i) = H_i(n, F_i, J(n, F_i, G_i, H_i))$. But $J(n, F_i, G_i, H_i)$ converges against $J(n, F, G, H)$ by assumption. Hence $H_i(n, F_i, J(n, F_i, G_i, H_i))$ converges against $H(n, F, J(n, F, G, H))$ in virtue of the continuity of H, whence the statement follows.

II. $\boldsymbol{S}$ is closed against barrecursion of higher type

2.1. *Parameters; the barrecursive equations*

a) Below it is convenient to use the following notation: 1) if $F \in \boldsymbol{S}$ is of type $(\sigma_1, \cdots, \sigma_s/\tau)$ then $F(X^1_{\sigma_1}, \cdots, X^s_{\sigma_s})$ denotes a functional of type τ

depending on the parameters $X^1_{\sigma_1}, \cdots, X^s_{\sigma_s}$, 2) if $F \in S$ is of type $(\sigma_1, \cdots, \sigma_s, \mu/\tau)$ then $\lambda BF(X^1_{\sigma_1}, \cdots, X^s_{\sigma_s}, B)$ denotes a functional H of type (μ/τ), depending on $X^1_{\sigma_1}, \cdots, X^s_{\sigma_s}$; for given values G_i, $i = 1, \cdots, s$ H assumes a certain value H_0 given by the equation $H_0(G) = F(G_1, \cdots, G_s, G)$ for all $G \in S(\mu)$, 3) single parameters and lists of parameters will also be denoted by such symbols as Z, z etc; lists of types will be denoted by capital Greek letters Σ, Λ etc. Thus if we say that Z is of type Σ we mean eg. that Z is $X^1_{\sigma_1}, \cdots, X^s_{\sigma_s}$ and that Σ is $\sigma_1, \cdots, \sigma_s$. The lists Z and Σ may be empty. Below we often omit types and assume tacitly that all functionals, variables and parameters appearing in equations or other contexts are provided in a compatible way with types.

Definition 1. A functional $\varphi \in \boldsymbol{S}$ is called a barrecursive functional if it has the following properties: 1) $\varphi(Z, G, H, Y, x, \bar{\alpha}(x)) = \varphi(Z, G, H, Y, x, \alpha)$, 2) for all functionals G, H, Y and all values of the parameters Z the following equations are satisfied:

$$\text{I. } \varphi(Z, G, H, Y, x, \bar{\alpha}(x)) = G(Z, x, \bar{\alpha}(x)) \text{ if } Y(Z, \bar{\alpha}(x)) < k+x,$$

$$\text{II. } \varphi(Z, G, H, Y, x, \bar{\alpha}(x)) = H(Z, x, \bar{\alpha}(x), \lambda s\varphi(Z, G, H, Y, x+1, \bar{\alpha}(x) * s)) \text{ if } Y(Z, \bar{\alpha}(x)) \geqq x+k.$$

This equations are supposed to hold for all $\alpha \in \boldsymbol{S}$ and x. The natural number k is arbitrary but fixed and the only constant which enters into the equations. The types of Z, G, H, Y, α and x are in that order: Σ, $(\Sigma, 0, (0/\sigma)/\tau)$, $(\Sigma, 0, (0/\sigma), (\sigma/\tau)/\tau)$, $(\Sigma, (0/\sigma)/0)$, $(0/\sigma)$, 0.

Notation. In order not to overburden the notation we often write $\varphi(x, \alpha)$ in place of $\varphi(Z, G, H, Y, x, \alpha)$.

Below we often have to study equations I, II with Z, G, H, Y held constant. φ is then a function of x and α only. In this case we call φ a solution of I, II with respect to G, H, Y and Z. If only G, H, Y are held constant then we call φ a solution with parameters Z of I, II with respect to G, H, Y. If finally Z, G, H, Y are considered as parameters then we call φ a solution of I, II with Z, G, H, Y as parameters; φ is then by definition a barrecursive functional.

There is a variant of definition 1, which will be of importance below, namely

Definition 2. Let G, H, Y, containing parameters Z, be given; their types are as in def. 1. Then φ is said to be a solution with parameters Z up to $\bar{\alpha}(x)$ of I, II with respect to G, H, Y if the following holds: 1) $\varphi(Z, y, \beta) = \varphi(Z, y, \bar{\beta}(y))$, 2) for all arguments Z, y, β the following equations are satisfied:

$$\text{I}^* \quad \varphi(Z, y, \bar{\beta}(y)) = G(Z, x+y, \bar{\alpha}(x) * \bar{\beta}(y)) \qquad \text{if } Y(Z, \bar{\alpha}(x) * \bar{\beta}(y)) < x+y+k$$

$$\text{II}^* \quad \varphi(Z, y, \bar{\beta}(y)) = H(Z, x+y, \bar{\alpha}(x) * \bar{\beta}(y), \lambda s \varphi(Z, y+1, \bar{\beta}(y) * s)) \qquad \text{otherwise.}$$

Remark. A solution φ of I*, II* is nothing else than a solution with parameters Z of I, II but with respect to certain functionals G', H', Y' different from G, H, Y. The parameters Z in I*, II* may of course be absent. In connection with I*, II* we use the same terminology as with I, II. A conparison of definitions 1, 2 shows: a solution φ up to $\bar{\alpha}(0)$ of I, II with respect to G, H, Y is a solution of I, II with respect to G, H, Y according to def. 1.

2.2. *Transfinite induction*

Lemmas 3, 4 permit us to use the principle of bar induction with respect to functionals $Y \in S$ of type $((0/\sigma)/0)$.

Definition 3. Let $Y \in \boldsymbol{S}$ be of type $((0/\sigma)/0)$ and k an arbitrary number. A finite sequence $\{f_0, \cdots, f_{x-1}\}$ of elements $\in \boldsymbol{S}$ of type σ is called 'secured' with respect to Y, k (to Y if $k = 0$) iff the following holds: if $\alpha \in S(0/\sigma)$ is such that $\alpha(i) = f_i$ for $i < x$ then $Y(\alpha) < x+k$. We call $\{f_0, \cdots, f_{x-1}\}$ unsecured otherwise.

Remark. In this connection we use a somewhat unprecise way of speaking: we call $\bar{\alpha}(x)$ secured if $\{\alpha(0), \cdots, \alpha(x-1)\}$ is secured, and unsecured otherwise. Then we have the following principle of bar induction (or transfinite induction, as we sometimes say): if $A(\bar{\alpha}(x))$ holds for all $\bar{\alpha}(x)$ secured with respect to Y, k, if moreover $(s)A(\bar{\beta}(y) * s) \supset A(\bar{\beta}(y))$ is true for all $\bar{\beta}(y)$ then $A(\bar{\gamma}(x))$ holds for all $\bar{\gamma}(x)$.

2.3. *Some lemmas*

Lemma 5. *Let $F \in \boldsymbol{S}$ be of type $(\Sigma, (0/\sigma)/\tau)$. Then $F(Z, \bar{\alpha}(x) * s)$ depends continuously on all its arguments, that is on Z, α, s (and x).*

Proof. Follows from the fact that $\boldsymbol{S}$ is closed against substitution.

Lemma 6. *Let F_n $n = 1, 2, \cdots$* and *F be of type $(\Sigma, \sigma/\mu)$ and assume $F_n \to F$. Then $\lambda BF_n[Z, B] \to \lambda BF[Z, B]$.*

Proof. We have to show: if $Z_n \to Z$ then $\lambda BF_n(Z_n, B) \to \lambda BF(Z, B)$. Hence let $X_n \to X$ be true. Then $\lambda BF_n(Z_n, B)(X_n) = F_n(Z_n, X_n)$ and $\lambda BF(Z, B)(X) = F(Z, X)$. But $F_n(Z_n, X_n) \to F(Z, X)$ in virtue of $F_n \to F$. Hence the lemma follows.

Remark. In virtue of our notation conventions we can read the last lemma also as: $\lambda BF_n(Z_n, B) \to \lambda BF(Z, B)$ whenever $Z_n \to Z$.

LEMMA 7: *Let F_n, F in* **S** *be of type* $(\Sigma, \sigma/\mu)$ and let Z_n, $n = 1, 2, \cdots$ *and Z be particular values of the parameters. Assume that for all $X_n \to X$ $F_n(Z_n, X_n)$ converges against $F(Z, X)$. Then $\lambda BF_n(Z_n, B) \to \lambda BF(Z, B)$.*

PROOF. This is a straightforward consequence of our definition of induced convergence.

LEMMA 8. *The system I, II of equations without parameters Z admits at most one solution φ with respect to G, H, Y.*

The proof is by a straightforward bar induction with respect to Y, k and is omitted. Of crucial importance is the following

LEMMA 9. *Let G_n, H_n, Y_n, $n = 1, 2, \cdots$ and G, H, Y be functionals (of suitable types) without parameters Z having the following properties:* 1) *for every n there is a solution φ_n of I, II with respect to G_n, H_n, Y_n,* 2) *there exists a solution φ of I, II with respect to G, H, Y,* 3) *G_n, H_n, Y_n converge against G, H, Y respectively. Then $\varphi_n \to \varphi$.*

PROOF. We proceed by transfinite induction with respect to Y, k: if $\bar{\alpha}_n(x)$ converges against $\bar{\alpha}(x)$ then $\varphi_n(x, \bar{\alpha}_n(x)) \to \varphi(x, \bar{\alpha}(x))$. *Case* 1: $\bar{\alpha}(x)$ is secured with respect to Y, k. Then $\varphi(x, \bar{\alpha}(x)) = G(x, \bar{\alpha}(x))$ by definition. Since $Y_n \to Y$, $\bar{\alpha}_n(x) \to \bar{\alpha}(x)$ it follows that $Y_n(\bar{\alpha}_n(x)) = Y(\bar{\alpha}(x))$ for almost all n's. Hence $Y_n(\bar{\alpha}_n(x)) < x+k$ for almost all n's. Hence $\varphi_n(x, \bar{\alpha}_n(x)) = G_n(x, \bar{\alpha}_n(x))$ for almost all n's. But $G_n(, \bar{\alpha}_n(x)) \to G(x, \bar{\alpha}(x))$ in virtue of $G_n \to G$ whence the statement follows. *Case* 2: The statement holds for all $\bar{\alpha}(x) * a$. We distinguish two subcases. *Subcase* 1: $Y(\bar{\alpha}(x)) < x+k$. Then we proceed as in Case 1. *Subcase* 2: $Y(\bar{\alpha}(x)) \geqq x+k$. Then $\varphi(x, \bar{\alpha}(x)) = H(x, \bar{\alpha}(x), \lambda s\varphi(x, \bar{\alpha}(x) * s))$ by definition. As before $Y_n(\bar{\alpha}_n(x)) = Y(\bar{\alpha}(x))$ for almost all n's. Hence $\varphi_n(x, \bar{\alpha}_n(x))$ is $H_n(x, \bar{\alpha}_n(x), \lambda s\varphi_n(x, \bar{\alpha}_n(x) * s))$ for almost all n's. The inductive assumption is: for all a, if $\bar{\beta}_n(x+1) \to \bar{\alpha}(x) * a$, then $\varphi_n(x+1, \bar{\beta}_n(x+1)) \to \varphi(x+1, \alpha(x) * a)$. From this we infer: whenever $\bar{\alpha}_n(x) \to \bar{\alpha}(x)$ and $a_n \to a$ then $\varphi_n(x+1, \bar{\alpha}_n(x) * a_n) \to \varphi(x+1, \bar{\alpha}(x) * a)$. According to lemma 7 (or 6) this implies: if $\bar{\alpha}_n(x) \to \bar{\alpha}(x)$ then $\lambda s\varphi_n(x+1, \bar{\alpha}_n(x) * s) \to \lambda s\varphi(x+1, \bar{\alpha}(x) * s)$. In virtue of $H_n \to H$ this implies the convergence of $H_n(x, \bar{\alpha}_n(x), \lambda s\varphi_n(x+1, \bar{\alpha}_n(x) * s))$ against $H(x, \bar{\alpha}(x), \lambda s\varphi(x+1, \bar{\alpha}(x) * s))$ whence the statement follows.

LEMMA 10. *Let G_n, H_n, Y_n and G, H, Y be functionals of suitable types all containing the parameters Z. Assume that the following holds:* 1) *for every n, Z there exists a solution φ^n_Z of I, II with respect to G_n, H_n, Y_n, Z,* 2) *for every Z there exists a solution φ_Z of I, II with respect to G, H, Y, Z,* 3) *$G_n \to G$, $H_n \to H$, $Y_n \to Y$. If $Z_n \to Z$ then $\varphi^n_{Z_n} \to \varphi_Z$.*

PROOF. Consider Z_n, $n = 1, 2, \cdots$ and Z as fixed and assume $Z_n \to Z$.

Define G'_n, H'_n, Y'_n and G', H', Y' as follows: $G'_n(x, \alpha) = G_n(Z_n, x, \alpha)$, $H'_n(x, \alpha, \xi) = H_n(Z_n, x, \alpha, \xi)$, $Y'_n(\alpha) = Y_n(Z_n, \alpha)$, $G'(x, \alpha) = G(Z, x, \alpha)$, $H'(x, \alpha, \xi) = H(Z, x, \alpha, \xi)$, $Y'(\alpha) = Y(Z, \alpha)$ for all x, α, ξ. Then $G'_n \to G'$, $H'_n \to H'$ and $Y'_n \to Y'$ in virtue of assumption 3). But $\varphi^n_{Z_n}$ is a solution of I, II with respect G'_n, H'_n, Y'_n while φ_Z is a solution of I, II with respect to G', H', Y'. Hence the statement follows from lemma 9.

LEMMA 11. *Assume that for every G, H, Y and all values of the parameters Z there exists a solution of I, II with respect to G, H, Y, Z. Then there exists a solution $\varphi(Z, G, H, Y, x, \alpha)$ of I, II; in other words, a solution of I, II with G, H, Y, Z as parameters exists (everything with respect to a fixed compatible list of types).*

PROOF. Denote by $\varphi\{G, H, Y, Z\}$ the solution of I, II with respect to G, H, Y, Z. From the last lemma we infer: if $G_n \to G$, $H_n \to H$, $Y_n \to Y$ and $Z_n \to Z$ then $\varphi\{G_n, H_n, Y_n, Z_n\} \to \varphi\{G, H, Y, Z\}$. This means: if $G_n, H_n, Y_n, Z_n, \alpha_n$ converges against G, H, Y, Z, α then $\varphi\{G_n, H_n, Y_n, Z_n\}(x, \alpha_n) \to \varphi\{G, H, Y, Z\}(x, \alpha)$. Hence by defining $\varphi(Z, G, H, Y, x, \alpha) = \varphi\{G, H, Y, Z\}(x, \alpha)$ for all G, H, Y, Z, x, α we get the desired solution φ of I, II with G, H, Y, Z as parameters.

COROLLARY. *Under the assumptions of lemma* 11 *it follows that the barrecursive functional (of a certain type) exists.*

PROOF. This is but a restatement of lemma 11.

Hence theorem 2 is proved as soon as we can prove

THEOREM 3. *For every G, H, Y without parameters and every $\bar{\alpha}(x)$ there exists a solution φ of I, II up to $\bar{\alpha}(x)$, that is a solution of I*, II* with respect to G, H, Y and $\bar{\alpha}(x)$.*

PROOF. We proceed by induction with respect to Y, k. *Case* 1: $\bar{\alpha}(x)$ is secured with respect Y, k, that is $Y(\bar{\alpha}(x) * \beta) < x+k$ for all β. Put $\varphi(y, \beta) = G(x+y, \bar{\alpha}(x) * \bar{\beta}(y))$. Then $Y(\bar{\alpha}(x) * \bar{\beta}(y)) < x+k \leqq x+k+y$ and φ thus defined is indeed a solution up to $\bar{\alpha}(x)$ of I, II with respect to G, H, Y. *Case* 2: Now assume that $\bar{\alpha}(x)$ is such that for every z of type σ the following assumption holds: there exists a solution φ_Z of I, II up to $\bar{\alpha}(x) * z$. We want to construct a solution φ of I, II up to $\bar{\alpha}(x)$. By assumption φ_Z satisfies the following conditions: 1) $\varphi_Z(y, \beta) = \varphi_Z(y, \bar{\beta}(y))$, 2) I*: $\varphi_Z(y, \bar{\beta}(y)) = G(x+y+1, \bar{\alpha}(x) * z * \bar{\beta}(y))$ if $Y(\bar{\alpha}(x) * z * \bar{\beta}(y)) < x+y+k+1$, and II*: $= H(x+y+1, \bar{\alpha}(x) * z * \bar{\beta}(y), \lambda s \varphi_Z(y+1, \bar{\beta}(y) * s))$ otherwise. Define $\hat{G}$, $\hat{H}$, $\hat{Y}$ as follows: 1) $\hat{G}(z, y, \beta) = G(x+y+1, \bar{\alpha}(x)*z*\bar{\beta}(y))$, 2) $\hat{H}(z, y, \beta, \xi) = H(x+y+1, \bar{\alpha}(x)*z*\bar{\beta}(y), \xi)$, 3) $\hat{Y}(z, \beta) = Y(\bar{\alpha}(x) * z * \beta)$. The functionals $\hat{G}$, $\hat{H}$, $\hat{Y}$ are clearly in $\boldsymbol{S}$. Now φ_Z is clearly a solution of I, II with respect to $\hat{G}$, $\hat{H}$, $\hat{Y}$, z but with $x+k+1$ in place of

k. In virtue of lemma 10 we have: if $z_n \to z$, $\beta_n \to \beta$ then $\varphi_{Z_n}(y, \beta_n) \to \varphi_Z(y, \beta)$. Define $\hat{\varphi}$ as follows: $\hat{\varphi}(z, y, \beta) = \varphi_Z(y, \beta)$. Obviously $\hat{\varphi} \in \boldsymbol{S}$. It follows in particular that $\hat{\varphi}(z, 0, 0)$ depends continuously on z and hence that $\lambda s \hat{\varphi}(s, 0, 0)$ is an element of $\boldsymbol{S}$ (eg. by specializing lemma 6). Now we define a functional φ as follows: α) $\varphi(y+1, z * \beta) = \hat{\varphi}(z, y, \bar{\beta}(y))$, β_1) $\varphi(0, \beta) = G(x, \bar{\alpha}(x))$ if $Y(\bar{\alpha}(x)) < x+k$, β_2) $\varphi(0, \beta) = H(x, \bar{\alpha}(x), \lambda s \hat{\varphi}(s, 0, 0))$ if $Y(\bar{\alpha}(x)) \geqq x+k$. That φ thus defined is an element in $\boldsymbol{S}$ follows immediately from the fact that $\hat{\varphi}$ is in $\boldsymbol{S}$. It remains to verify that φ is indeed a solution up to $\bar{\alpha}(x)$ of I, II. We distinguish four cases. $A : y > 0$ and $Y(\bar{\alpha}(x) * \bar{\beta}(y)) < x+y+k$. Then $\bar{\beta}(y)$ can be written as $z * \bar{\gamma}(y-1)$. We have: $Y(\bar{\alpha}(x) * z * \bar{\gamma}(y-1)) < (x+1)+(y-1)+k$. According to our inductive assumptions about φ_Z, written in terms of $\hat{\varphi}$, we have: $\hat{\varphi}(z, y-1, \bar{\gamma}(y-1)) = G(\bar{\alpha}(x) * z * \bar{\gamma}(y-1), x+1+y-1)$. Using definition α) of φ we infer that equation I* is indeed satisfied by φ. $B : y > 0$ and $Y(\bar{\alpha}(x) * \bar{\beta}(y)) \geqq x+y+k$. Again we write $\bar{\beta}(y)$ in the form $z * \bar{\gamma}(y-1)$. Thus $Y(\bar{\alpha}(x) * z * \bar{\gamma}(y-1)) \geqq x+1+y-1+k$ holds. In view of our inductive assumption about φ_Z, written in terms of $\hat{\varphi}$, we have: $\hat{\varphi}(z, y-1, \bar{\gamma}(y-1)) = H(x+1+y-1, \bar{\alpha}(x) * z * \bar{\gamma}(y-1), \lambda s \hat{\varphi}(z, y, \bar{\gamma}(y-1) * s))$. Using again definition α) of φ we infer that equation II* is satisfied by φ. $C : y = 0$ and $Y(\bar{\alpha}(x)) < x+k$. According to β_1) we have $\varphi(0, 0) = G(x, \bar{\alpha}(x))$, that is φ is indeed a solution of I*. $D : y = 0$ and $Y(\bar{\alpha}(x)) \geqq x+k$. By definition β_2) we have $\varphi(0, 0) = H(x, \bar{\alpha}(x), \lambda s \hat{\varphi}(s, 0, 0))$. Now $\varphi(1, z * \beta) = \hat{\varphi}(z, 0, 0)$ according to α), therefore $\varphi(1, \bar{\alpha}(0) * z) = \hat{\varphi}(z, 0, 0)$. Hence $\varphi(0, \bar{\alpha}(0)) = H(x, \bar{\alpha}(x), \lambda s \varphi(1, \bar{\alpha}(0) * s))$. That is, φ is indeed a solution up to $\bar{\alpha}(x)$ of I, II with respect to G, H, Y.

III. Application to an equational calculus

3.1. *Syntax*

a) In what follows we set up an equational calculus, more or less along the lines of [3], pg. 225. In order to save symbols we use the elements of $\boldsymbol{S}$ and the terms constructed with the aid of them as basic symbols of the calculus. The calculus is only apparently nonconstructive; the reader will easily recognize that the calculus defined below can be explained in an entirely finitistic way. We begin with the construction of a subset $\boldsymbol{E}$ of terms by means of the inductive clauses below. T1) $S_0 \subseteq \boldsymbol{E}$, T2) the successorfunction s, given by $s(i) = i+1$, is in $\boldsymbol{E}$, T3) free variables are in $\boldsymbol{E}$, T4) the barrecursive functionals of all types, determined by the barresursive equations I, II with $k = 0$, belong to $\boldsymbol{E}$, T5) the induction functionals of all types belong to $\boldsymbol{E}$, T6) if T_i of type σ_i, $i = 1, \cdots, s$ and T of type $(\sigma_1, \cdots, \sigma_s/\tau)$ are in $\boldsymbol{E}$ then $T[T_1, \cdots, T_s]$ is in $\boldsymbol{E}$, T7) if T is in $\boldsymbol{E}$ then

$(\lambda S_X^B T)$ is in $\boldsymbol{E}$, T8) if T is in $\boldsymbol{E}$ and if its free variables are among $Z_1, \cdots, Z_N$ then $(T/Z_1, \cdots, Z_N)^*$ is in $\boldsymbol{E}$. It is easy to see that all primitive recursive functionals of Goedels system $\boldsymbol{T}$ belong to $\boldsymbol{E}$. In particular all functionals listed under f)–k) in I, 1.5, belong to $\boldsymbol{E}$. Without danger of confusion we often write $\bar{\alpha}[t]$ in place of $\Delta_1[\alpha, t]$ and $\bar{\alpha}[t] * q$ in place of $\Delta_2[\alpha, t, q]$ (for terms t, α, q of appropriate types). Let $\doteq$ be a new sign. By an equation we understand an expression $T \doteq R$ with T, R terms in $\boldsymbol{E}$ of the same type. Let $Z_1, \cdots, Z_N$ be the free variables which occur in T or in R. Here and below we denote by $T/Q_1, \cdots, Q_N$ the result obtained by replacing every occurrence of Z_i in T by Q_i, $i = 1, \cdots, N$:, similarly with $R/Q_1, \cdots, Q_N$. The equation $T \doteq R$ is said to be true if $(T/F_1, \cdots, F_N)^* = (R/F_1, \cdots, F_N)^*$ for all $F_i \in S$ (where F_i and Q_i have the same type as Z_i). We now take some equations as axioms, according to the clauses below. *eq* 1: $s[t] = t+1$ is an axiom (t of type 0). *eq* 2: $J[0, F, G, H] = G[F]$ and $J[t+1, F, G, H] = H[t, F, J[t, F, G, H]]$ are axioms, with F a list of one or more terms as parameters. *eq* 3: Let the free variables of $T \in \boldsymbol{E}$ be among $Z_1, \cdots, Z_N$. Then $(T/Z_1, \cdots, Z_N)_*[Q_1, \cdots, Q_N] = T/Q_1, \cdots, Q_N$ is an axiom, where the Q_i's are arbitrary terms (of the right types of course) and where $T/Q_1, \cdots, Q_N$ has the same meaning as above. *eq* 4: Let X be a free variable different from $Z_1, \cdots, Z_N$. Let T be a term whose free variables are among $Z_1, \cdots, Z_N, X$; let $Q_1, \cdots, Q_N, Q$ be arbitrary terms. Then $((\lambda BS_X^B T)/Q_1, \cdots, Q_N)[Q] \doteq T/Q_1, \cdots, Q_N, Q$ is an axiom. *eq* 5: $T \doteq T$ is an axiom.

Besides the axioms, we have also the rules of inference R, B1, B2.

*R*1: If $T_1 \doteq T_2$ and $a \doteq b$ have been proved then we are permitted to infer $T_1' \doteq T_2'$ where $T_1' \doteq T_2'$ is obtained from $T_1 = T_2$ by replacing an occurrence of a by b or conversely.

*B*1: If $Y[F, \bar{\alpha}[t]]+q+1 \doteq t$ has been derived then

$$\varphi[F, G, H, Y, t, \alpha] \doteq G[F, t, \bar{\alpha}[t]]$$

can be derived. Here F denotes a list (of type Σ) of one or several terms as parameters; α is a term of type $(0/\sigma)$ for suitable σ, t and q are terms of type 0 and φ, G, H, Y are assumed to have the correct type structures.

*B*2: If $Y[F, \bar{\alpha}[t]] \doteq t+q$ has been derived, then

$$\varphi[F, G, H, Y, t, \alpha] \doteq H[F, t, \bar{\alpha}[t], \lambda s\varphi[F, G, H, Y, t+1, \bar{\alpha}[t] * s]]$$

can be derived.

In order to indicate that an equation $t \doteq q$ is provable from our axioms by means of the given rules we write $\vdash t \doteq q$. The equational calculus thus obtained is denoted by $\boldsymbol{E}$. The main theorem, which connects truth and provability is

THEOREM 4. *If* $\vdash T \doteq R$ *then* $T \doteq R$ *is true.*

To put it in more familiar terms, theorem 4 states that S is a model of E. The proof of th. 4 is not trivial in the proper sense but it requires only routine verifications which are based heavily on lemmas 1, 2; since the proof is quit lengthy we omit it in order to save space.

3.2. *Reducible terms*

a) By a constant we understand from now on a constant term of length one, that is an element of S. A constant term is a term without free variables. As in [3] we introduce

DEFINITION 4. 1) A constant term t of type 0 is reducible if there is a number m such that $\vdash t \doteq m$ holds. 2) A constant term T of type $(\sigma_1, \cdots, \sigma_s/\tau)$ is reducible if for all reducible terms $Q_1, \cdots, Q_s$ of types $\sigma_1, \cdots, \sigma_s$ respectively $T[Q_1, \cdots, Q_s]$ is reducible (all terms belonging to E by assumption). Thus, reducible terms do not contain free variables. Our aim is to prove the following

THEOREM 5. *All constant terms are reducible.*

PROOF. Our proof follows very closely the considerations in [3]. We proceed in steps. 1) If $\vdash a \doteq b$ with a, b constant terms, if b is reducible, then a is reducible. We proceed by induction with respect to types. If 0 is the type of a, b then $\vdash b \doteq m$ for some m by assumption. Hence by rule R: $\vdash a \doteq m$,. Thus a is reducible. Assume the statement to be true for types $\sigma_1, \cdots, \sigma_s, \tau$ and let a, b have type $(\sigma_1, \cdots, \sigma_s/\tau)$; let $c_1, \cdots, c_s$ be reducible terms of types $\sigma_1, \cdots, \sigma_s$ respectively. From $a[c_1, \cdots, c_s] \doteq a[c_1, \cdots, c_s]$ and $\vdash a \doteq b$ we infer by rule R: $\vdash a[c_1, \cdots, c_s] \doteq b[c_1, \cdots, c_s]$. Since b is reducible by assumption, $b[c_1, \cdots, c_s]$ is reducible and hence $a[c_1, \cdots, c_s]$ is reducible according to the induction hypothesis. Thus a is reducible in virtue of the arbitrariness of the c_i's. 2) Let T be a constant term of type $(0, \sigma_1, \cdots, \sigma_s/\tau)$. If $T[n, Q_1, \cdots, Q_s]$ is reducible for all reducible terms Q_i of type σ_i, $i = 1, \cdots, s$ and every n then T is reducible. Let $Q_1, \cdots, Q_s$ be reducible; let t of type 0 be reducible. Hence $\vdash t \doteq m$ for some m. Therefore $T[t, Q_1, \cdots, Q_s] \doteq T[m, Q_1, \cdots, Q_s]$ by rule R. Using our assumption and clause 1) it follows that $T[t, Q_1, \cdots, Q_s]$ is reducible. 3) Let T be a term such that every constant occurring in T is reducible. If $Q_1, \cdots, Q_N$ are reducible terms then $T/Q_1, \cdots, Q_N$ is reducible (with $T/Q_1, \cdots, Q_N$ the result of replacing Z_i by Q_i, $i = 1, \cdots, N$ where the free variables in T are among $Z_1, \cdots, Z_N$). We proceed by induction with respect to the length of T. *Case* 1: T has length 1. Then the statement is obvious. *Case* 2: T is composed according to clause 3) of term formation (I, 1.5). Let T eg. be $P[L]$; put $P_1 = P/Q_1, \cdots, Q_N$, $L_1 = L/Q_1, \cdots, Q_N$. By assumption P_1, L_1 and hence $P_1[L_1]$ are reducible. Thus the statement holds in virtue of the arbitrariness of the Q_i's. *Case* 3: T is $\lambda BS_X^B P$ where X is a free variable different from $Z_1, \cdots, Z_N$. Put $P_1 = P/Q_1, \cdots, Q_N$ and $T_1 = T/Q_1, \cdots, Q_N$. Obviously $T_1 = \lambda BS_X^B P_1$. We have to show that $T_1[L]$ is reducible for every reducible L. Now $(\lambda BS_X^B P_1)[L] \doteqdot S_X^L P_1$ is an axiom, hence provable. Since P has less symbols than T it follows from the inductive assumption that $S_X^L P_1$ is reducible, and hence from clause 1) above that $T_1[L]$ is reducible. In virtue of the arbitrariness of $Q_1, \cdots, Q_N$, L the statement holds also in this case. 4) In order to prove that every constant term is reducible it follows from 3) that it is sufficient to prove that every constant is reducible. This is done in the same way as in [3], that is first we show that the basic constants s, J, φ are reducible and afterwards by an induction over the clauses defining $\boldsymbol{E}$ that every constant which is defined in terms of previous ones, is reducible. The only clause among those defining $\boldsymbol{E}$, which introduces a new constant in terms of others already defined is T8). The verification that s, the induction functionals J, and $(T/Z_1, \cdots, Z_N)^*$ are reducible, provided the constants of T are reducible, is exactly the same as in [3] and hence omitted. It follows from this that all primitive recursive functionals are reducible; we mention in particular Δ_1, Δ_2, Δ_3 and 0_σ (I, 1.5). Hence we concentrate on the proof that the barrecursive functionals are reducible. Let φ be a fixed bar recursive functional. In virtue of 2) we have to show: if F, G, H, Y and α are reducible terms then $\varphi[F, G, H, Y, n, \alpha]$ is reducible for all n. For simplicity we have assumed that G, H, Y and hence φ contain exactly one extra parameter (F); the case of more parameters is treated in exactly the same way. Let F, G, H, Y be arbitrary but fixed reducible terms; the type of Y is supposed to be $(\mu, (0/\sigma)/0)$, that of F hence μ. We are through if we can prove A): for every n, if α is reducible then $\varphi[F, G, H, Y, n, \alpha]$ is reducible. Before proceeding further, we note that the following statement B) holds: for every constant term α, $(Y[F, \bar{\alpha}[n]])^* = Y^*(F^*, \bar{\alpha}^*(n))$. This is an immediate consequence of our definition of $(T[Q_1, \cdots, Q_s]/Z_1, \cdots, Z_N)^*$ for the case where no free variables occur in T, $Q_1, \cdots, Q_s$ and where $N = 0$. Now define Y_F as follows: $Y_F(\xi) = Y^*(F^*, \xi)$ for all $\xi \in S(0/\sigma)$. Clearly $Y_F \in \boldsymbol{S}$. Now let P be the one place predicate, whose range of definition is the set of finite sequences of elements from $S(\sigma)$ and which applies to $f_0, \cdots, f_{n-1}$ (in signs $P(\langle f_0, \cdots, f_{n-1}\rangle)$) if and only if the following holds: if α is a reducible term such that $\alpha^*(i) = f_i$ for $i < n$ then $\varphi[F, G, H, Y, n, \alpha]$ is reducible. The proof of A) thus reduces to the proof of C): for all $f_0, \cdots, f_{n-1} \in S(\sigma)$, the statement $P(\langle f_0, \cdots, f_{n-1}\rangle)$ is true. We prove C) by barinduction over Y_F, 0.

I. Let $f_0, \cdots, f_{n-1}$ be secured with respect to Y_F: if $\xi(i) = f_i$ for $i < n$ then $Y_F(\xi) < n$. Let α be a reducible term such that $\alpha^*(i) = f_i$ for $i < n$.

Then $Y^*(F^*, \bar{\alpha}^*(n)) < n$. But $Y^*(F^*, \bar{\alpha}^*(n)) = (Y[F, \bar{\alpha}[n]])^*$ by B). Since $Y, F, \Delta_1, \alpha, n$ are all reducible it follows that $Y[F, \bar{\alpha}[n]]$ is reducible ($\bar{\alpha}[n]$ being an abbreviation for $\Delta_1[\alpha, n]$), that is $\vdash Y[F, \bar{\alpha}(n]] \doteq m$ for some m. Since every provable equation is true, necessarily $m < n$ and hence we obtain by some primitive recursive manipulations $\vdash Y[F, \bar{\alpha}[n]] + j+1 \doteq n$. From rule B we obtain: $\vdash \varphi[F, G, H, Y, n, \alpha] \doteq G[F, n, \bar{\alpha}[n]]$. However $G, F, n, \bar{\alpha}[n]$ are all reducible hence the righthandside of the last equation is reducible and hence according to clause 1) the lefthandside is reducible, what settles this case.

II. Now assume that for all $f \in S(\sigma)$ the statement $P(\langle f_0, \cdots, f_{n-1}, f\rangle)$ holds. We have to infer $P(\langle \mathrm{f}_0, \cdots, \mathrm{f}_{n-1}\rangle)$. That is, we have: for all $f \in S(\sigma)$, if β is a reducible term with $\beta^*(i) = f_i$ for $i < n$ and $\beta^*(n) = f$ then $\varphi[F, G, H, Y, n+1, \beta]$ is reducible. Now let α be an arbitrary reducible term with $\alpha^*(i) = f_i, i = 0, \cdots, n-1$. From this we infer a): if a is a reducible term of type σ then $\Delta_2[\alpha, n, a]$ (that is $\bar{\alpha}[n] * a$) is reducible. It follows from our inductive assumption, applied to $\langle f_0, \cdots, f_{n-1}, a^*\rangle$ that $\varphi[F, G, H, Y, n+1, \bar{\alpha}[n] * a]$ is reducible. We have thereby used that $(\bar{\alpha}[n] * a)^*(i) = f_i,\ i < n$ and $(\bar{\alpha}[n] * a)^*(n) = a^*$ holds. Now $(\lambda B\varphi[F, G, H, Y, n+1, \alpha[n] * B])[b] \doteq \varphi[F, G, H, Y, n+1, \bar{\alpha}[n] * b]$ is an axiom (eq 4). If b is reducible then the righthandside is reducible according to our arguments above, hence we infer from clause 1) that the lefthandside is reducible for all reducible terms b. Hence $\lambda B\varphi[F, G, H, Y, n+1, \alpha[n] * B]$ is itself reducible. As before we infer that $Y[F, \bar{\alpha}[n]]$ is reducible. We distinguish two cases. *Case* 1: $(Y[F, \bar{\alpha}[n]])^* < n$. Then we infer exactly as under I that $\varphi[F, G, H, Y, n, \alpha]$ is reducible. *Case* 2: $(Y[F, \bar{\alpha}[n]])^* \geqq n$. Using the same reasoning as under I we infer that $Y[F, \bar{\alpha}[n]] \doteq n+j$ is provable for some j. By means of rule B2 we infer $\vdash \varphi[F, G, H, Y, n, \alpha] \doteq H[F, n, \bar{\alpha}[n], \lambda B\varphi[F, G, H, Y, n+1, \bar{\alpha}[n] * B]]$. However $H, F, \bar{\alpha}[n], G, Y$ are all reducible and $\lambda B\varphi[F, G, H, Y, n+1, \bar{\alpha}[n] * B]$ has just been proved to be reducible, hence the righthandside of the last equation is reducible, and so the lefthandside according to 1) which concludes the proof.

The reader recognizes that the definition of barrecursive functionals has not been fully formalized within our calculus. Despite this the calculus is already strong enough to 'compute' the values of the constant terms of type 0 (in E). It is not difficult to see that our calculus is essentially contained in Spectors Σ_4 (apart from a slightly more general type structure). Moreover Σ_4 has essentially the same constant terms as our equational calculus (λ-abstraction can be defined within Σ_4). Hence we obtain the

COROLLARY. *For every constant term T of type* 0 *of Spectors system Σ_4 there is a number m such that $\Sigma_4 \vdash T = m$ holds.*

IV. Constructive elements

4.1. *The subset of constructive elements.*

In the last chapter we have developed a certain calculus in order to evaluate the constant terms of the set $\boldsymbol{E}$. In this chapter we will evaluate the constant terms of $\boldsymbol{E}$ by means of an other method which is more general and which will serve as basis for the next chapter. By induction with respect to types we introduce for each σ a subset $C(\sigma)$ of $S(\sigma)$, called the constructive elements of type σ, and with each $f \in C(\sigma)$ we associate a certain nonempty set $G(f)$ of natural numbers, called the Goedelnumbers of f.

DEFINITION 5. a) $C(0) = S(0) = \boldsymbol{N}$ and for $n \in \boldsymbol{N}$ we put $G(n) = \{n\}$. b) Assume that $C(\sigma_i) \subseteq S(\sigma_i)$ and $C(\tau) \subseteq S(\tau)$, $i = 1, \cdots, s$ is already known and that for each $f \in \bigcup_i C(\sigma_i) \cup C(\tau)$ a nonempty set $G(f)$ of natural numbers, (the Goedelnumbers of f) is given. By definition a functional $f \in S(\sigma)$, $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$, belongs to $C(\sigma)$ if and only if the following conditions are satisfied: 1) if $g_i \in C(\sigma_i)$, $i = 1, \cdots, s$, then $f(g_1, \ldots, g_s) \in C(\tau)$, 2) there is a partial recursive function μ with the property: if $e_i \in G(g_i)$, $i = 1, \cdots, s$, then $\mu(e_1, \cdots, e_s)$ is defined and in $G(f(g_1, \cdots, g_s))$. A number e belongs to $F(f)$ iff it is the Goedelnumber of a partial recursive function μ which satisfies clause 2) above. So much for the definition. The set $\boldsymbol{C} = \bigcup_\sigma C(\sigma)$ is called the set of constructive functionals.

4.2. *Closure properties of* $\boldsymbol{C}$.

The main properties of $\boldsymbol{C}$ are described by

THEOREM 6. 1) *$\boldsymbol{C}$ is closed against λ-abstraction*, 2) *$\boldsymbol{C}$ is closed against substitution*, 3) *$\boldsymbol{C}$ is closed against permutation* (I, 1.5. a)). 4) *the projections, the functionals E, 0, Δ_1, Δ_2, Δ_3 and the successor functions belong to $\boldsymbol{C}$*, 5) *the induction functionals J of all types belong to $\boldsymbol{C}$*, 6) *the barrecursive functionals φ of all types belong to $\boldsymbol{C}$.*

The proofs of 1)–5) reduce to quite elementary applications of the fixed point theorem and will be omitted. The only nontrivial part of the theorem is 6). In the proof of 6) below, we will make use of 1)–5) whenever we find it necessary.

PROOF OF 6). Let φ be a barrecursive functional of suitable type. In order to prove 6) we have to prove two things: 1) If Z, G, H, Y, α are in $\boldsymbol{C}$ then $\varphi(Z, G, H, Y, n, \alpha)$ is in $\boldsymbol{C}$, 2) there is a partialrecursive function $\mu(z, g, h, j, n, a)$ with the property: if z, g, h, j, a are the Goedelnumbers of Z, G, H, Y, α respectively, then $\mu(z, g, h, j, n, a)$ is a Goedelnumber of $\varphi(Z, G, H, Y, n, \alpha)$. Here z denotes a list of Goedelnumbers $z_1, \cdots, z_s$

associated with the members $F_1, \cdots, F_s$ of the list Z of parameters. We proceed in steps.

A. As follows from 3) of the theorem, Δ_1 and Δ_2 belong to $\boldsymbol{C}$. Let d and k be fixed Goedelnumbers of Δ_1 and Δ_2 respectively. Next, let Z, G, H, Y be elements from $\boldsymbol{C}$ whose types are Σ, $\sigma_1 = (\Sigma, 0, (0/\sigma)/\tau)$, $\sigma_2 = (\Sigma, 0, (0/\sigma), (\sigma/\tau)/\tau)$ and $\sigma_3 = (\Sigma, (0/\sigma)/0)$; hence φ has type $(\Sigma, \sigma_1, \sigma_2, \sigma_3, 0, (0/\sigma)/\tau)$. The list G, H, Y will be abbreviated by F. Let z, g, h, j be the Goedelnumbers of Z, G, H, J; we denote the list g, h, j by t.

B. According to a well known theorem in recursion theory there exists a primitive recursive function ϕ with $\{\phi(e, z, t, x, a)\}(s) = \{e\}(z, t, \{d\}(\{k\}(a, x, s), x+1), x+1)$.

C. Below, the following will be used: $\Delta_2(\alpha, x, r) = \Delta_2(\Delta_1(\alpha, x), x, r)$.

D. Now we are looking for a Goedelnumber e of a partialrecursive function $\{e\}(z, t, a, x)$ which satisfies the following conditions:

(U) $\{e\}(z, t, a, x) = \{g\}(z, x, a)$ if $\{j\}(z, a) < x+k$,

(V) $\{e\}(z, t, a, x) = \{h\}(z, x, a, \phi(e, z, t, x, a))$ if $x+k \leqq \{j\}(z, a)$.

According to the fixed point theorem there exists a Goedelnumber e having the following properties: 1) if $\{j\}(z, a)$ and $\{g\}(z, x, a)$ are defined and if $\{j\}(z, a) < x+k$ then $\{e\}(z, t, a, x)$ is defined and equal to $\{g\}(z, x, a)$, 2) if $\{j\}(z, a)$ and $\{h\}(z, x, a, \phi(e, z, t, x, a))$ are defined and $x+k \leqq \{j\}(z, a)$ then $\{e\}(z, t, a, x)$ is defined and its value equal to $\{h\}(z, x, a, \phi(e, z, t, x, a))$.

E. Define Y' of type $((0/\sigma)/0)$ as follows: $Y'(\alpha) = Y(Z, \alpha)$ for all α. Below we prove by barinduction over Y' the following statement $P(\langle f_0, \cdots, f_{n-1}\rangle)$ for all finite sequences $f_0, \cdots, f_{n-1}$ of elements $f_i \in S(\sigma)$, $i = 1, \cdots, n-1$: if $\alpha \in C(0/\sigma)$, if $\alpha(i) = f_i$ for $i < n$, if $a \in G(\alpha)$ then $\varphi(Z, G, H, Y, n, \alpha)$ is constructive and $\{e\}(z, t, \{d\}(a, n), n)$ a Goedelnumber of it.

F. Before coming to the proof of $P(\langle f_0, \cdots, f_{n-1}\rangle)$ we note that clause 6) follows as soon as we have proved $P(\langle f_0, \cdots, f_{n-1}\rangle)$ for all finite sequences $f_0, \cdots, f_{n-1}$. To see this, it is sufficient to apply a second time the theorem mentioned under B): there exists a primitive recursive function $\Gamma(x, y)$ such that $\{\Gamma(x, y)\}(z, t, n, a) = \{x\}(z, t, \{y\}(a, n))$. Then $\Gamma(e, d)$ is a Goedelnumber of φ.

G. Now to the proof of $P(\langle f_0, \cdots, f_{n-1}\rangle)$. Assume $\alpha \in C(0/\sigma)$, $a \in G(\alpha)$ and $\alpha(i) = f_i$ for $i < n$. *Case* 1: $f_0, \cdots, f_{n-1}$ is secured with respect to Y', k. This means $Y'(\bar{\alpha}(n)) < n+k$, that is $Y(Z, \bar{\alpha}(n)) < n+k$. Since Y, Z, α and so $\bar{\alpha}(n)$ are constructive it follows that $\{j\}(z, \{d\}(a, n))$ is defined and equal to $Y(Z, \bar{\alpha}(n))$, hence smaller than $n+k$. On the other hand, g is Goedelnumber of the constructive element G, hence $\{g\}(z, n, \{d\}(a, n))$ is defined and a Goedelnumber of $G(Z, n, \bar{\alpha}(n))$, which is constructive. By definition $\varphi(Z, F, n, \bar{\alpha}(n))$ is equal to $G(Z, n, \bar{\alpha}(n))$, hence it is constructive. Moreover it follows from the fixed point properties of e that $\{e\}(z, t, \{d\}(a, n), n)$ is defined and equal to $\{g\}(z, n, \{d\}(a, n))$. Thus $P(\langle f_0, \cdots . f_{n-1}\rangle)$ holds in this case. *Case* 2: Assume that for all $f \in S(\sigma)$ $P(\langle f_0, \cdots, f_{n-1}, f\rangle)$ is true. *Subcase* 1: $Y'(\bar{\alpha}(n)) < n+k$. Then we proceed as under case 1. *Subcase* 2: $Y'(\bar{\alpha}(n)) \geqq n+k$. Then we conclude as before that $\{j\}(z, \{d\}(a, n))$ is defined and $\geqq n+k$. By definition $\varphi(Z, F, n, \bar{\alpha}(n)) = H(Z, n, \bar{\alpha}(n), \lambda s\varphi(Z, F, n+1, \bar{\alpha}(n) * s))$. According to our inductive assumption, the following statement I holds for every $f \in S(\sigma)$: if $\beta \in C(0/\sigma)$ is such that $\beta(i) = f_i$, $i < n$ and $\beta(n) = f$ then $\varphi(Z, F, n+1, \bar{\beta}(n+1))$ is constructive and $\{e\}(z, t, \{d\}(b, n+1), n+1)$ a Goedelnumber of it, where b is any Goedelnumber of β. Since α is constructive we infer from C): if $u \in C(\sigma)$ then $\Delta_2(\alpha, n, u) \in C(0/\sigma)$, that is $\Delta_2(\Delta_1(\alpha, n), n, u) \in C(0/\sigma)$. If m is any Goedelnumber of u, then $\{k\}(\{d\}(a, n), n, m)$ is a Goedelnumber of $\Delta_2(\Delta_1(\alpha, n), n, u)$, that is of $\Delta_2(\alpha, n, u)$. If we replace β in statement I by $\Delta_2(\alpha, n, u)$ and b by $\{k\}(\{d\}(a, n), n, m)$ then it follows that $\varphi(Z, F, n+1, \bar{\alpha}(n) * u)$ is constructive with $\{e\}(z, t, \{d\}(\{k\}(\{d\}(a, n), n, m), n+1), n+1)$ as a Goedelnumber. Using the identity presented under B) it follows that $\{\phi(e, z, t, n, \{d\}(a, n))\}(m)$ is a Goedelnumber of $\varphi(Z, F, n+1, \bar{\alpha}(n) * u)$. Hence $\lambda B\varphi(Z, F, n+1, \bar{\alpha}(n) * B)$ belongs to $C(\sigma/\tau)$ and $\phi(e, z, t, n, \{d\}(a, n))$ is a Goedelnumber of it. Since H is constructive it follows that $H(Z, n, \bar{\alpha}(n), \lambda B\varphi(Z, F, n+1, \bar{\alpha}(n) * B))$ is constructive and that $q = \{h\}(z, n, \{d\}(a, n), \phi(e, z, t, n, \{d\}(a, n)))$ is one of its Goedelnumbers. Hence it follows that $\{e\}(z, t, \{d\}(a, n), n)$ is defined and equal to q, that is a Goedelnumber of $H(Z, n, \bar{\alpha}(n), \lambda B\varphi(Z, F, n+1, \bar{\alpha}(n) * B))$. That is $\varphi(Z, G, H, Y, n, \bar{\alpha}(n))$ is constructive and $\{e\}(z, g, h, j, \{d\}(a, n), n)$ indeed a Goedelnumber of it.

COROLLARY. *If T is a constant term belonging to* **E** *then $T^* \in \boldsymbol{C}$.*

The proof of this corollary requires only little additional work: it amounts to show that $(T/Z_1, \cdots, Z_N)^* \in \boldsymbol{C}$ provided the constants of T are in $\boldsymbol{C}$. This however is an easy consequence of clauses 1)–5) of theorem 6; we omit the details.

V. Barrecursion and classical analysis

5.1. *Remarks on the topological structure of* **S**

It was the aim of Spector to give a consistency proof of classical analysis via barrecursion of higher types. Conversely, one should expect that barrecursion itself is not stronger than classical analysis. If we look at our model **S** then we see that it is not suitable for this purpose since it was constructed within whole Zermelo-Fraenkel set theory. Looking at the definition of $\boldsymbol{S}$ it is not at all clear how to reduce this model to classical analysis. The following reason supports this view: the functionals of higher types are themselves objects which are not directly accessible to classical analysis, since the power set has to be used in their definition. Thus $\boldsymbol{S}$ would be accessible to classical analysis only if we could reduce $\boldsymbol{S}$ in some way or the other to the countable. To this end one would try to prove among others that $S(\sigma)$ has a countable basis. That is we are obliged to study the topological structure of $\boldsymbol{S}$. Now the author was unable to prove anything of significance about this topological structure; in particular he was not able to prove that $S(\sigma)$ has a countable basis. Hence something else had to be tried. An indication of what can be done is given by

THEOREM 7. *Let $S(\sigma)^*$ be the following subset of $S(\sigma) : f \in S(\sigma)^*$ iff there exists a sequence $f_0, f_1, \cdots$ of elements from $C(\sigma)$ such that $f_n \to f$. Then $\boldsymbol{S}^* = \bigcup_\sigma S(\sigma)^*$ is a model for barrecursion of higher type.*

The proof of this theorem is not straightforward, however we omit it because it is very similar to the proof of another theorem to be given below. Rather than to give further comments on theorem 7 we pass to the construction of another model $\boldsymbol{K} = \bigcup_\sigma K(\sigma)$, which can be reduced to classical analysis. The reader will recognize that theorem 7 serves as a guide to the construction of $K(\sigma)$.

5.2. *Remarks on classical analysis*

Standard systems of classical analysis are systems whose basic language is that of second order arithmetic and which are obtained from formal number theory by suitably formalized versions of the following axiom schemas: 1) bar induction of type 0, 2) transfinite induction of type 0, 3) comprehension axiom, 4) axiom of choice. There is another system of classical analysis, which is equivalent to any of the systems listed above, namely ZF^-, that is Zermelo-Fränkel minus powerset axiom. This fact is discussed at some length by G. Kreisel in the introduction to chapter I of the Stanford report, vol. I ([5]). If we add Goedels axiom $V = L$ to ZF^-, then we obtain the theory $ZF^- + V = L$ which can be interpreted in ZF^-. The author has not found a published proof of this fact, however it seems to be contained implicitly in the literature, as the author learned from G. Kreisel (communication by letter). Anyway, a routine inspection of [0] shows, that Goedels construction can be used in a straightforward way to obtain a reduction of $ZF^- + V = L$ to ZF^-. Actually we could perform the construction to be presented below without the aid of $V = L$; in this case however we would have to replace equality by an equivalence relation and this would increase work and space considerably. Another remark about the reduction of the announced model $\boldsymbol{K}$ to $ZF^- + V = L$ is appropriate. To this end, define h by 1) $h(0) = 1$, 2) $h((\sigma_1, \cdots, \sigma_s/\tau)) = \sum_i h(\sigma_i) + h(\tau)$. Now we cannot prove within $ZF^- + V = L$ a statement like '$\boldsymbol{K}$ is closed against primitive recursion and barrecursion'. However, as will be seen, we can prove in $ZF^- + V = L$ for every N the statement '$\bigcup_\sigma K(\sigma)$, $h(\sigma) \leqq N$, is closed against primitive recursion and barrecursion provided that the types σ which have to be considered satisfy $h(\sigma) \leqq N$'. Finally a few words about the formalization itself. We do not present a formalization of a proof of the last statement within $ZF^- + V = L$ in the proper sense of the word; this would require too much space and work. We will present the arguments in an intuitive fashion, but in such a way that it will be clear to the reader that the proof could be reproduced within the system $ZF^- + V = L$.

Definition 6. The order $Od(x)$ of any element is the smallest ordinal α such that $F_C\alpha = x$ ([0]).

5.3. *The model* $K = \bigcup K(\sigma)$

a) By induction with respect to types we introduce for each σ a class $K(\sigma)$, a subset $C(\sigma) \subseteq K(\sigma)$, for each $f \in C(\sigma)$ a nonempty set $G(f)$ of natural numbers, the Goedelnumbers of f, and finally a convergence notion $\to_\sigma$, mostly written simply as $\to$, if no danger of confusion arises.

Case 1: $\sigma = 0$. Then $K(0) = C(0) =$ set of natural numbers and for $n \in C(0)$ we put $G(n) = \{n\}$. By definition $n_i \to n$, $i = 1, 2, \cdots$ if $n_i = n$ with the exception of finitely many i's.

Case 2: $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$. Assume that for every type μ from the set $\{\sigma_1, \cdots, \sigma_s, \tau\}$ we have defined a class $K(\mu)$, a subset $C(\mu)$, for each $f \in C(\mu)$ a nonempty set $G(f)$ of Goedelnumbers and a convergence notion $\to$, having the following properties: 1) axioms 1)–4) of Kuratowski hold for $\to$ (I, 1.2), 2) for every $f \in K(\mu)$ there is a sequence $f_i \in C(\mu)$, $i = 1, 2, \cdots$ such that $f_i \to f$, $i = 1, 2, \cdots$, 3) if $f \neq f'$ then $G(f) \cap G(f') = \phi$. By $C(\sigma)'$ we denote the set of mappings from $C(\sigma_1) \times \cdots \times C(\sigma_s)$ into $C(\tau)$ which have the properties α), β), γ) listed below. α) For $f \in C(\sigma)'$ there exists a partial recursive function $\phi(x_1, \cdots, x_s)$ such that $\phi(e_1, \cdots, e_s)$ is defined and in $G(f(f_1, \cdots, f_s))$ whenever $f_i \in C(\sigma_i)$ and $e_i \in G(f_i)$. β) For $f \in C(\sigma)'$ and every s-tupel $\langle f_1, \cdots, f_s \rangle \in K(\sigma_1) \times \cdots \times K(\sigma_s)$ there is a g in $K(\tau)$ having the property: if $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ and if $f_i^n \in C(\sigma_i)$ for all i, n then $f(f_1^n, \cdots, f_s^n) \to g$. According to Kuratowski's axiom 4) and property 2) of the $K(\mu)$'s it follows that there exists exactly one such g. Furthermore it follows from axioms 1)–3) of Kuratowski that if $f_1, \cdots, f_s$ happen to be in $C(\sigma_1), \cdots, C(\sigma_s)$ respectively that then necessarily $f(f_1, \cdots, f_s) = g$. Hence we can extend

f in an unambiguous way to whole $K(\sigma_1) \times \cdots \times K(\sigma_s)$. The element g, uniquely determined by $f_1, \cdots, f_s$ is denoted by $\bar{f}(f_1, \cdots, f_s)$, that is $\bar{f}$ is the just mentioned extension of f. γ) $\bar{f}$ is continuous on $K(\sigma_1) \times \cdots \times K(\sigma_s)$ that is if $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ then $\bar{f}(f_1^n, \cdots, f_s^n) \to \bar{f}(f_1, \cdots, f_s)$. A number e is called a Goedelnumber of f if e is the Goedelnumber of a partialrecursive function ϕ which satisfies the clauses stated under α); the set of Goedelnumbers of f is again denoted by $G(f)$. Before proceeding further we note the

LEMMA 12. *For $f, f' \in C(\sigma)'$, if $f \neq f'$ then $G(f) \cap G(f') = \phi$.*

PROOF. Assume $e \in G(f) \cap G(f')$ and $f \neq f'$. Then there are elements $f_i \in C(\sigma_i)$, $i = 1, \cdots, s$ such that $f(f_1, \cdots, f_s) \neq f'(f_1, \cdots, f_s)$. Assume $e_i \in G(f_i)$, $i = 1, \cdots, s$. Then $\{e\}(e_1, \cdots, e_s) \in G(f(f_1, \cdots, f_s)) \cap G(f'(f_1, \cdots, f_s))$, contradicting assumption 3) about the $C(\mu)$'s.

An immediate consequence of this is

COROLLARY. *$C(\sigma)'$ is a set.*

Next we define a class $K(\sigma)'$. As elements we take the denumerable sequences $G = \{F_1, F_2, \cdots\}$ of elements from $C(\sigma)'$ having properties u), v) listed below.

u) For every s-tupel $\langle f_1, \cdots, f_s \rangle \in K(\sigma_1) \times \cdots \times K(\sigma_s)$ there exists a $g \in K(\tau)$ with the property: whenever $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ then $\bar{F}_n(f_1^n, \cdots, f_s^n) \to g$. According to axiom 4) of Kuratowski the element g is uniquely determined by $f_1, \cdots, f_s$). Hence we may denote it by $G(f_1, \cdots, f_s)$.

v) G is continuous, that is if $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ then $G(f_1^n, \cdots, f_s^n) \to G(f_1, \cdots, f_s)$. Two elements from $K(\sigma)'$ will be called equivalent, in symbols $G_1 \simeq G_2$ if $G_1(f_1, \cdots, f_s) = G_2(f_1, \cdots, f_s)$ for all $f_1, \cdots, f_s$ from $K(\sigma_1), \cdots, K(\sigma_s)$ respectively. That $\simeq$ is indeed an equivalence relation is easily seen. Now we can define $K(\sigma)$. By definition $K(\sigma)$ is that subclass of $K(\sigma)'$ which contains from every equivalence class precisely one element, namely that one of lowest order. Prior to the definition of $C(\sigma)$ we note in this connection

LEMMA 13. *If $F \in C(\sigma)'$ then $G = \{F, F, \cdots\}$ is in $K(\sigma)'$ and $G(f_1, \cdots, f_s) = \bar{F}(f_1, \cdots, f_s)$ for all $f_i \in K(\sigma_i)$, $i = 1, \cdots, s$.*

We omit the obvious proof. Now to $C(\sigma)$. As $C(\sigma)$ we take the subset of $K(\sigma)$ given as follows: a sequence $P = \{F_1, F_2, \cdots\}$ from $K(\sigma)$ belongs to $C(\sigma)$ if and only if P is equivalent to a sequence $\{F, F, \cdots\}$ for some $F \in C(\sigma)'$. As set $G(P)$ of Goedelnumbers of such a $P \in C(\sigma)$ we take the set $G(F)$. Finally, it remains to define $\to$ on $K(\sigma)$. We put $G_n \to G$, $n = 1, 2, \cdots$ ($G_n, G \in K(\sigma)$) if and only if $G_n(f_1^n, \cdots, f_s^n) \to G(f_1, \cdots, f_s)$ for all f_i^n, f_i such that $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$.

This concludes the definition of the objects $K(\sigma)$, $C(\sigma)$ and $\to$.

b) It remains to verify that they have properties 1), 2), 3).

LEMMA 14. *$K(\sigma)$, $C(\sigma)$, $\to$ and $G(x)$ have properties* 1), 2), 3).

PROOF. a) First we note that $K(\sigma)$ is a class, whose elements are (or rather represent) continuous mappings from $K(\sigma_1)\times\cdots\times K(\sigma_s)$ into $K(\tau)$. The convergence notion $\to$ on $K(\sigma)$ is that one given by Kuratowski in [2], pg. 94. The proof that $\to$ indeed satisfies the axioms 1), 2), 3) of Kuratowski is word by word the same as that one given in [2], pg. 94. Hence let us verify 4). Assume $G_n - G$ and $G_n \to G'$. Then we infer in particular that for any s-tupel $f_1, \cdots, f_s$ from $K(\sigma_1), \cdots, K(\sigma_s)$ respectively $G_n(f_1, \cdots, f_s) \to G(f_1, \cdots, f_s)$ and $G_n(f_1, \cdots, f_s) \to G'(f_1, \cdots, f_s)$ holds. Since 4) holds in $K(\tau)$ we conclude $G(f_1, \cdots, f_s) = G'(f_1, \cdots, f_s)$, that is $G \simeq G'$. Hence both are in the same equivalence class. Hence they are equal.

b) In order to verify 2), let $G = \{F_1, F_2, \cdots\}$ be an element of $K(\sigma)$, with F_i by definition in $C(\sigma)'$. Then, by definition $\bar{F}_n(f_1^n, \cdots, f_s^n) \to G(f_1, \cdots, f_s)$ for all s-tuples $f_1^n, \cdots, f_s^n$ and $f_1, \cdots, f_s$ from $K(\sigma_1)\times \cdots \times K(\sigma_s)$ such that $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$. Now let P_n be the unique element from $C(\sigma)$ with the property: $P_n \simeq \{F_n, F_n, \cdots\}$. According to lemma 13 we have $P_n(f_1, \cdots, f_s) = \bar{F}_n(f_1, \cdots, f_s)$ for all $f_i \in K(\sigma_i)$, $i = 1, \cdots, s$. Therefore $P_n(f_1^n, \cdots, f_s^n) \to G(f_1, \cdots, f_s)$ for all $f_i^n, f_i \in K(\sigma_i)$ with $f_i^n \to f_i$, $i = 1, \cdots, s$, $n = 1, 2, \cdots$. Hence $P_n \to G$ by definition.

c) Finally we consider 3). Let G_1, G_2 be two elements from $C(\sigma)$ with $G_1 \neq G_2$. By definition there exists elements F_1, F_2 in $C(\sigma)'$ such that $G_1 \simeq \{F_1, F_1, \cdots\}$, $G_2 \simeq \{F_2, F_2, \cdots\}$ holds. Also by definition $G(G_1) = G(F_1)$, $G(G_2) = G(F_2)$. It is clear that $F_1 = F_2$ would imply $G_1 = G_2$ by lemma 13. Hence $F_1 \neq F_2$. From lemma 12 we infer $G(F_1) \cap G(F_2) = \phi$, which concludes the proof.

REMARKS. If we formalize the definition of $K(\sigma)$, $C(\sigma)$, $\to_\sigma$ within $ZF^- + V = L$ then $K(\sigma)$ is a class; more precisely there is for each σ a formula $K_\sigma(x)$ with exactly one free variable expressing that x is an element of $K(\sigma)$. The class $C(\sigma)$ however, being the range of a certain function (G^{-1}), whose domain is a subset of natural numbers, turns out to be a set. The elements of $K(\sigma)$ are not functions in the proper sense of the word; but there is a formula $V_\sigma(x, y_1, \cdots, y_s, t)$ without other free variables, than those indicated, which expresses that x is an element from $K(\sigma)$ and that t is the value of x for the arguments $y_1, \cdots, y_s$. The convergence notion $\to_\sigma$ finally is represented by a formula $L_\sigma(x, y)$ with precisely x, y as free variables, the first of which runs over denumerable sequences of elements from $K(\sigma)$ while the second runs over elements from $K(\sigma)$; the formula expresses that the denumerable sequence 'x' converges against the element y.

5.4. *Some properties of* $\boldsymbol{K} = \bigcup K(\sigma)$

In this section we list some properties of $\boldsymbol{K}$ in the form of statements denoted by $L0, L1, \cdots$. Since most of these properties are easy consequences of the definition of $\boldsymbol{K}$ we omit the proofs in many cases or content ourselves with a hint.

L0: If $G(f_1, \cdots, f_s) = G'(f_1, \cdots, f_s)$ for all $f_1, \cdots, f_s$ then $G = G'$.

Proof. Obvious from our construction of $K(\sigma)$.

L1: If $G \in C(\sigma)$ with $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$ then $G(f_1, \cdots, f_s) \in C(\tau)$ for $f_i \in C(\sigma_i)$, $i = 1, \cdots, s$.
This is an immediate consequence of the definition of $C(\sigma)$.

L2: Let α be an arbitrary mapping from the set N of natural numbers into $K(\sigma)$. Then there is a unique element $G_\alpha \in K(0/\sigma)$ with $G_\alpha(n) = \alpha(n)$, $n \in \boldsymbol{N}$.

Proof. Let f be an arbitrary but fixed element from $C(\sigma)$. Put $\alpha(i) = f_i$ and let f_i^n, $n = 1, 2, \cdots$, be a list of elements from $C(\sigma)$ such that $f_i^n \to f_i$, $n = 1, 2, \cdots$ for all i. Define a mapping α_n from N into $C(\sigma)$ as follows: 1) if $i < n$ then $\alpha_n(i) = f_i^n$, 2) if $i \geqq n$ then $\alpha_n(i) = f_n^n$. One easily verifies $\alpha_n \in C(0/\sigma)'$. Consider $G = \{\alpha_1, \alpha_2, \cdots\}$. It follows from the definition of $K(0/\sigma)'$ that G is an element thereof; moreover it follows from the definition of G and the α_n's that $G(m) = \alpha(m)$ holds for all m. Then G_α is obviously the element of smallest order equivalent to G.

Notation. Without danger of confusion we write α in place of G_α.

L3: Let $Y \in \boldsymbol{K}$ be of type $((0/\sigma)/0)$. Then there exists for every $\alpha \in K(0/\sigma)$ an $n \in \boldsymbol{N}$ such that the following holds: if $\beta \in K(0/\sigma)$ and $\beta(i) = \alpha(i)$ for $i < n$ then $Y(\beta) = Y(\alpha)$.

Proof. Assume the contrary. Then there exists for every n *a* β_n with $\beta_n(i) = \alpha(i)$ for $i < n$ and $Y(\beta_n) \neq Y(\alpha)$. But this implies $\beta_n \to \alpha$ according to the definition of $\to$ in $K(0/\sigma)$. This in turn implies $Y(\beta_n) = Y(\alpha)$ for allmost all n's, contradicting $Y(\beta_n) \neq Y(\alpha)$, $n = 1, 2, \cdots$.

L4: Each functional $F \in K(\sigma_1, \cdots, \sigma_s/\tau)$ is continuous with respect to $\to_{\sigma_i}$, $i = 1, \cdots, s$ and $\to_\tau$.

Proof. Obvious, since this is one of the conditions which has to be satisfied by elements F from $K(\sigma_1, \cdots, \sigma_s/\tau)$.

L5: $F(f_1, \cdots, f_s)$, considered as a function of $F, f_1, \cdots, f_s$ is continuous with respect to $\to_\sigma$, $\to_{\sigma_i}$, $i = 1, \cdots, s$ and $\to_\tau$.

PROOF. Obvious, since $\to_\sigma$ has been defined in that way that the statement is true (where $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$).

L6: Let $L(x_1, \cdots, x_s, y)$ be a relation (or rather a formula) and $\sigma_1, \cdots, \sigma_s, \tau$ types having the following properties: 1) for every s-tupel $f_i \in K(\sigma_i)$, $i = 1, \cdots, s$ there exists exactly one $g \in K(\tau)$ with $L(f_1, \cdots, f_s, g)$, 2) if $f_i \in C(\sigma_i)$, $i = 1, \cdots, s$ and $L(f_1, \cdots, f_s, h)$ then $h \in C(\tau)$, 3) if $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ and $L(f_1^n, \cdots, f_s^n, h_n)$ then $h_n \to h$, 4) there exists a partialrecursive function ϕ such that $\phi(e_1, \cdots, e_s)$ is defined and in $G(h)$ whenever $f_i \in C(\sigma_i)$, and $e_i \in G(f_i)$, $i = 1, \cdots, s$. Then there exists a G in $C(\sigma)$ with the property: $G(f_1, \cdots, f_s) = h$ iff $L(f_1, \cdots, f_s, h)$ (with $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$).

PROOF. Define the mapping F from $C(\sigma_1) \times \cdots \times C(\sigma_s)$ into $C(\tau)$ as follows: $F(f_1, \cdots, f_s) = h$ iff $L(f_1, \cdots, f_s, h)$ holds. One easily verifies that F satisfies α), β), γ) in the definition of $C(\sigma)'$; hence $F \in C(\sigma)'$. Similarly we infer: $\bar{F}(f_1, \cdots, f_s) = h$ iff $L(f_1, \cdots, f_s, h)$ (with $\bar{F}$ as before the extension of F to whole $K(\sigma_1) \times \cdots \times K(\sigma_s)$). It is evident that the G in question is the element of smallest order equivalent to $\{F, F, \cdots\}$.

L7: Let $L(x_1, \cdots, x_s, y)$ be a relation having properties 1), 3) of L6 and the following additional property α): there exists a sequence $G_n \in C(\sigma)$ such that $G_n(f_1^n, \cdots, f_s^n) \to g$ whenever $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$. Then there is a $G \in K(\sigma)$ such that $G(f_1, \cdots, f_s) = h$ iff $L(f_1, \cdots, f_s, h)$.

PROOF. Let $F_n \in C(\sigma)'$ ($\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$) be such that $G_n \simeq \{F_n, F_n, \cdots\}$. From 2), 4) of L6 and α) of L7 we easily infer that $\{F_1, F_2, \cdots\}$ is in $K(\sigma)'$. The G we are looking for is the element of smallest order equivalent to $\{F_1, F_2, \cdots\}$.

L8: If $F \in C(\mu_1, \cdots, \mu_t, \sigma_1, \cdots, \sigma_s/\tau)$, if $f_i \in C(\mu_i)$, $i = 1, \cdots, t$, then $F(f_1, \cdots, f_t, x_1, \cdots, x_s)$, considered as a function of $x_1, \cdots, x_s$ alone is in $C(\sigma_1, \cdots, \sigma_s/\tau)$.

PROOF. The straightforward proof is via L6, taking for $L(x_1, \cdots, x_s, y)$ the relation $F(f_1, \cdots, f_s, x_1, \cdots, x_s) = y$.

L9: If $F \in K(\mu_1, \cdots, \mu_t, \sigma_1, \cdots, \sigma_s/\tau)$, $f_i \in K(\mu_i)$, $i = 1, \cdots, t$ then $F(f_1, \cdots, f_t, x_1, \cdots, x_s)$, considered as a function of $x_1, \cdots, x_s$ alone, is in $K(\sigma_1, \cdots, \sigma_s/\tau)$.

Proof. Let f_i^n and F_n be elements from $C(\mu_i)$ and $C(\mu_1, \cdots, \mu_t, \sigma_1, \cdots, \sigma_s/\tau)$ respectively such that $f_i^n \to f_i$, $F_n \to F$, $n = 1, 2, \cdots$, $i = 1, \cdots, t$, holds. Let $G_n \in C(\sigma_1, \cdots, \sigma_s/\tau)$ be such that $G_n(h_1, \cdots, h_s) = F_n(f_1^n, \cdots, f_t^n, h_1, \cdots, h_s)$ for all $h_1, \cdots, h_s$; that such G_n exist follows from L8. Let L be the relation: $F(f_1, \cdots, f_t, x_1, \cdots, x_s) = y$. One quickly verifies that L satisfies 1), 3) of L6 and that L and the G_n's satisfy α) of L7. Then apply L7.

L10: If $F \in C(\mu, \sigma_1, \cdots, \sigma_s/\tau)$ then there is a $G \in C(\sigma_1, \cdots, \sigma_s/(\mu/\tau))$ such that $G(f_1, \cdots, f_s)(h) = F(h, f_1, \cdots, f_s)$ for all f_i and h.

Proof. Let $L(x_1, \cdots, x_s, y)$ be the relation $(z)(y(z) = F(z, x_1, \cdots, x_s))$. From L8, L9 we infer that 1), 2) of L6 are satisfied. Assume $f_i^n \to f_i$, $n = 1, 2, \cdots, i = 1, \cdots, s$; let G and G_n be such that $G(u) = F(u, f_1, \cdots, f_s)$ and $G_n(u) = F(u, f_1^n, \cdots, f_s^n)$ hold for all u (in $K(\mu)$). Then $G_n \to G$ iff $G_n(h_n) \to G(h)$ for all $h_n \to h$. Hence assume $h_n \to h$. Then $F(h_n, f_1^n, \cdots, f_s^n) \to F(h, f_1, \cdots, f_s)$ according to the continuity of F. Hence $G_n \to G$ in virtue of the arbitrariness of the h_n's and h. Thus 3) of L6 is satisfied. The verification of 4) is an easy exercise in recursion theory and omitted. The existence of G is ensured by L6.

Notation. We denote the G in L10 by $\lambda BF[B, x_1, \cdots, x_s]$.

L11: If $F \in K(\mu, \sigma_1, \cdots, \sigma_s/\tau)$ then there is a $G \in K(\sigma_1, \cdots, \sigma_s/(\mu/\tau))$ (also denoted by $\lambda BF[B, x_1, \cdots, x_s]$) such that $G(f_1, \cdots, f_s)(h) = F(h, f_1, \cdots, f_s)$ for all $f_1, \cdots, f_s, h$.

Proof. Let $L(x_1, \cdots, x_s, y)$ be $(z)(y(z) = F(z, x_1, \cdots, x_s))$. According to L9 clause 1) of L6 is satisfied. Clause 3) of L6 follows in the same way as in the proof of L10. In order to verify α) of L7 assume $F_n \in C(\mu, \sigma_1, \cdots, \sigma_s/\tau)$ and $F_n \to F$ and put $G_n = \lambda BF_n[B, x_1, \cdots, x_s]$; we claim that L and G_n satisfy α) of L7. To this end assume $f_i^n \to f_i$, $n = 1, 2, \cdots, \; i = 1, \cdots, s$ and $L(f_1, \cdots, f_s, G)$, that is, $G(u) = F(u, f_1, \cdots, f_s)$ for all u. We have to prove: $G_n \to G$. Hence assume $h_n \to h$. Then $G_n(h_n) = F_n(h_n, f_1^n, \cdots, f_s^n) \to F(h, f_1, \cdots, f_s) = G(h)$, that is $G_n \to G$ holds indeed. The statement now follows from L7.

Along these lines we can develop the whole theory of Goedels primitive recursive functionals. We omit the details of construction which do not present any difficulties and content ourself by stating the main result.

Theorem 8. *$\boldsymbol{K} = \bigcup K(\sigma)$ and $\boldsymbol{C} = \bigcup C(\sigma)$ are closed under substitution and λ-abstraction. The primitive recursive functionals from Goedels system $\boldsymbol{T}$ all belong to $\boldsymbol{C} = \bigcup C(\sigma)$. Among these we mention in particular P, E, 0, Δ_1, Δ_2, Δ_3 and the induction functionals J. $\boldsymbol{K}$ and $\boldsymbol{C}$ are closed under permutation.*

This theorem will be used below several times without explicit mention.

5.5. *The barrecursive functionals are in* $\boldsymbol{C} = \bigcup C(\sigma)$

a) In order to show that the barrecursive functionals belong to $\boldsymbol{C}$ we retain the notions, notations and definitions used in chapters I, II and IV as they stand. The barrecursive equations in particular (I and II) and the definition of solution remains the same as before and parameters are handled in the same way; similarly with the equations I*, II* and solutions up to $\bar{\alpha}(x)$, etc. The proofs of the lemmas and theorems are often mere repetitions of other proofs given earlier. Therefore we will only work out those parts which require new arguments and treat the other parts summarily.

b) Next a remark concerning bar induction. According to L2, all mappings from N into $K(\sigma)$ belong to $K(0/\sigma)$; according to L3 all functionals of type $((0/\sigma)/0)$ are continuous. It follows from this that barinduction with respect to functionals of type $((0/\sigma)/0)$ is available in $ZF^- + V = L$. This will be of basic importance below.

c) For later use we note a few additional lemmas.

L12: If $F_n, n = 1, 2, \cdots$ and F are of type $(\mu, \sigma_1, \cdots, \sigma_s/\tau)$ and if $F_n \to F$ then $\lambda BF_n[B, x_1, \cdots, x_s] \to \lambda BF[B, x_1, \cdots, x_s]$.

L13: With F_n, F as in L12, if $f_i^n \to f_i$, $n = 1, 2, \cdots$, $i = 1, \cdots, s$ then $\lambda BF_n[B, f_1^n, \cdots, f_s^n] \to \lambda BF[B, f_1, \cdots, f_s]$.

L14: If $F \in \boldsymbol{K}$ is of type $(\Sigma, (0/\sigma)/\tau)$ then $F(Z, \bar{\alpha}(x) * s)$ depends continuously on all arguments.

L15: Let $F_n, n = 1, 2, \cdots$ be a family of elements from $K(\sigma_1, \cdots, \sigma_s/\tau)$. Then there is a $G \in K(0, \sigma_1, \cdots, \sigma_s/\tau)$ such that $G(n, x_1, \cdots, x_s) = F_n(x_1, \cdots, x_s)$.

The proof of L15 is very similar to that of L2; we omit it.

L16: The system I, II of equations without parameters and with G, H, Y held constant admit at most one solution.

The proof is by a straightforward transfinite induction.

L17: Let G_n, H_n, Y_n and G, H, Y, $n = 1, 2, \cdots$ be functionals of suitable types without parameters Z, having the following properties: 1) for every n there is a solution φ_n of I, II with respect to G_n, H_n, Y_n, 2) there is a solution φ of I, II with respect to G, H, Y, 3) $G_n \to G$, $H_n \to H$, $Y_n \to Y$. Then $\varphi_n \to \varphi$.

PROOF. Exactly the same as that of lemma 9.

L18: Let G_n, H_n, Y_n and G, H, Y, $n = 1, 2, \cdots$ be functionals of suitable types, all containing parameters Z. Assume that the following holds: 1) for every Z, n there exists a solution φ_Z^n of I, II with respect to

G_n, H_n, Y_n, Z, 2) for every Z there exists a solution φ_Z of I, II with respect to G, H, Y, Z, 3) G_n, H_n, Y_n converge against G, H, Y respectively. If $Z_n \to Z$ then $\varphi^n_{Z_n} \to \varphi_Z$.

PROOF. Follows from L17 as L10 from L9.

d) The major tool which we need in addition to the considerations of chapters II and IV is provided by a theorem, which is formulated below. In this theorem d and k are Goedelnumbers of the constructive elements Δ_1 and Δ_2 respectively which both belong to $\mathbf{C}$; Z is a list of parameters belonging to $\mathbf{C}$ and z is a list of Goedelnumbers associated with them. The functionals G, H, Y are assumed to belong to $\mathbf{C}$ and to have g, h, j as Goedelnumbers; the list g, h, j is abbreviated by t.

THEOREM 9. *There exists a Goedelnumber e of a partialrecursive function which has the following property. Let Z be a list of parameters, all belonging to* $\mathbf{C}$ *and G, H, Y functionals of appropriate types belonging to* $\mathbf{C}$. *Let z be a list of Goedelnumbers of the parameters Z and g, h, j Goedelnumbers of G, H, Y respectively. Assume that there exists a solution* $\varphi(x, \alpha)$ *of the barrecursive equations I, II with respect to G, H, Y and Z. Then* φ *belongs to* $\mathbf{C}$ *and* $\{e\}(z, t, \{d\}(a, n)), n)$ *is a Goedelnumber of* $\varphi(x, \alpha)$ *if* $\alpha \in C(0/\sigma)$ *and a a Goedelnumber of* α *(with t denoting g, h, j).*

PROOF. The partialrecursive function is the same as that one constructed with the aid of the fixed point theorem in part D of the proof of theorem 6, clause 6). The verification that e thus constructed has the properties required by our theorem, is by transfinite induction with respect to Y_0, k where Y_0 is given as follows: $Y_0(\xi) = Y(Z, \xi)$ for all ξ. The details of this transfinite induction are exactly the same as those presented in the proof of theorem 6), clause 6); hence we omit it.

As consequence of this theorem we obtain

THEOREM 10. *Let G, H, Y be functionals from* $\mathbf{C}$, *containing the list Z of parameters. Assume the following: for every value of Z there exists a solution* φ_Z *of the barrecursive equations I, II with respect to G, H, Y, Z. Then there exists a solution* $\varphi \in \mathbf{C}$ *with parameters Z of I, II with respect to G, H, Y with the property:* $\varphi(Z, x, \alpha) = \varphi_Z(x, \alpha)$.

PROOF. Let $L(Z, \alpha, x, y)$ be the relation 'there is a solution φ_Z of I, II with respect to G, H, Y, Z and $\varphi_Z(x, \alpha) = y$'. We show that L satisfies 1)–4) of L6. 1) is satisfied in virtue of L16 and the assumptions of the theorem. Clause 2) on the other hand is a consequence of theorem 9. The existence of the partialrecursive function required by clause 4) is easily inferred from the partialrecursive function provided by theorem 9. Clause 3) finally is a particular case of L17. The existence of φ is thus guaranteed by L6.

The last among the preliminary theorems needed is

THEOREM 11. *Let G_n, H_n, Y_n and G, H, Y have the following properties: 1) they contain no parameters, 2) $G_n \to G$, $H_n \to H$, $Y_n \to Y$, 3) for every n there exists a solution φ_n of equations I, II with respect to G_n, H_n, Y_n, 4) G_n, H_n, Y_n belong to* **C**. *Then there exists a solution φ up to $\bar{\alpha}(x)$ of I, II with respect to G, H, Y for every $\bar{\alpha}(x)$.*

PROOF. The proof is by transfinite induction with respect to Y, k. *Case* 1: $\bar{\alpha}(x)$ is secured. Put $\varphi(y, \beta) = G(x+y, \bar{\alpha}(x) * \bar{\beta}(y))$. The statement is then easily verified. *Case* 2: The statement holds for all $\bar{\alpha}(x) * a$. By the barinductive assumption there exists a solution $\varphi_a(y, \beta)$ for every a of:

1) $\varphi_a(y, \bar{\beta}(y)) = \varphi_a(y, \beta)$,

2) I*: $\varphi_a(y, \bar{\beta}(y)) = G(x+1+y, \bar{\alpha}(x) * a * \bar{\beta}(y))$ if $Y(\bar{\alpha}(x) * a * \bar{\beta}(y)) < x+y+k+1$,

II*: $= H(x+y+1, \bar{\alpha}(x) * a * \bar{\beta}(y), \lambda s \varphi_a(y+1, \bar{\beta}(y) * s))$ otherwise.

Let $L(a, y, \beta, z)$ be the predicate 'there exists a solution φ_a of I*, II* and $z = \varphi_a(y, \bar{\beta}(y))$'. We apply L7 to L, with a, y, β in place of $x_1, \cdots, x_s$ and z in place of y. From L16 and L18 we infer that 1), 3) of L6 are satisfied. In order to verify α) of L7 let $\bar{\alpha}_n(x)$ be a sequence of constructive elements converging against $\bar{\alpha}(x)$. For x, α_n fixed denote $\varphi_n(x+y+1, \bar{\alpha}_n(x) * a * \bar{\beta}(y))$ by $\tilde{\varphi}_n$: $\tilde{\varphi}_n(a, y, \beta) = \varphi_n(x+y+1, \bar{\alpha}_n(x) * a * \bar{\beta}(y))$. Since G_n, H_n, Y_n are constructive by assumption 4) one immediately infers from theorem 9 that φ_n and hence $\tilde{\varphi}_n$ is constructive. We show that the sequence $\tilde{\varphi}_n$ and L satisfy α) of L7. Define G'_n, H'_n, Y'_n as follows: 1) $Y'_n(a, \beta) = Y_n(\bar{\alpha}_n(x) * a * \beta)$, 2) $G'_n(a, y, \beta) = G_n(x+y+1, \bar{\alpha}_n(x) * a * \bar{\beta}(y))$, 3) $H'_n(a, y, \beta, \xi) = H_n(x+y+1, \bar{\alpha}_n(x) * a * \bar{\beta}(y), \xi)$. Define G', H', Y' similarly but with G, H, Y in place of G_n, H_n, Y_n respectively on the righthandside of 1)–3). It follows from assumption 2) that $G'_n \to G'$, $H'_n \to H'$ and $Y'_n \to Y'$ holds. Obviously $\tilde{\varphi}_n$ is a solution of I, II with respect to G'_n, H'_n, Y'_n with a as parameter. φ_a on the other hand is a solution of I, II with respect to G', H', Y' and a. It follows from L18 that the following holds: if $a_n \to a$, $\beta_n \to \beta$ then $\tilde{\varphi}_n(a_n, y, \beta_n) \to \varphi_a(y, \beta)$. Hence $\tilde{\varphi}_n$, $n = 1, 2, \cdots$ and L satisfy indeed α) of L7. Hence we infer from L7 that there is a $\hat{\varphi} \in K$ such that $\hat{\varphi}(a, y, \beta) = z$ iff $L(a, y, \beta, z)$. Now define φ as follows: a) $\varphi(y+1, a * \beta) = \hat{\varphi}(a, y, \beta)$, b$_1$) $\varphi(0, \beta) = G(x, \bar{\alpha}(x))$ iff $Y(\bar{\alpha}(x)) < x+k$, b$_2$) $\varphi(0, \beta) = H(x, \bar{\alpha}(x), \lambda s \hat{\varphi}(s, 0, 0))$ otherwise. The verification that φ thus defined is indeed a solution up to $\bar{\alpha}(x)$ of I, II with respect to G, H, Y is now precisely the same as in the proof of theorem 2.

For $x = 0$ we obtain the

COROLLARY 1. *Let* G_n, H_n, Y_n *and* G, H, Y *not contain parameters. Assume*: 1) $G_n \to G$, $H_n \to H$, $Y_n \to Y$, 2) G_n, H_n, Y_n *are all in* $\boldsymbol{C}$, 3) *for every n there exists a solution* φ_n *of I, II with respect to* G_n, H_n, Y_n. *Then there exists a solution of I, II with respect to* G, H, Y.

COROLLARY 2. *Let* G, H, Y *be elements from* $\boldsymbol{C}$ *containing parameters* Z. *Assume that for all constructive values of* Z *there exists a solution* φ_Z *of I, II with respect to* G, H, Y. *Then there exists a solution* φ *with parameters* Z *of I, II with respect to* G, H, Y; *this solution belongs to* $\boldsymbol{C}$.

PROOF. Let Z be arbitrary and let Z_n, $n = 1, 2, \cdots$ be a sequence of values of the parameters, all belonging to $\boldsymbol{C}$; assume $Z_n \to Z$. By assumption there exists a solution φ_n of I, II with respect to G, H, Y, Z_n. Define G_n, H_n, Y_n without parameters as follows: 1) $G_n(x, \alpha) = G(Z_n, x, \alpha)$, 2) $H_n(x, \alpha, \xi) = H(Z_n, x, \alpha, \xi)$, 3) $Y_n(\alpha) = Y(Z_n, \alpha)$. Define G', H', Y' without parameters as follows: 1) $G'(x, \alpha) = G(Z, x, \alpha)$, 2) $H'(x, \alpha, \xi) = H(Z, x, \alpha, \xi)$, 3) $Y'(x, \alpha) = Y(Z, x, \alpha)$. In virtue of the continuity of G, H, Y we have $G_n \to G'$, $H_n \to H'$, $Y_n \to Y'$. The elements G_n, H_n, Y_n are clearly all in $\boldsymbol{C}$ since the Z_n's are in $\boldsymbol{C}$. Obviously φ_n is a solution of I, II with respect to G_n, H_n, Y_n. But now we are precisely in the situation of the last corollary, that is we can infer that a solution φ' of I, II with respect to G', H', Y' exists. Hence a solution φ' of I, II with respect to G, H, Y and Z exists. Since Z was arbitrary we infer that for every Z a solution φ_Z of I, II with respect to G, H, Y, Z exists. But G, H, Y are constructive and so the theorem follows from theorem 10.

The main result from which everything else follows is

THEOREM 12. *Let* G, H, Y *be constructive without parameters. If* $\bar{\alpha}(x)$ *is constructive then there exists a solution* φ *up to* $\bar{\alpha}(x)$ *of I, II with respect to* G, H, Y.

PROOF. We proceed by transfinite induction with respect to Y, k. *Case* 1: $\bar{\alpha}(x)$ is secured. Then we proceed as under case 1 in the proof of theorem 2. *Case* 2: for every constructive a the statement of the theorem holds, that is there exists a solution φ_a of:

1) $\varphi_a(y, \beta) = \varphi_a(y, \bar{\beta}(y))$,

2) I*: $\varphi_a(y, \beta) = G(x+y+1, \bar{\alpha}(x) * a * \bar{\beta}(y))$ if $Y(\bar{\alpha}(x) * a * \bar{\beta}(y)) < x+k+1+y$,

II*: $= H(x+y+1, \bar{\alpha}(x) * a * \bar{\beta}(y), \lambda s \varphi_a(y, \bar{\beta}(y) * s))$ otherwise.

Define G', H', Y' as follows: 1) $G'(a, y, \beta) = G(x+y+1, \bar{\alpha}(x) * a * \bar{\beta}(y))$, 2) $Y'(a, \beta) = Y(\bar{\alpha}(x) * a * \beta)$, 3) $H'(a, y, \beta, \xi) = H(x+y+1,$

$\bar{\alpha}(x) * a * \bar{\beta}(y), \xi)$. The elements G', H', Y' are obviously constructive (since $\bar{\alpha}(x)$ is constructive by assumption) and for constructive a φ_a is a solution of I, II with respect to G', H', Y' and a. But this are precisely the assumptions stated in corollary 2) of theorem 11. Hence, according to this corollary there exists a single solution $\hat{\varphi} \in C$ with a as parameter of I, II with respect to G', H', Y'. In other words, $\hat{\varphi}(a, y, \beta)$, considered as function of y, β only, is a solution up to $\bar{\alpha}(x) * a$ of I, II with respect to G, H, Y and this for every a, not only for constructive ones. Now define φ as follows: a) $\varphi(y+1, a * \beta) = \hat{\varphi}(a, y, \beta)$, b$_1$) $\varphi(0, \beta) = G(x, \bar{\alpha}(x))$ if $Y(\bar{\alpha}(x)) < x+k$, b_2) $\varphi(0, \beta) = H(x, \bar{\alpha}(x), \lambda s\hat{\varphi}(s, 0, 0))$ otherwise. The verification that φ thus defined is indeed a solution up to $\bar{\alpha}(x)$ of I, II with respect to G, H, Y is again the same as in the proof of theorem 2.

Corollary 1. *If G, H, Y are constructive without parameters then there exists a solution φ of I, II with respect to G, H, Y which belongs to $\boldsymbol{C}$.*

Proof. Put $x = 0$ in the last theorem and apply the theorem.

Theorem 13. *If G, H, Y are arbitrary without parameters then there exists a solution φ of I, II with respect to G, H, Y.*

Proof. Let G_n, H_n, Y_n be a sequence of elements from $\boldsymbol{C}$ which converge against G, H, Y respectively and then combine corollary 1 of theorem 11 with corollary 1 of theorem 12.

Now we come to the final result.

Theorem 14. *The barrecursive functionals belong to $\boldsymbol{C}$.*

Proof. Let $L(Z, G, H, Y, y)$ be the relation expressing: 'y is the solution of I, II with respect to G, H, Y, Z'. According to L16 and the last theorem 1) of L6 holds. Clause 2) of L6 is satisfied in virtue of theorem 9. The partial recursive function required by 4) of L6 is provided by the partial recursive function mentioned in theorem 9. Clause 3) of L6 finally follows from L18. Hence there is a $\varphi \in \boldsymbol{C}$ such that $\varphi(Z, G, H, Y, x, \alpha)$, considered as a function of x, α only, is a solution of I, II with respect to G, H, Y, Z. That is the barrecursive functionals exist and belong to $\boldsymbol{C}$.

5.6. *Remarks*

The theory of functionals $\bigcup_\sigma K(\sigma)$ can be developed in $ZF^- + V = L$ in the sense mentioned in 5.2. A meticulous formalization would require lengthy work but we hope that it is clear from our intuitively presented theory that such a formalization is possible in principle. In order to formulate a particular consequence of this, let Σ_4^N be Spectors system restricted to formulas containing only variables and constants of types σ with $h(\sigma) \leqq N$. With the aid of a suitable truth definition it is not difficult to prove with the aid of the model $\boldsymbol{K}$

THEOREM 15. *In $ZF^- + V = L$ we can prove the consistency of Σ_4^N.*

5.6. *Some strange extensions of classical analysis*

The particular formalization of classical analysis which we have in mind is the system $Z_\sigma + DC_\sigma$, considered in appendix 1 of [1]. The language of this system contains variables for functionals of all types. Z_σ is just classical number theory, formulated within that language. DC_σ is the axiom of dependent choices, which looks as follows:

$$(X)(EY)A(X, Y) \supset (Z)(EF)(n)(F(0) = Z \wedge A(F(n), F(n+1))$$
$$(X, Y, Z \text{ variables of type } \sigma).$$

THEOREM 16. *$\boldsymbol{K}$ and $\boldsymbol{S}$ are models of $Z_\sigma + DC_\sigma$.*

PROOF. Assume eg. that $A(X, Y)$ has only X, Y free and that $(X)(EY)A(X, Y)$ is true in $\boldsymbol{K}$. Let F be in $K(\sigma)$. Define F_n as follows: 1) $F_0 = F$, 2) F_{n+1} is an element such that $A(F_n, F_{n+1})$ is true; the existence of such an F_{n+1} follows from the truth of $(X)(EY)A(X, Y)$. Consider the mapping $\hat{F}(x)$ from $\boldsymbol{N}$ into $K(\sigma)$ given by: $\hat{F}(n) = F_n$. Then clearly $\hat{F}(0) = F$ and $A(\hat{F}(n), \hat{F}(n+1))$ is true for all n. According to L2 $\hat{F}$ belongs to $K(0/\sigma)$. Hence the righthandside of DC_σ is indeed satisfied. Similarly with $\boldsymbol{S}$ in place of $\boldsymbol{K}$.

Next let Y be a variable of type $((0/\sigma)/0)$. Let $\mathrm{Eq}_\sigma(U, V)$ be a formula with exactly two free variables U, V, both of type σ, defined as follows: 1) $\mathrm{Eq}_0(U, V)$ is $U = V$, 2) if $\sigma = (\sigma_1, \cdots, \sigma_s/\tau)$ then $\mathrm{Eq}_\sigma(U, V)$ is $(X_1, \cdots, X_s)\mathrm{Eq}_\tau(U(X_1, \cdots, X_s), V(X_1, \cdots, X_s))$. Briefly, $\mathrm{Eq}_\sigma(U, V)$ expresses that U, V are equal. Finally let $Ct_\sigma(Y)$ be the formula $(\xi)(\mathrm{E}x)(\eta)(\mathrm{Eq}_\mu(\bar{\xi}(x), \bar{\eta}(x)) \supset Y(\xi) = Y(\eta))$, (with $\mu = (0/\sigma)$); hence $Ct_\sigma(Y)$ expresses the continuity of Y. Since both in $\boldsymbol{S}$ and $\boldsymbol{K}$ functionals of type $((0/\sigma)/0)$ are continuous it follows that $(Y)CT_\sigma(Y)$ is true in both models. Hence we have

THEOREM 17. *$Z_\sigma + DC_\sigma + (Y)Ct_\sigma(Y)$ is consistent.*

With the aid of K one can reduce the consistency of $Z_\sigma + DC_\sigma + (Y)Ct_\sigma(Y)$ to the consistency of $ZF^- + V = L$; there are of course simpler ways to obtain such a reduction. We can go even further and add to $Z_\sigma + DC_\sigma$ a denumerable list of axioms which in their essence state that every functional is an element of $\boldsymbol{K}$; the result is a rather strange extension of classical analysis $(Z_\sigma + DC_\sigma + \boldsymbol{K})$ in which we can develop the whole theory presented in this chapter. The consistency of $Z_\sigma + DC_\sigma +$

$\boldsymbol{K}$ can again be reduced to that of $ZF^{-}+V=L$. We content ourselves with these indications since the interest of $Z_\sigma+DC_\sigma+\boldsymbol{K}$ lies only in its curiosity.

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