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AN ALGEBRAIC TREATMENT OF MAL'CEV'S THEOREMS CONCERNING NILPOTENT LIE GROUPS AND THEIR LIE ALGEBRAS

by

I. N. Stewart

This paper, and the one immediately following it, form the 1st, 2nd, 3rd, and 5th chapters of the author's Ph.D. thesis (Warwick 1969). Part of chapter 4 has already appeared in [17] and parts of later chapters are in the process of publication [18, 19].

The present paper gives a 'bare-hands' method of constructing, for any locally nilpotent Lie algebra over the rational field, the corresponding Lie Group (in an obvious sense), which is a complete locally nilpctent torsion-free group. Thus we have a purely algebraic version of the theorems of A. I. Mal'cev [14]. The paper following this one develops two lines of thought. The first is a Lie algebra version of a theorem of Roseblade [16] on groups in which all subgroups are subnormal. The other deals with certain chain conditions in special classes of Lie algebras, and should be considered as a sequel to [17].

These results arose from a general attack on the structure of infinite-dimensional Lie algebras, in the spirit of infinite group theory. For completeness, and to put them into a suitable context, we first give a brief outline of the main results of the thesis. (Unfortunately, since certain theorems have already been extracted as separate papers, it has not been possible to publish the thesis as a whole.)

Abstract of Thesis

Chapter 1 sets up notation.

Chapter 2 gives an algebraic treatment of Mal'cev's correspondence between complete locally nilpotent torsion-free groups and locally nilpotent Lie algebras over the rational field. This enables us to translate certain of our later results into theorems about groups. As an application we prove a theorem about bracket varieties.

Chapter 3 considers Lie algebras in which every subalgebra is an n-step subideal and shows that such algebras are nilpotent of class bounded in terms of n. This is the Lie-theoretic analogue of a theorem of J. E. Roseblade about groups.

Chapter 4 considers Lie algebras satisfying certain minimal conditions on subideals. We show that the minimal condition for 2-step subideals implies Min-si, the minimal condition for *all* subideals, and that any Lie algebra satisfying Min-si is an extension of a $\mathfrak{T}$-algebra by a finite-dimensional algebra (a $\mathfrak{T}$-algebra is one in which every subideal is an ideal). We show that algebras satisfying Min-si have an ascending series of ideals with factors simple or finite-dimensional abelian, and that the type of such a series may be made any given ordinal number by suitable choice of Lie algebra. We show that the Lie algebra of all endomorphisms of a vector space satisfies Min-si. As a by-product we show that every Lie algebra can be embedded in a simple Lie algebra (and a similar result holds for associative algebras). Not every Lie algebra can be embedded as a subideal in a perfect Lie algebra.

Chapter 5 considers chain conditions in more specialised classes of Lie algebras. Thus a locally soluble Lie algebra satisfying Min-si must be soluble and finite-dimensional. A locally nilpotent Lie algebra satisfying the maximal condition for ideals is nilpotent and finite-dimensional. A similar result does not hold for the minimal condition for ideals. All of these results can be combined with the Mal'cev correspondence to give theorems about groups.

Chapter 6 develops the theory of $\mathfrak{T}$-algebras, and in particular classifies such algebras under conditions of solubility (over any field) or finite-dimensionality (characteristic zero). We also classify locally finite Lie algebras, every subalgebra of which lies in $\mathfrak{T}$, over algebraically closed fields of characteristic zero.

Chapter 7 concerns various radicals in Lie algebras, analogous to certain standard radicals of infinite groups. We show that not every Baer algebra is Fitting, answering a question of B. Hartley [5]. As a consequence we can exhibit a torsion-free Baer goup which is not a Fitting group (previous examples have all been periodic). We show that under certain circumstances Baer *does* imply Fitting (both for groups and Lie algebras). The last section considers Gruenberg algebras.

Chapter 8 is an investigation parallelling those of Hall and Kulatilaka [4, 10] for groups. We ask: when does an infinite-dimensional Lie algebra have an infinite-dimensional abelian subalgebra? The answer is: not always. Under certain conditions of generalised solubility the answer is 'yes' and we can make the abelian subalgebra in question have additional properties (e.g. be a subideal). The answer is also shown to be 'yes' if the algebra is locally finite over a field of characteristic zero. This implies that any infinite-dimensional associative algebra over a field of characteristic zero contains an infinite-dimensional commutative subalgebra. It also implies that a locally finite Lie algebra over a field of characteristic zero satisfies the minimal condition for subalgebras if and only if it is finite-dimensional.

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1. Notation and Terminology

Throughout this paper we shall be dealing mainly with infinite-dimensional Lie algebras. Notation and terminology in this area are non-standard; the basic concepts we shall need are dealt with in this preliminary chapter. In any particular situation all Lie algebras will be over the same fixed (but arbitrary) field $\mathfrak{k}$; though on occasion we may impose further conditions on $\mathfrak{k}$.

1.1 *Subideals*

Let L be a Lie algebra (of finite or infinite dimension) over an arbitrary field $\mathfrak{k}$. If $x, y \in L$ we use square brackets $[x, y]$ to denote the Lie product of x and y. If H is a (Lie) subalgebra of L we write $H \leqq L$, and if H is an ideal of L we write $H \lhd L$. The symbol $\subseteq$ will denote set-theoretic inclusion.

A subalgebra $H \leqq L$ is an *ascendant subalgebra* if there exists an ordinal number σ and a collection $\{H_\alpha : 0 \leqq \alpha \leqq \sigma\}$ of subalgebras of L such that $H_0 = H$, $H_\sigma = L$, $H_\alpha \lhd H_{\alpha+1}$ for all $0 \leqq \alpha < \sigma$, and $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for limit ordinals $\lambda \leqq \sigma$. If this is the case we write $H \lhd^\sigma L$. $H \operatorname{asc} L$ will denote that $H \lhd^\sigma L$ for some σ. If $H \lhd^n L$ for a finite ordinal n we say H is a *subideal* of L and write $H \operatorname{si} L$. If we wish to emphasize the rôle of the integer n we shall refer to H as an *n-step subideal* of L.

If $A, B \leqq L$, $X \subseteq L$, and $a, b \in L$ we define $\langle X \rangle$ to be the subalgebra of L generated by X; $[A, B]$ to be the subspace spanned by all products $[a, b]$ $(a \in A, b \in B)$ $[A, {}_nB] = [[A, {}_{n-1}B], B]$ and $[A, {}_0B] = A$; $[a, {}_nb] = [[a, {}_{n-1}b], b]$ and $[a, {}_0b] = a$. We let $\langle X^A \rangle$ denote the *ideal closure* of X under A, i.e. the smallest subalgebra of L which contains X and is invariant under Lie multiplication by elements of A.

1.2 *Derivations*

A map $d : L \to L$ is a *derivation* of L if it is linear and, for all $x, y \in L$,

$$[x, y]d = [xd, y] + [x, yd].$$

The set of all derivations of L forms a Lie algebra under the usual vector space operations, with Lie product $[d_1, d_2] = d_1 d_2 - d_2 d_1$. We denote this algebra by $\operatorname{der}(L)$ and refer to it as the *derivation algebra* of L. If $x \in L$ the map $\operatorname{ad}(x) : L \to L$ defined by

$$y \cdot \operatorname{ad}(x) = [y, x] \qquad (y \in L)$$

is a derivation of L. Such derivations are called *inner derivations*. The map $x \to \operatorname{ad}(x)$ is a Lie homomorphism $L \to \operatorname{der}(L)$.

1.3 *Central and Derived Series*

L^n will denote the n-th term of the lower central series of L, so that $L^1 = L$, $L^{n+1} = [L^n, L]$. $L^{(\alpha)}$ (for ordinals α) will denote the α-th term of the (transfinite) derived series of L, so that $L^{(0)} = L$, $L^{(\alpha+1)} = [L^{(\alpha)}, L^{(\alpha)}]$, and $L^{(\lambda)} = \bigcap_{\alpha<\lambda} L^{(\alpha)}$ for limit ordinals λ. $\zeta_\alpha(L)$ will denote the α-th term of the (transfinite) upper central series of L, so that $\zeta_1(L)$ is the centre of L, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta_1(L/\zeta_\alpha(L))$, $\zeta_\lambda(L) = \bigcup_{\alpha<\lambda} \zeta_\alpha(L)$ for limit ordinals λ.

L^n, $L^{(\alpha)}$, and $\zeta_\alpha(L)$ are all *characteristic ideals* of L in the sense that they are invariant under derivations of L. We write $I \operatorname{ch} L$ to mean that I is a characteristic ideal of L. The important property of characteristic ideals is that $I \operatorname{ch} K \lhd L$ implies $I \lhd L$ (see Hartley [5] p. 257).

L is *nilpotent* (of *class* $\leqq n$) if $L^{n+1} = 0$, and is *soluble* (of *derived length* $\leqq n$) if $L^{(n)} = 0$.

1.4 *Classes of Lie Algebras*

We borrow from group theory the very useful ‘Calculus of Classes and Closure Operations’ of P. Hall [2].

By a *class of Lie algebras* we shall understand a class $\mathfrak{X}$ in the usual sense, whose elements are Lie algebras, with the further properties

C1) $\{0\} \in \mathfrak{X}$,
C2) $L \in \mathfrak{X}$ and $K \cong L$ implies $K \in \mathfrak{X}$.

Familiar classes of Lie algebras are:

$\mathfrak{D}$ = the class of all Lie algebras
$\mathfrak{A}$ = abelian Lie algebras
$\mathfrak{N}$ = nilpotent Lie algebras
$\mathfrak{N}_c$ = nilpotent Lie algebras of class $\leqq c$
$\mathfrak{F}$ = finite-dimensional Lie algebras
$\mathfrak{F}_m$ = Lie algebras of dimension $\leqq m$
$\mathfrak{G}$ = finitely generated Lie algebras
$\mathfrak{G}_r$ = Lie algebras generated by $\leqq r$ elements.

We shall introduce other classes later on, and will maintain a fixed symbolism for the more important classes. The symbols $\mathfrak{X}$, $\mathfrak{Y}$ will be reserved for arbitrary classes of Lie algebras. Algebras belonging to the class $\mathfrak{X}$ will often be called *$\mathfrak{X}$-algebras.*

A (non-commutative non-associative) binary operation on classes of Lie algebras is defined as follows: if $\mathfrak{X}$ and $\mathfrak{Y}$ are any two classes let $\mathfrak{X}\mathfrak{Y}$ be the class of all Lie algebras L having an ideal I such that $I \in \mathfrak{X}$ and $L/I \in \mathfrak{Y}$. Algebras in this class will sometimes be called *$\mathfrak{X}$-by-$\mathfrak{Y}$-algebras.* We extend this definition to products of n classes by defining

$$\mathfrak{X}_1 \cdots \mathfrak{X}_n = ((\mathfrak{X}_1 \cdots \mathfrak{X}_{n-1})\mathfrak{X}_n).$$

We may put all $\mathfrak{X}_i = \mathfrak{X}$ and denote the result by $\mathfrak{X}^n$. Thus in particular $\mathfrak{A}^n$ is the class of soluble Lie algebras of derived length $\leqq n$.

(0) will denote the class of 0-dimensional Lie algebras.

1.5 *Closure Operations*

A *closure operation* A assigns to each class $\mathfrak{X}$ another class $\mathrm{A}\mathfrak{X}$ (the A-*closure* of $\mathfrak{X}$) in such a way that for all classes $\mathfrak{X}$, $\mathfrak{Y}$ the following axioms are satisfied:

01) $\mathrm{A}(0) = (0)$
02) $\mathfrak{X} \leqq \mathrm{A}\mathfrak{X}$
03) $\mathrm{A}(\mathrm{A}\mathfrak{X}) = \mathrm{A}\mathfrak{X}$
04) $\mathfrak{X} \leqq \mathfrak{Y}$ implies $\mathrm{A}\mathfrak{X} \leqq \mathrm{A}\mathfrak{Y}$

($\leqq$ will denote ordinary inclusion for classes of Lie algebras). $\mathfrak{X}$ is said to be A-*closed* if $\mathfrak{X} = \mathrm{A}\mathfrak{X}$. It is often easier to define a closure operation A by specifying which classes are A-closed. Suppose $\mathscr{S}$ is a collection of classes such that $(0) \in \mathscr{S}$ and $\mathscr{S}$ is closed under arbitrary intersections. Then we can define, for each class $\mathfrak{X}$, the class

$$\mathrm{A}\mathfrak{X} = \cap \{\mathfrak{Y} \in \mathscr{S} : \mathfrak{X} \leqq \mathfrak{Y}\}$$

(where the empty intersection is the universal class $\mathfrak{D}$). It is easily seen that A is a closure operation, and that $\mathfrak{X}$ is A-closed if and only if $\mathfrak{X} \in \mathscr{S}$. Conversely if A is a closure operation the set $\mathscr{S}$ of all A-closed classes contains (0), is closed under arbitrary intersections, and determines A.

Standard examples of closure operations are $\mathrm{S}, \mathrm{I}, \mathrm{Q}, \mathrm{E}, \mathrm{N}_0, \mathrm{L}$ defined as follows: $\mathfrak{X}$ is S-closed (I-closed, Q-closed) according as every subalgebra (ideal, quotient algebra) of an $\mathfrak{X}$-algebra is always an $\mathfrak{X}$-algebra. $\mathfrak{X}$ is E-closed if every extension of an $\mathfrak{X}$-algebra by an $\mathfrak{X}$-algebra is an $\mathfrak{X}$-algebra, equivalently if $\mathfrak{X} = \mathfrak{X}^2$. $\mathfrak{X}$ is N_0-closed if $I, J \lhd L$, $I, J \in \mathfrak{X}$ implies $I + J \in \mathfrak{X}$. Finally $L \in \mathrm{L}\mathfrak{X}$ if and only if every finite subset of L is contained in an $\mathfrak{X}$-subalgebra of L. $\mathrm{L}\mathfrak{X}$ is the class of *locally $\mathfrak{X}$-algebras.*

Clearly $\mathrm{S}\mathfrak{X}$ consists of all subalgebras of $\mathfrak{X}$-algebras, $\mathrm{I}\mathfrak{X}$ consists of all *sub*ideals of $\mathfrak{X}$-algebras, and $\mathrm{Q}\mathfrak{X}$ consists of all epimorphic images of $\mathfrak{X}$-algebras; while $\mathrm{E}\mathfrak{X} = \bigcup_{n=1}^{\infty}\mathfrak{X}^n$ and consists of all Lie algebras having a finite series of subalgebras

$$0 = L_0 \leqq L_1 \leqq \cdots \leqq L_n = L$$

with $L_i \lhd L_{i+1}$ $(0 \leqq i \leqq n-1)$ and $L_{i+1}/L_i \in \mathfrak{X}$ $(0 \leqq i \leqq n-1)$.

Thus $\mathrm{E}\mathfrak{A}$ is the class of soluble Lie algebras, $\mathrm{L}\mathfrak{N}$ the class of locally nilpotent Lie algebras, and $\mathrm{L}\mathfrak{F}$ the class of locally finite (-dimensional) Lie algebras.

Suppose A and B are two closure operations. Then the product AB defined by $\mathrm{AB}\mathfrak{X} = \mathrm{A}(\mathrm{B}\mathfrak{X})$ need not be a closure operation — 03 may fail to hold. We can define $\{\mathrm{A}, \mathrm{B}\}$ to be the closure operation whose closed classes are those classes $\mathfrak{X}$ which are both A-closed and B-closed. If we partially order operations on classes by writing $\mathrm{A} \leqq \mathrm{B}$ if and only if $\mathrm{A}\mathfrak{X} \leqq \mathrm{B}\mathfrak{X}$ for any class $\mathfrak{X}$, then $\{\mathrm{A}, \mathrm{B}\}$ is the smallest closure operation greater than both A and B. It is easy to see (as in Robinson [15] p. 4) that $\mathrm{AB} = \{\mathrm{A}, \mathrm{B}\}$ (and is consequently a closure operation) if and only if $\mathrm{BA} \leqq \mathrm{AB}$. From this it is easy to deduce that ES, EI, QS, QI, LS, LI, EQ, LQ are closure operations.

2. A Correspondence between Complete Locally Nilpotent Torsion-free Groups and Locally Nilpotent Lie Algebras

In [14] A. I. Mal'cev proves the existence of a connection between locally nilpotent torsion-free groups and locally nilpotent Lie algebras over the rational field, which relates the normality structure of the group to the ideal structure of the Lie algebra. This connection is essentially the standard Lie group – Lie algebra correspondence in an infinite-dimensional situation. Mal'cev's treatment is of a topological nature, involving properties of nilmanifolds; but since the results can be stated in purely algebraic terms, it is of interest to find algebraic proofs. In [12, 13] M. Lazard outlines an algebraic treatment of Mal'cev's results, using 'typical sequences' (suites typiques) in a free group. Here we present a third approach, via matrices.

2.1 *The Campbell-Hausdorff Formula*

Let G be a finitely generated nilpotent torsion-free group. It is well-known (Hall [3] p. 56 lemma 7.5, Swan [20]) that G can be embedded in a group of (upper) unitriangular $n \times n$ matrices over the integers $\boldsymbol{Z}$ for some integer $n > 0$. This in turn embeds in the obvious manner in the group T of unitriangular $n \times n$ matrices over the rational field $\boldsymbol{Q}$. Let U

denote the set of $n \times n$ zero-triangular matrices over $\boldsymbol{Q}$. With the usual operations U forms an associative $\boldsymbol{Q}$-algebra, and this is nilpotent; indeed $U^n = 0$.

For any $t \in T$ we may use the logarithmic series to define

$$\log(t) = \log(1+(t-1)) \tag{1}$$
$$= (t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \cdots$$

for if $t \in T$ then $t-1 \in U$ so $(t-1)^n = 0$, and the series (1) has only finitely many non-zero terms. If $t \in T$ then $\log(t) \in U$.

Conversely if $u \in U$ we may use the exponential series to define

$$\exp(u) = 1 + u + \frac{u^2}{2!} + \frac{u^2}{3!} + \cdots \tag{2}$$

and exp $(u) \in T$ if $u \in U$.

Standard computations reveal that the maps log: $T \to U$ and exp : $U \to T$ are mutual inverses; in particular they are bijective.

U can be made into a Lie algebra over $\boldsymbol{Q}$ by defining a Lie product

$$[u, v] = uv - vu \qquad (u, v \in U). \tag{3}$$

As usual we define $[u_1, \cdots, u_m]$ $(u_i \in U, i = 1, \cdots, m)$ inductively to be $[[u_1, \cdots, u_{m-1}], u_m]$ $(m \geqq 2)$.

LEMMA 2.1.1 (*Campbell-Hausdorff Formula*)
If $x, y \in U$ then

$$\log(\exp(x) \cdot \exp(y)) = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}[x, y, y] + \cdots$$

where each term is a rational multiple of a Lie product $[z_1, \cdots, z_m]$ of length m such that each z_i is equal either to x or to y, and such that only finitely many products of any given length occur.

The proof is well-known, and can be found in Jacobson [6] p. 173.

COROLLARY

1) *If $a, b \in U$ and $ab = ba$ then* $\log(\exp(a)\exp(b)) = a + b$.

2) *If $t \in T$, $n \in \boldsymbol{Z}$ then* $\log(t^n) = n \cdot \log(t)$. *These may also be proved directly.*

A group H is said to be *complete* (in the sense of Kuroš [11] p. 233) of for every $n \in \boldsymbol{Z}$, $h \in H$ there exists $g \in H$ with $g^n = h$. H is an *R-group* (Kuroš [11] p. 242) if $g, h \in H$ and $n \in \boldsymbol{Z}$, together with $g^n = h^n$, imply $g = h$.

If H is a complete R-group, $h \in H$, and $q \in \boldsymbol{Q}$, then it is easy to see that we may define h^q as follows: if $q = m/n$, $m, n \in \boldsymbol{Z}$, then h^q is the unique

$g \in H$ for which $g^n = h^m$. Further, if $h \in H$, $q, r \in \boldsymbol{Q}$, we can show that $(h^q)^r = h^{qr}$, $h^{q+r} = (h^q)(h^r)$.

LEMMA 2.1.2
T is a complete R-group.

PROOF:
1) T is complete: let $t \in T$, $n \in \boldsymbol{Z}$. Define $s = \exp(1/n \log(t))$ and use corollary to lemma 2.1.1 to show that $s^n = t$.
2) T is an R-group: suppose $s, t \in T$, $n \in \boldsymbol{Z}$, and $s^n = t^n$. Then $n \cdot \log(s) = n \cdot \log(t)$ so $s = t$.
This gives us easy proofs of two known results:

PROPOSITION 2.1.3
Let H be a finitely generated nilpotent torsion-free group. Then H is an R-group, and can be embedded in a complete R-group (which may be taken to be a group of unitriangular matrices over $\boldsymbol{Q}$).

PROOF:
It suffices to note that a subgroup of an R-group is itself an R-group.

2.2 *The Matrix Version*

Suppose T is as above, and let G be a complete subgroup of T. Let U be equipped with the Lie algebra structure defined by (3). Define two maps $\flat$, $\sharp$ as follows:

$$\flat : G \to U, \; g^\flat = \log(g) \qquad (g \in G). \tag{4}$$

Let $L = G^\flat = \{g^\flat : g \in G\}$:

$$\sharp : L \to G, \; l^\sharp = \exp(l) \qquad (l \in L). \tag{5}$$

The aim of this section is to prove

THEOREM 2.2.1
With the above notation,

1) *The maps $\flat$, $\sharp$ are mutual inverses.*
2) *If H is a complete subgroup of G then $H^\flat$ is a Lie subalgebra of L. In particular L is a Lie algebra.*
3) *If M is a subalgebra of L then $M^\sharp$ is a complete subgroup of G.*
4) *If H is a complete normal subgroup of a complete subgroup K of G, then $H^\flat$ is an ideal of $K^\flat$.*
5) *If M is an ideal of a subalgebra N of L, then $M^\sharp$ is a complete normal subgroup of $N^\sharp$.*

The proof requires several remarks:

REMARK 2.2.2

L is contained in a nilpotent Lie algebra, since U is nilpotent as an associative algebra and hence as a Lie algebra.

REMARK 2.2.3

Let $g \in G$, $\lambda \in \boldsymbol{Q}$, and define g^λ as suggested immediately before lemma 2.1.2. Then $(g^\lambda)^\flat = \lambda g^\flat$. For let $\lambda = m/n$, $m, n \in \boldsymbol{Z}$. By definition $(g^\lambda)^n = g^m$. Taking logs and using part 2 of the corollary to lemma 2.1.1 we find $n \cdot \log(g^\lambda) = m \cdot \log(g)$. Thus we have $(g^\lambda)^\flat = \log(g^\lambda) = m/n \log(g) = \lambda g^\flat$.

REMARK 2.2.4

Denoting group commutators by round brackets (to avoid confusion with Lie products) thus:

$$(x, y) = x^{-1}y^{-1}xy$$

and inductively $(x_1, \cdots, x_m) = ((x_1, \cdots, x_{m-1}), x_m)$ then the Campbell-Hausdorff Formula implies that for $g_1, \cdots, g_m \in G$,

$$(g_1, \cdots, g_m)^\flat = [g_1^\flat, \cdots, g_m^\flat] + \Sigma_w P_w$$

where each P_w is a rational linear combination of products $[g_{i_1}^\flat, \cdots, g_{i_w}^\flat]$ with $w > m$ and $i_\lambda \in \{1, \cdots, m\}$ for $1 \leqq \lambda \leqq w$, such that each of $1, \cdots, m$ occurs at least once among the i_λ $(1 \leqq \lambda \leqq w)$. The exact form of the P_w is determined by the Campbell-Hausdorff Formula. The proof is by induction on m and can be found in Jennings [7] 6.1.6.

REMARK 2.2.5

We now describe a special method of manipulating expressions with terms of the form $h^\flat$, where h lies in some sub*set* H of G, which will be needed in the sequel. Suppose we have an expression

$$h^\flat + \Sigma\lambda_j C_j \qquad (\lambda_j \in \boldsymbol{Q}) \tag{6}$$

where each C_j is a Lie product of length $\geqq r$ of elements of $H^\flat$. We can write this as

$$h^\flat + \Sigma\mu_j D_j + \Sigma\nu_i E_i \qquad (\mu_j, \nu_i \in \boldsymbol{Q})$$

where the D_j are of length r, the E_i of length $\geqq r+1$. Take one of the terms D_j, say

$$D = D_1 = [h_1^\flat, \cdots, h_r^\flat].$$

By remark 2.2.4 we may replace D by the expression

$$(h_1, \cdots, h_r)^\flat + \Sigma\alpha_k F_k \qquad (\alpha_k \in \boldsymbol{Q})$$

where each F_k is a product of length $\geqq r+1$ of elements of $H^\flat$. Let

$(h_1, \cdots, h_r) = g \in G$. By the Campbell-Hausdorff Formula and remark 2.2.3

$$(hg^\lambda)^\flat = h^\flat + \lambda g^\flat + \Sigma \beta_l G_l \qquad (\lambda, \beta_l \in \boldsymbol{Q})$$

where the G_l are products of length $\geqq 2$ of elements equal either to $h^\flat$ or to $g^\flat$. But $g^\flat = D - \Sigma\, \alpha_k F_k$, each term of which is a product of $\geqq r$ elements of $H^\flat$.

Thus we may remove the terms D_j one by one to obtain a new expression for (6), of the form

$$(hg_1^{\lambda_1} \cdots g_s^{\lambda_s})^\flat + \Sigma \gamma_i H_i \qquad (\lambda_j, \gamma_i \in \boldsymbol{Q})$$

where the g_j are group commutators of length r in elements of H, and the H_i are products of length $\geqq r+1$ in elements of $H^\flat$.

We are now ready for the

Proof of theorem 2.2.1

1) Follows from the definitions of $\flat$, $\sharp$.

2) Any element of the Lie algebra generated by $H^\flat$ is of the form (6) with $r = 1$, $h = 0$. Using remark 2.2.5 over and over again, we can express this element as

$$(h')^\flat + \Sigma \delta_i J_i \qquad (\delta_i \in \boldsymbol{Q})$$

where, since H is a subgroup of G and is complete, $h' \in H$; and the J_i are products of length $> c$, the class of nilpotency of U. But then $J_i = 0$, and the element under consideration has been expressed as an element of $H^\flat$. Thus $H^\flat$ is a Lie algebra. In particular so is $L = G^\flat$.

3) Let $m, n \in M$, $\lambda \in \boldsymbol{Q}$. We must show that $(m^\sharp)^\lambda$ and $m^\sharp n^\sharp$ are elements of $M^\sharp$. Now $(m^\sharp)^\lambda = (\lambda m)^\sharp \in M^\sharp$. Further, the Campbell-Hausdorff Formula implies that $(m^\sharp n^\sharp)^\flat = m + n + \frac{1}{2}[m, n] + \cdots \in M$. By part (1) of this theorem $m^\sharp n^\sharp \in M^\sharp$.

4) Let $h \in H$, $k \in K$. We must show that $[h^\flat, k^\flat] \in H^\flat$. We prove, using descending induction on r, that any product of the form $[a_1^\flat, \cdots, a_r^\flat]$ with $a_j \in K$ for all j and at least one $a_i \in H$ is a member of $H^\flat$. This is trivially true for $r > c$, the class of nilpotency of U. The transition from $r+1$ to r follows from remark 2.2.4, noting that if a group commutator $(k_1, \cdots, k_m)$ with all $k_j \in K$ has some element $k_i \in H$, then the whole commutator lies in H (since H is a normal subgroup of K). The case $r = 2$ gives the result required.

5) Let $m \in M$, $n \in N$. Then $(m^\sharp, n^\sharp)^\flat = [m, n] +$ products of length $\geqq 3$ of elements of M and N, each term containing at least one element of M (Remark 2.2.4). Since M is an ideal of N each such term lies in M, so that $(m^\sharp, n^\sharp)^\flat \in M$. By part (1) $(m^\sharp, n^\sharp) \in M^\sharp$, whence $M^\sharp$ is normal in $N^\sharp$.

2.3 *Inversion of the Campbell-Hausdorff Formula*

A given finitely generated nilpotent torsion-free group can in general be embedded in a unitriangular matrix group in many ways. In order to extend our results to *locally* nilpotent groups and Lie algebras we need a more 'natural' correspondence. This comes from a closer examination of the matrix situation; the method used is to effect what Lazard [13] refers to as 'inversion of the Campbell-Hausdorff formula'. To express the result concisely we must briefly discuss infinite products in locally nilpotent groups. The set-up is analogous to that in Lie algebras with regard to infinite sums (such as the right-hand side of the Campbell-Hausdorff formula) which make sense provided the algebra is locally nilpotent; for then only finitely many terms of the series are non-zero.

Suppose we have a *finite* set of variables $\{x_1, \cdots, x_f\}$. A formal infinite product

$$\omega(x_1, \cdots, x_f) = \prod_{i=0}^{\infty} K_i^{\lambda_i}$$

is said to be an *extended word* in these variables if

E1) $\lambda_i \in \boldsymbol{Q}$ for all i,

E2) Each K_i is a commutator word $K_i(x_1, \cdots, x_f) = (x_{j_1}, \cdots, x_{j_r})$ (r depending on i) in the variables $x_1, \cdots, x_f$,

E3) Only finitely many terms K_i have any given length r.

Suppose G is a complete locally nilpotent torsion-free group, and $g_1, \cdots, g_f \in G$. G is a complete R-group (Proposition 2.1.3) so that

$$(K_i(g_1, \cdots, g_f))^{\lambda_i} = (g_{j_1}, \cdots, g_{j_r})^{\lambda_i}$$

is defined in G. The group H generated by $g_1, \cdots, g_f$ is nilpotent of class c (say) so if K_i has length $> c$ $K_i(g_1, \cdots, g_f) = 1$. Thus only finitely many values of $(K_i(g_1, \cdots, g_f))^{\lambda_i} \neq 1$ and we may define $\omega(g_1, \cdots, g_f)$ to be the product (in order) of the non-1 terms. Thus if $\omega(x_1, \cdots, x_f)$ is an extended word, and G is any complete locally nilpotent torsion-free group, then we may consider ω to be a function $\omega : G^f \to G$.

Similarly we may define an *extended Lie word* to be a formal sum

$$\zeta(w_1, \cdots, w_e) = \sum_{i=0}^{\infty} \mu_j J_j$$

where

D1) $\mu_j \in \boldsymbol{Q}$ for all j,

D2) Each J_j is a Lie product $J_j(w_1, \cdots, w_e) = [w_{i_1}, \cdots, w_{i_s}]$ (s depending on j) in the variables $w_1, \cdots, w_e$,

D3) Only finitely many terms J_j have any given length s.

Then if L is any locally nilpotent Lie algebra over $\boldsymbol{Q}$, we may consider ζ to be a function $\zeta : L^e \to L$.

Let us now return to the matrix group/matrix algebra correspondence of section 2.2. Suppose we 'lift' the Lie operations from L to G by defining

$$\begin{aligned} \lambda g &= (\lambda g^\flat)^\sharp \\ g+h &= (g^\flat + h^\flat)^\sharp \\ [g, h] &= [g^\flat, h^\flat]^\sharp \end{aligned}$$

$(g, h \in G, \lambda \in \boldsymbol{Q})$. Then G with these operations forms a Lie algebra which we shall denote by $\mathscr{L}(G)$. Similarly we may 'drop' the group operations from G to L by defining

$$\begin{aligned} lm &= (l^\sharp m^\sharp)^\flat \\ l^\lambda &= (l^{\sharp\lambda})^\flat \end{aligned}$$

$(l, m \in L, \lambda \in \boldsymbol{Q})$. L with these operations forms a complete group $\mathscr{G}(L)$. $\mathscr{L}(G)$ is isomorphic to L and $\mathscr{G}(L)$ is isomorphic to G.

The crucial observation we require is that these operations can be expressed as extended words (resp. extended Lie words). This is Lazard's 'inversion'.

LEMMA 2.3.1

Let G be a complete subgroup of T, and let $L = G^\flat$ as described in section 2.2. Then there exist extended words $\varepsilon_\lambda(x)$ $(\lambda \in \boldsymbol{Q})$, $\sigma(x, y)$, $\pi(x, y)$ such that for $g, h \in G$, $\lambda \in \boldsymbol{Q}$,

$$\begin{aligned} \lambda g &= \varepsilon_\lambda(g) \\ g+h &= \sigma(g, h) \\ [g, h] &= \pi(g, h) \end{aligned}$$

(where the operations on the left are those defined above).

Further there exist extended Lie words $\delta_\lambda(x)$ $(\lambda \in \boldsymbol{Q})$, $\mu(x, y)$, $\gamma(x, y)$ such that

$$\begin{aligned} l^\lambda &= \delta_\lambda(l) \\ lm &= \mu(l, m) \\ (l, m) &= \gamma(l, m) \end{aligned}$$

$(l, m \in L, \lambda \in \boldsymbol{Q})$ (operations on left as above).

These words can be taken to be independent of the particular G, L chosen.

PROOF:

1) ε_λ:

$(\lambda g^\flat)^\sharp = \exp(\lambda \cdot \log(g)) = g^\lambda$, so $\varepsilon_\lambda(x) = x^\lambda$ has the required properties.

2) σ:

Here we must do more work. We show that there exist words $\sigma_i(x, y)$ satisfying

$$\sigma_{i+1}(x, y) = \sigma_i(x, y)\gamma_{i+1}(x, y)$$
$$\sigma_0(x, y) = 1$$

where γ_{i+1} is a word of the form

$$K_1^{\lambda_1} \cdots K_u^{\lambda_u} \qquad (\lambda_j \in \boldsymbol{Q}\ 1 \leqq j \leqq u)$$

with each K_j a commutator word $(z_{j_1}, \cdots, z_{j_{i+1}})$ of length $i+1$ with $z_{j_k} = x$ or y $(1 \leqq k \leqq i+1)$; such that if G is a complete subgroup of the group of $c \times c$ unitriangular matrices over $\boldsymbol{Q}$ $(c \geqq 1)$ then

$$g+h = \sigma_c(g, h) \qquad (g, h \in G).$$

The existence of these words is a consequence of the manipulation process described in remark 2.2.5. This enables us to take an expression of the form

$$h^\flat + \Sigma\lambda_j C_j \qquad (\lambda_j \in \boldsymbol{Q}) \tag{7}$$

where h lies in some subset H of G, and the C_j are Lie products of length $\geqq r$ in elements of $H^\flat$, and replace it by an expression

$$(hg_1^{\mu_1} \cdots g_m^{\mu_m}) + \Sigma\gamma_i H_i \qquad (\mu_j, \gamma_i \in \boldsymbol{Q})$$

where the g_j are commutator words in elements of H of length r, and the H_i are Lie products of elements of $H^\flat$ of length $\geqq r+1$.

We obtain the σ_i by systematically applying this procedure to the expression $g^\flat + h^\flat$. We choose a total ordering $\ll$ of the left-normed Lie products in x, y in such a way that the length is compatible with the ordering. Next we apply the process of section 2.2.5 to the expression $g^\flat + h^\flat$ (with g playing the role of h in (7), $\lambda_1 = 1$, $C_1 = h^\flat$) and at each stage in the process

1) Express all Lie products in $g^\flat$, $h^\flat$ as sums of left-normed commutators (using anticommutativity and the Jacobi identity),

2) Collect together all multiples of the same left-normed product,

3) Operate on the term D (in the notation of Remark 2.2.5) which is smallest in the ordering $\ll$.

At the i-th stage we will have expressed $g+h$ in the form

$$(\sigma_i(g, h))^\flat + \Sigma\theta_k I_k \qquad (\theta_k \in \boldsymbol{Q})$$

where σ_i is a word in g, h and the terms I_k are Lie products in $g^\flat$, $h^\flat$ of length $> i$. At the $(i+1)$-th stage this will have been modified to

$$(\sigma_i(g, h) \cdot g_1^{\lambda_1} \cdots g_m^{\lambda_m})^\flat + \Sigma\phi_l J_l \qquad (\phi_l \in \boldsymbol{Q})$$

where the g_i are group commutators in g, h of length $i+1$, the $\lambda_i \in \boldsymbol{Q}$, and the J_l are Lie products in $g^\flat$, $h^\flat$ of length $> i+1$.

We put

$$\begin{aligned}\gamma_{i+1}(g, h) &= g_1^{\lambda_1} \cdots g_m^{\lambda_m},\\ \sigma_{i+1}(g, h) &= \sigma_i(g, h)\gamma_{i+1}(g, h)\\ \sigma_0(g, h) &= 1.\end{aligned}$$

It is clear from the way that the process 2.2.4 operates that the form of the words σ_i, γ_i depends only on the ordering $\ll$ (and the Campbell-Hausdorff formula) so that we can define the required words $\sigma_i(x, y)$ and $\gamma_i(x, y)$ independently of G.

Now if G consists of $c \times c$ matrices, then at the c-th stage we have

$$g^\flat + h^\flat = (\sigma_c(g, h))^\flat + \Sigma \psi_p K_p \qquad (\psi_p \in \boldsymbol{Q})$$

where the terms K_p are of length $> c$ so are 0. Thus

$$g+h = (g^\flat + h^\flat)^\# = \sigma_c(g, h)$$

as claimed.

We now define

$$\sigma(x, y) = \prod_{i=0}^{\infty} \rho_i(x, y).$$

If G is a complete group of unitriangular $c \times c$ matrices over $\boldsymbol{Q}$, then G is nilpotent of class $\leqq c$, so for all $j > 0$ $\sigma_{c+j}(g, h) = 1$, so $\sigma(g, h) = \sigma_c(g, h)$. Hence for any such G we have $g+h = \sigma(g, h)$ as required.

3) π:

Similar proof. Work on the expression

$$1^\flat + [g^\flat, h^\flat]$$

(which equals $[g^\flat, h^\flat]$) with 1 playing the role of h in (7), $\lambda_1 = 1$, $C_1 = [g^\flat, h^\flat]$.

4) δ_λ:

$l^\lambda = (l^{\#\lambda})^\flat = \log(\exp(l)^\lambda) = \lambda l$ $(l \in L)$ so $\delta_\lambda(x) = \lambda x$ will do.

5) μ:

Put $\mu(x, y) = x+y+\frac{1}{2}[x, y]+ \cdots$ as in the Campbell-Hausdorff formula.

6) γ:

Follows at once from the existence of δ_λ and μ. The lemma is proved.

To illustrate the method, we calculate the function σ up to terms of length 3. To this length the Campbell-Hausdorff formula becomes

$$(gh)^\flat = g^\flat + h^\flat + \tfrac{1}{2}[g^\flat, h^\flat] + \tfrac{1}{12}([g^\flat, h^\flat, h^\flat] + [h^\flat, g^\flat, g^\flat])$$

and thus

$$(x, y)^\flat = [x^\flat, y^\flat] + \tfrac{1}{2}([x^\flat, y^\flat, x^\flat] + [x^\flat, y^\flat, y^\flat]).$$

We choose left-normed commutators as follows:

$$a^\flat \ll b^\flat \ll [a^\flat, b^\flat] \ll [a^\flat, b^\flat, a^\flat] \ll [a^\flat, b^\flat, b^\flat].$$

Now $(a+b)^\flat = a^\flat + b^\flat$ by definition

$$\begin{aligned}
&= (ab)^\flat - \tfrac{1}{2}[a^\flat, b^\flat] - \tfrac{1}{12}([a^\flat, b^\flat, b^\flat] - [a^\flat, b^\flat, a^\flat]) \\
&= (ab)^\flat - \tfrac{1}{2}\{(a, b)^\flat - \tfrac{1}{2}([a^\flat, b^\flat, a^\flat] + [a^\flat, b^\flat\ b^\flat])\} \\
&\quad + \tfrac{1}{12}([a^\flat, b^\flat, a^\flat] - [a^\flat, b^\flat, b^\flat]) \\
&= (ab(a, b)^{-\frac{1}{2}})^\flat - \tfrac{1}{2}([(ab)^\flat, (a, b)^{-\frac{1}{2}\flat}]) \\
&\quad + \tfrac{1}{4}([a^\flat, b^\flat, a^\flat] + [a^\flat, b^\flat, b^\flat]) \\
&\quad + \tfrac{1}{12}([a^\flat, b^\flat, a^\flat] - [a^\flat, b^\flat, b^\flat]) \\
&= (ab(a, b)^{-\frac{1}{2}})^\flat - \tfrac{1}{2}([a^\flat + b^\flat, -\tfrac{1}{2}[a^\flat, b^\flat]]) \\
&\quad + \tfrac{1}{4}([a^\flat, b^\flat, a^\flat] + [a^\flat, b^\flat, b^\flat]) \\
&\quad + \tfrac{1}{12}([a^\flat, b^\flat, a^\flat] - [a^\flat, b^\flat, b^\flat]) \\
&= (ab(a, b)^{-\frac{1}{2}})^\flat + \tfrac{1}{12}[a^\flat, b^\flat, a^\flat] + \tfrac{1}{12}[a^\flat, b^\flat, b^\flat] \\
&= (ab(a, b)^{-\frac{1}{2}}(a, b, a)^{1/12}(a, b, b)^{-1/12})^\flat.
\end{aligned}$$

Thus up to terms of length 3

$$\sigma(a, b) = ab(a, b)^{-\frac{1}{2}}(a, b, a)^{1/12}(a, b, b)^{-1/12}.$$

Similarly we find

$$\pi(a, b) = (a, b)(a, b, a)^{-\frac{1}{2}}(a, b, b)^{-\frac{1}{2}}.$$

2.4 *The General Version*

As remarked in section 2.3, if $\omega\ (x_1, \cdots, x_f)$ is an extended word and G any complete locally nilpotent torsion-free group, then ω can be considered as a function $G^f \to G$. Similarly for extended Lie words and locally nilpotent Lie algebras over $\boldsymbol{Q}$. On this basis we can establish a general version of Mal'cev 's correspondence as follows:

THEOREM 2.4.1

Let G be a complete locally nilpotent torsion-free group. Define operations on G as follows:

If $\lambda \in \boldsymbol{Q}$, $g, h \in G$ set

$$\lambda g = \varepsilon_\lambda(g)$$
$$g+h = \sigma(g, h)$$
$$[g, h] = \pi(g, h).$$

With these operations G becomes a Lie algebra over $\boldsymbol{Q}$, which we denote by $\mathscr{L}(G)$. $\mathscr{L}(G)$ is a locally nilpotent Lie algebra.

Conversely, let L be a locally nilpotent Lie algebra over $\boldsymbol{Q}$. Define, for $\lambda \in \boldsymbol{Q}$, $l, m \in L$, operations:

$$l^\lambda = \delta_\lambda(l)$$
$$lm = \mu(l, m).$$

With these operations L becomes a complete locally nilpotent torsion-free group, which we denote by $\mathscr{G}(L)$.

PROOF:

The axioms for a Lie algebra can be expressed as certain relations between the functions ε_λ, σ, π involving at most 3 variables. Thus if these relations can be shown to hold in any 3-generator subgroup of G, they hold throughout G. But, as remarked earlier, any finitely generated nilpotent torsion-free group can be embedded in a group of unitriangular $c \times c$ matrices over $\boldsymbol{Q}$ for some integer $c > 0$ (Hall [3], Swan [20]). But the required relations certainly hold in this situation, since by the construction of ε_λ, σ, π they express the fact that the logarithms of these matrices form a Lie algebra under the usual operations – a fact which is manifest.

Any finitely generated subalgebra of $\mathscr{L}(G)$ is the image under $\mathscr{L}$ of the completion $\bar{H}$ of some finitely generated subgroup H of G. H is nilpotent, so by Kuroš [11] p. 258. $\bar{H}$ is also nilpotent. The form of the words $\varepsilon_\lambda, \sigma, \pi$ now ensures that the original finitely generated subalgebra of $\mathscr{L}(G)$ is nilpotent. Hence $\mathscr{L}(G)$ is locally nilpotent.

In a similar way the axioms for a complete group hold in L if they hold in any finitely generated subalgebra. Now a finitely generated nilpotent Lie algebra is finite-dimensional (Hartley [5] p. 261) and any finite-dimensional nilpotent Lie algebra over $\boldsymbol{Q}$ can be embedded in a Lie algebra of zero-triangular matrices over $\boldsymbol{Q}$ (Birkhoff [1]). We may therefore proceed analogously to complete the proof.

We next consider the relation between the structure of G and that of $\mathscr{L}(G)$; also L and $\mathscr{G}(L)$.

THEOREM 2.4.2

Let G, H be complete locally nilpotent torsion-free groups; *let L be a locally nilpotent Lie algebra over $\boldsymbol{Q}$. Then*

1) $\mathscr{G}(\mathscr{L}(G)) = G, \quad \mathscr{L}(\mathscr{G}(L)) = L.$

2) *H is a subgroup of G if and only if*

$$\mathscr{L}(H) \leqq \mathscr{L}(G).$$

3) *H is a normal subgroup of G if and only if*

$$\mathscr{L}(H) \lhd \mathscr{L}(G).$$

4) $\phi : G \to H$ *is a group homomorphism if and only if* $\phi : \mathscr{L}(G) \to \mathscr{L}(H)$ *is a Lie homomorphism. The kernel of* ϕ *is the same in both cases.*

5) *If H is a normal subgroup of G, then*

$$\mathscr{L}(G/H) = \mathscr{L}(G)/\mathscr{L}(H).$$

(*Note*: *using part* (1) *we can easily recast parts* (2), (3), (4), (5) *in a* '$\mathscr{G}$' *form instead of an* '$\mathscr{L}$' *form.*)

PROOF:

1) Let $g, h \in G$. We must show that for $\lambda \in \boldsymbol{Q}$

$$g^{\lambda} = \delta_{\lambda}(g)$$
$$gh = \mu(g, h)$$

where δ_{λ}, μ are defined in terms of the Lie operations of $\mathscr{L}(G)$. Now $\delta_{\lambda}(g) = \lambda g = \varepsilon_{\lambda}(g) = g^{\lambda}$. To show that $gh = \mu(g, h)$ we may confine our attention to the completion of the group generated by g and h. Thus without loss of generality G is a group of unitriangular matrices over $\boldsymbol{Q}$.

Now by definition

$$\mu(g, h) = g+h+\tfrac{1}{2}[g, h]+ \cdots$$

and $+$, $[\ ,\ ]$ are defined in $\mathscr{L}(G)$ by

$$g+h = (g^{\flat}+h^{\flat})^{\sharp}$$
$$[g, h] = [g^{\flat}, h^{\flat}]^{\sharp}$$

so

$$\begin{aligned}\mu(g, h)^{\flat} &= g^{\flat}+h^{\flat}+\tfrac{1}{2}[g^{\flat}, h^{\flat}]+ \cdots \\ &= (gh)^{\flat} \text{ by Campbell-Hausdorff}\end{aligned}$$

so $\mu(g, h) = gh$ as required.

The converse is similar and will be omitted.

2) and 3) are clear from the form of the functions ε_{λ}, π, σ, δ_{λ}, μ, γ.

4) Follows from the observation that group homomorphisms (resp. Lie homomorphisms) preserve extended words (resp. extended Lie words). The kernels are the same since the identity element of G is the zero element of $\mathscr{L}(G)$.

5) We first show that H-cosets in G are the same as $\mathscr{L}(H)$-cosets in $\mathscr{L}(G)$.

Let $x \in G$, $z \in Hx$. Then $z = hx$ for some $h \in H$, and $hx = h+x+\frac{1}{2}[h, x]+ \cdots \in \mathscr{L}(H)+x$ since $h \in \mathscr{L}(H)$ which is an ideal of $\mathscr{L}(G)$. Thus $Hx \subseteq \mathscr{L}(H)+x$.

Now let $y \in \mathscr{L}(H)+x$. Then $y = h+x$ for some $h \in H$, and $h+x = h \cdot x \cdot (h, x)^{-\frac{1}{2}} \cdots \in Hx$ since H is a normal subgroup of G. Therefore $\mathscr{L}(H)+x \subseteq Hx$.

Hence $Hx = \mathscr{L}(H)+x$. The operations on the cosets are defined by the same extended words, and the result follows.

REMARK

In categorical guise, let $\mathscr{C}_{\mathscr{G}}$ denote the category of complete locally nilpotent torsion-free groups and group homomorphisms, $\mathscr{C}_{\mathscr{L}}$ the category of locally nilpotent Lie algebras over $\boldsymbol{Q}$ and Lie homomorphisms. Then

$$\mathscr{L} : \mathscr{C}_{\mathscr{G}} \to \mathscr{C}_{\mathscr{L}}$$
$$\mathscr{G} : \mathscr{C}_{\mathscr{L}} \to \mathscr{C}_{\mathscr{G}}$$

are covariant functors, defining an isomorphism between the two categories.

Observe, however, that our definition of $\mathscr{L}$ and $\mathscr{G}$ is stronger than a purely category-theoretic one – as far as the underlying *sets* are concerned they are both identity maps.

We shall now develop a few more properties of the correspondence, which we need later. But first let us recall the definition of a *centraliser* in a Lie algebra: suppose $H \subseteq X \subseteq L$, $H \leqq L$, and $H \lhd X$. Then

$$C_L(X/H) = \{c \in L : [c, X] \leqq H\}.$$

There is a similar definition for groups.

LEMMA 2.4.3

Let G, H be complete locally nilpotent torsion-free groups, with $H \leqq G$, $H \lhd X \subseteq G$. Then

$$\mathscr{L}(C_G(X/H)) = C_{\mathscr{L}(G)}(\mathscr{L}(X)/\mathscr{L}(H))$$

(*where the notation $\mathscr{L}(X)$ indicates the set X considered as a subset of $\mathscr{L}(H)$*).

PROOF:

Let $c \in C = C_G(X/H)$. Then for any $x \in X$, $[c, x] = (c, x)(c, x, c)^{-\frac{1}{2}} \cdots \in H$ (from the definition of C and since $H \lhd X$). Consequently $c \in C_{\mathscr{L}(\mathscr{G})}(\mathscr{L}(X)/\mathscr{L}(H))$. The converse inclusion is similar.

COROLLARY 1

1) $\mathscr{L}(C_G(X)) = C_{\mathscr{L}(G)}(\mathscr{L}(X))$ (put $H = 0$)

2) $\mathscr{L}(N_G(H)) = I_{\mathscr{L}(G)}(\mathscr{L}(H))$ (put $X = H$).

(Here N_G denotes the *normaliser* in G, and $I_{\mathscr{L}(G)}$ the *idealiser* in $\mathscr{L}(G)$ (also called the normaliser in Jacobson [6] p. 57, but we prefer the alternative terminology)).

COROLLARY 2

Letting $\zeta_\alpha(G)$ denote the α-th term of the upper central series of G, then

$$\mathscr{L}(\zeta_\alpha(G)) = \zeta_\alpha(\mathscr{L}(G)).$$

PROOF:

Use transfinite induction on α and lemma 2.4.3.

COROLLARY 3

The upper central series of G and $\mathscr{L}(G)$ become stationary at the same ordinal α. In particular if either G or $\mathscr{L}(G)$ is nilpotent then so is the other and their classes of nilpotency are equal.

PROOF:

Immediate from Corollary 2.

Suppose G is a complete locally nilpotent torsion-free group, and H is any subgroup. Then the *completion* $\bar{H}$ *of* H *in* G is the smallest complete subgroup of G which contains H. The next lemma collects some known facts about completions.

LEMMA 2.4.4

Suppose G is a complete locally nilpotent torsion-free group, and $H \leqq K \leqq G$.

1) *If $H \lhd K$ then $\bar{H} \lhd \bar{K}$.*

2) *$\bar{K}$ is equal to the isolator of K in G, which is the set of all $g \in G$ such that $g^n \in K$ for some $n \in \mathbf{Z}$.*

PROOF:

1) see Kuroš [11] p. 254.

2) see Kuroš [11] pp. 249, 255.

LEMMA 2.4.5

Let G be a complete locally nilpotent torsion-free group, H a complete subgroup of G. Then $H \lhd^\alpha G$ if and only if $\mathscr{L}(H) \lhd^\alpha \mathscr{L}(G)$.

PROOF:

There is a normal series

$$H = H_0 \lhd H_1 \lhd \cdots H_\beta \lhd H_{\beta+1} \lhd \cdots H_\alpha = G$$

from H to G, such that $H_\lambda = \bigcup_{\mu<\lambda} H_\beta$ at limit ordinals λ. Let $L_\beta = \mathscr{L}(\overline{H}_\beta)$ (bars denoting completions in G). Then $\mathscr{L}(H) = L_0$, $\mathscr{L}(G) = L_\alpha$. By lemma 2.4.4.1 and theorem 2.4.2.3 we have $L_\beta \lhd L_{\beta+1}$ for all $\beta < \alpha$. Lemma 2.4.4.2 easily shows that at limit ordinals λ $L_\lambda = \bigcup_{\beta<\lambda} L_\beta$. The result follows.

In particular H is subnormal in G if and only if $\mathscr{L}(H)$ is a subideal of $\mathscr{L}(G)$; and H is ascendant in G if and only if $\mathscr{L}(H)$ is an ascendant subalgebra of $\mathscr{L}(G)$.

As an application of these results we will give a generalisation of a result of Yu. G. Fedorov (see Kuroš [11] p. 257) which states that a nilpotent torsion-free group and its completion have the same class of nilpotency. Our generalisation (proved in the next section) does not seem to have appeared in the literature.

2.5 *Bracket Varieties*

Let $\phi = \phi(x_1, \cdots, x_n)$ and $\psi = \psi(y_1, \cdots, y_m)$ be two group words. Following P. Hall we define the *outer commutator* word $(\phi, \psi)_0$ to be the word

$$\begin{aligned}&(\phi, \psi)_0(x_1, \cdots, x_{n+m}) = \\ &(\phi(x_1, \cdots, x_n))^{-1}(\psi(x_{n+1}, \cdots, x_{n+m}))^{-1} \\ &(\phi(x_1, \cdots, x_n))(\psi(x_{n+1}, \cdots, x_{n+m})).\end{aligned}$$

We define *bracket words* inductively: the identity word $\iota(x_1) = x_1$ is a bracket word of *height* $h(\iota) = 1$. If ϕ, ψ are bracket words then $(\phi, \psi)_0$ is a bracket word of height $h(\phi)+h(\psi)$.

Thus for example (x, y), $((x, y), z)$ and $((x, y), (z, t))$ are bracket words.

Analogous definitions can be made for Lie algebras. In this case we denote the outer commutator by $[\phi, \psi]_0$, and the height again by h. To each group bracket word ϕ there corresponds in a natural way a Lie bracket word ϕ^* defined inductively by

$$\iota^* = \iota$$
$$(\phi, \psi)_0^* = [\phi^*, \psi^*]_0.$$

Clearly $h(\phi) = h(\phi^*)$, and ϕ^* is obtained from ϕ by changing all round brackets to square ones.

If G is a group and ϕ a group bracket word, the *verbal subgroup* corresponding to ϕ is

$$\phi(G) = \langle \phi(g_1, \cdots, g_n) : g_i \in G \quad 1 \leqq i \leqq n \rangle$$

and the *variety* $\mathfrak{V}_\phi$ determined by ϕ is the class of all groups G for which $\phi(G) = 1$; equivalently those G for which the relation $\phi(g_1, \cdots, g_n) = 1$ holds identically in G.

Similarly we define the *verbal subalgebra* $\phi^*(L)$ of a Lie algebra L determined by a Lie bracket word ϕ^*, and the *variety* $\mathfrak{V}_{\phi^*}$.

If G is a group and ϕ a group bracket word, then a *ϕ-value* in G is an element expressible as $\phi(g_1, \cdots, g_n)$ $(g_i \in G\ 1 \leqq i \leqq n)$. Similarly for Lie algebras.

LEMMA 2.5.1

Let ϕ, ψ be Lie bracket words, L any Lie algebra (over an arbitrary field). Then

1) *$\phi(L)$ is the vector subspace of L spanned by the ϕ-values in L.*
2) *$\phi(L) \lhd L$.*
3) *$[\phi, \psi]_0(L) = [\phi(L), \psi(L)]$.*

PROOF:

We prove (1) and (2) simultaneously by induction on the height of ϕ.

If $h(\phi) = 1$ then $\phi = \iota$ and (1) and (2) are trivial. If $h(\phi) > 1$ then there are bracket words ψ, χ such that $\phi = [\psi, \chi]_0$ and $h(\psi)$, $h(\chi) < h(\phi)$. Inductively we may suppose that (1) and (2) hold for ψ and χ. Let x be a ϕ-value in L. Then there exist $\underline{y} = (y_1, \cdots, y_n)$ and $\underline{z} = (z_1, \cdots, z_m)$ $(y_1, \cdots, y_n, z_1, \cdots, z_m \in L)$ such that $x = \phi(\underline{y}, \underline{z}) = [\psi(\underline{y}), \chi(\underline{z})]$. If $t \in L$ then $[x, t] = [[\psi(\underline{y}), \chi(\underline{z})], t] = [[\psi(\underline{y}), t], \chi(\underline{z})] + [\psi(\underline{y}), [\chi(\underline{z}), t]]$ by Jacobi. By part (2) inductively $[\psi(\underline{y}), t)]$ lies in $\psi(L)$; by part (1) it is a linear combination of ψ-values. Similarly for $[\chi(\underline{z}), t]$. Thus $[x, t]$ is a linear combination of $[\psi, \chi]_0$-values. Hence the subspace spanned by the ϕ-values is an ideal of L, and so is equal to $\phi(L)$. This proves parts (1) and (2).

Part (3) now follows at once from part (1).

Results analogous to parts (2) and (3) are well known for groups.

Let G be a locally nilpotent torsion-free group. Then it is known that G has a unique *completion* $\bar{G}$, that is a complete locally nilpotent torsion-free group containing G and such that the completion of G in $\bar{G}$ is the whole of $\bar{G}$. Note that we cannot use Mal'cev's work on completions to establish the existence of $\bar{G}$ since we are trying to produce algebraic proofs of our theorems. The whole of Mal'cev's theory of completions has been developed in a purely algebraic setting by Kargapolov [8, 9]; and a method is outlined in Hall [3] p. 46.

Under the Mal'cev correspondence $\bar{G}$ can also be considered to be a Lie algebra over $\boldsymbol{Q}$. Denote completions (in $\bar{G}$) of subgroups of $\bar{G}$ by overbars. Temporarily denote by $i\langle X\rangle$ the ideal of $\bar{G}$ generated by

X (considering $\bar{G}$ as a Lie algebra) and let $n\langle X\rangle$ denote the normal subgroup of $\bar{G}$ generated by X, for any subset X of G.

LEMMA 2.5.2

Let G be a locally nilpotent torsion-free group, $A, B \lhd G$. Then

$$\overline{(A, B)} = \overline{(\bar{A}, \bar{B})} = [\bar{A}, \bar{B}]$$

(*where in the third expression $\bar{A}$ and $\bar{B}$ are considered as subalgebras of* G).

PROOF:

Throughout let a run through A, b through B, and α, β through $\boldsymbol{Q}$. Then

$$\begin{aligned}\overline{(A, B)} &= \overline{n\langle(a, b)\rangle}\\ &= i\langle[a, b]\rangle\end{aligned}$$

since from the form of the words π, γ of lemma 2.3.1 it is clear that $(a, b) \in i\langle[a, b]\rangle$ and $[a, b] \in \overline{n\langle a, b\rangle}$

$$\begin{aligned}&= i\langle[\alpha a, \beta b]\rangle\\ &= i\langle[a^\alpha, b^\beta]\rangle \quad (*)\\ &= [\bar{A}, \bar{B}] \qquad \text{using lemma 2.4.4.2.}\end{aligned}$$

But also

$$\begin{aligned}(*) &= \overline{n\langle(a^\alpha, b^\beta)\rangle} \quad \text{(as above)}\\ &= \overline{(\bar{A}, \bar{B})} \qquad \text{using lemma 2.4.4.2.}\end{aligned}$$

The promised generalisation of Fedorov's result:

THEOREM 2.5.3

Let G be any locally nilpotent torsion-free group, $\bar{G}$ its completion (viewed also as a Lie algebra over $\boldsymbol{Q}$). Let ϕ be any group bracket word. Then

1) $\overline{\phi(G)} = \overline{\phi(\bar{G})} = \phi^*(\bar{G})$

2) $G \in \mathfrak{V}_\phi \Leftrightarrow \bar{G} \in \mathfrak{V}_\phi \Leftrightarrow \bar{G} \in \mathfrak{V}_{\phi^*}$.

PROOF:

1) Use induction on $h(\phi) = h(\phi^*)$. If $h(\phi) = 1$ the result is clear. If not, then $\phi = (\psi, \chi)_0$ and so $\phi^* = [\psi^*, \chi^*]_0$ where all of $h(\psi)$, $h(\chi)$, $h(\psi^*)$, $h(\chi^*)$ are less than $h(\phi)$. Thus

$$
\begin{aligned}
\overline{\phi(G)} &= \overline{(\psi, \chi)_0(G)} \\
&= \overline{(\psi(G), \chi(G))} \text{ (lemma 2.5.1.3 for groups)} \\
&= \overline{(\overline{\psi(G)}, \overline{\chi(G)})} \text{ (lemma 2.5.2)} \\
&= \overline{(\overline{\psi(\bar{G})}, \overline{\chi(\bar{G})})} \text{ (induction hypothesis) (*)} \\
&= \overline{(\psi(\bar{G}), \chi(\bar{G}))} \text{ (lemma 2.5.2)} \\
&= \overline{(\psi, \chi)_0(\bar{G})} \\
&= \overline{\phi(\bar{G})}.
\end{aligned}
$$

Also

$$
\begin{aligned}
(*) &= [\overline{\psi(\bar{G})}, \overline{\chi(\bar{G})}] \quad \text{(lemma 2.5.2)} \\
&= [\psi^*(\bar{G}), \chi^*(\bar{G})] \text{ (induction hypothesis)} \\
&= [\psi^*, \chi^*]_0(\bar{G}) \quad \text{(lemma 2.5.1.3)} \\
&= \phi^*(G)
\end{aligned}
$$

which proves part (1).

2)
$$
\begin{aligned}
G \in \mathfrak{B}_\phi &\Leftrightarrow \phi(G) = 1 \\
&\Leftrightarrow \overline{\phi(G)} = 1 \\
&\Leftrightarrow \overline{\phi(\bar{G})} = 1 \quad (**) \\
&\Leftrightarrow \phi(\bar{G}) = 1 \\
&\Leftrightarrow \bar{G} \in \mathfrak{B}_\phi.
\end{aligned}
$$

Also

$$
\begin{aligned}
(**) &\Leftrightarrow \phi^*(\bar{G}) = 0 \\
&\Leftrightarrow \bar{G} \in \mathfrak{B}_{\phi^*}.
\end{aligned}
$$

Corollary

Let $\mathfrak{X}$ be a union of bracket varieties of groups, $\mathfrak{X}^*$ the union of the corresponding Lie bracket varieties. Then

$$G \in \mathfrak{X} \Leftrightarrow \bar{G} \in \mathfrak{X} \Leftrightarrow \bar{G} \in \mathfrak{X}^*.$$

In particular we may take for $\mathfrak{X}$ the classes (using P. Hall's notation [2]):

$$\mathfrak{N}_c,\ \mathfrak{N},\ \mathfrak{A}^d,\ \mathrm{E}\mathfrak{A},\ \mathfrak{A}\mathfrak{N}.$$

(The case $\mathfrak{X} = \mathfrak{N}_c$ is Fedorov's theorem.)

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