COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 22, nº 2 (1970), p. 227-239

<http://www.numdam.org/item?id=CM_1970__22_2_227_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 2, 1970, pag. 227-239
Wolters-Noordhoff Publishing
Printed in the Netherlands

HARMONIC ANALYSIS WITH RESPECT TO ALMOST INVARIANT MEASURES

by

R. Larsen

1. Introduction

One of the central concerns of abstract harmonic analysis is the study of the commutative Banach algebra $L_1(G)$ consisting of all the equivalence classes of complex valued functions on the locally compact abelian (LCA) group G which are integrable with respect to Haar measure m on G and with the usual convolution product

$$f * g(s) = \int_G f(st^{-1})g(t)dm(t), \qquad f, g \in L_1(G).$$

Since m is invariant under translation by elements of G it is trivially apparent that $\{T_s m | s \in G\}$ spans a one dimensional space of measures, where $T_s m(E) = m(Es)$. This observation suggests the development of a theory involving algebras which arise in a manner analogous to the group algebra $L_1(G)$ but where Haar measure is replaced by an *almost invariant measure*, that is, by a regular complex valued Borel measure μ on G such that $\{T_s \mu | s \in G\}$ spans a finite dimensional space of measures. In this paper we shall construct such a class of algebras and examine various aspects of their structure.

Before proceeding to the construction of these algebras we wish to recall the characterization theorem for almost invariant measures, namely, μ is an almost invariant measure on G if and only if there exists a unique continuous almost invariant function h on G such that $d\mu = hdm$. A function h is almost invariant provided that $\{T_s h | s \in G\}$ spans a finite dimensional space of functions and where $T_s h(t) = h(ts)$. This and other results related to almost invariant measures are available in (4, 5). To insure that the algebras under consideration are nontrivial we shall always assume that the supremum norm $||h||_\infty$ of h is finite. In this case h will be a trigonometric polynomial and, in particular, an almost periodic function. The characterization theorem will also permit us to phrase our definitions and results in terms of the function h rather than the almost invariant measure $d\mu = hdm$, thus allowing a certain simplification of terminology.

2. The algebra $\mathscr{L}_1(h)$

Given a trigonometric polynomial h on an LCA group G, $h \not\equiv 0$, we shall define a new multiplication in the Banach space underlying the group algebra $L_1(G)$, that is, the Banach space of equivalence classes of functions f absolutely integrable with respect to Haar measure under the norm $||f|| = \int_G |f(t)| dm(t)$, in such a way that the new product involves h in an essential fashion and reduces to the convolution product when $h \equiv 1$. This is accomplished by defining $f \circ g = fh * gh$, $f, g \in L_1(G)$. The algebra so obtained will be denoted by $\mathscr{L}_1(h)$. The next theorem is easily established.

THEOREM 1. *Let G be a LCA group, $h \not\equiv 0$ a trigonometric polynomial on G, $||h||_\infty \leqq 1$. Then $\mathscr{L}_1(h)$ is a commutative nonassociative Banach algebra under the multiplication* $\circ$.

REMARKS. a) By a *nonassociative Banach algebra* we mean a Banach space equipped with a multiplication which satisfies all the requirements of a Banach algebra with the possible exception of the associative law of multiplication. In accordance with the usual terminology for such algebras (10) a nonassociative algebra may be associative. If the associative law fails then the algebra is said to be *not associative*.

b) It is not difficult to show that $\mathscr{L}_1(h)$ may be not associative. Indeed if G is any nontrivial LCA group and $\gamma \in \hat{G}$, the dual group of G, is not identically one then, appealing to well known properties of the Fourier transform, one can readily verify that $\mathscr{L}_1(h)$ is not associative when $h(t) = (t, \gamma)$. The question of associativity for $\mathscr{L}_1(h)$ will be further investigated below.

c) The restrictions that $||h||_\infty \leqq 1$ is only one of convenience. If $1 < ||h||_\infty < \infty$ then one defines $f \circ g = fh * gh/(||h||_\infty)^2$.

d) Among possible products involving h the one chosen above seems most amenable to study. For instance the product $f \circ g = f * gh$ leads, in general, to a noncommutative algebra.

e) In view of the translation invariance of Haar measure one might suspect that the appropriate linear space to begin with in the construction of the algebras $\mathscr{L}_1(h)$ should consist of all Borel measurable functions such that $\int_G |T_{s^{-1}} f(t) h(t)| dm(t) < \infty$, $s \in G$. However, one can show that this space is identical with the linear space $L_1(G)$. The proof depends on the almost periodicity of h.

f) Obviously when h is a constant then the algebras $\mathscr{L}_1(h)$ are essentially identical with the group algebra $L_1(G)$. Thus in what follows we shall always assume that h is nonconstant.

Since the algebra $\mathscr{L}_1(h)$ we wish to study may not be associative, it is not immediately apparent to what extent we can employ the theory of (associative) Banach algebras in our investigations. For example, the usual proofs of the Gelfand-Mazur theorem seem to depend on the associative law of multiplication. Consequently, most of the following theorems will be established by direct arguments rather than through an appeal to the general theory of Banach algebras.

On the other hand the majority of the concepts involved in the theory of Banach algebras can be transferred *verbatim* to the nonassociative context, and we shall do so without further comment.

3. The multiplicative linear functionals on $\mathscr{L}_1(h)$

Our first concern will be to describe the multiplicative linear functionals for the algebras $\mathscr{L}_1(h)$, that is, the continuous homomorphisms of $\mathscr{L}_1(h)$ into the complex numbers. As in the study of the group algebra $L_1(G)$ we first reduce the problem to the solution of a certain functional equation.

THEOREM 2. *Let G be a LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Then the following are equivalent.*

i) *F is a multiplicative linear functional on $\mathscr{L}_1(h)$.*

ii) *There exists a unique bounded continuous function α on G such that*

$$F(f) = \int_G f(t)\alpha(t)\,dm(t), \qquad f \in \mathscr{L}_1(h), \tag{$*$}$$

and

$$h(t)h(s)\alpha(ts) = \alpha(t)\alpha(s), \qquad t, s \in G. \tag{$**$}$$

PROOF. If F is a multiplicative linear functional on $\mathscr{L}_1(h)$, then clearly there exists a bounded measurable function α on G which satisfies $(*)$. Appealing to Fubini's theorem one deduces that for all $f, g \in \mathscr{L}_1(h)$ we have

$$\begin{aligned}\int_G f(t)\alpha(t)\left(\int_G g(s)\alpha(s)\,dm(s)\right) dm(t) &= F(f)F(g)\\ &= F(f \circ g)\\ &= \int_G f(t)h(t)\left(\int_G g(s)h(s)\alpha(ts)\,dm(s)\right) dm(t).\end{aligned}$$

Hence for each $g \in \mathscr{L}_1(h)$,

$$\alpha(t)\int_G g(s)\alpha(s)\,dm(s) = h(t)\int_G g(s)h(s)\alpha(ts)\,dm(s)$$

for almost all t in G. But since the right hand member of this identity is continuous, we may assume without loss of generality that α is continuous. A repetition of the previous deduction and the continuity of h and α reveals that α satisfies (**). Thus i) implies ii).

The converse assertion is easily verified.

Consequently the description of the multiplicative linear functionals on $\mathscr{L}_1(h)$ is equivalent to finding the continuous solutions of the functional equation (**). We have discussed this equation elsewhere (6) and the main theorem of that paper immediately provides us with the following result. By a *radical algebra* we mean an algebra whose only multiplicative linear functional is the zero homomorphism. For any function k on a LCA group G we set $z(k) = \{t|k(t) = 0\}$ and $\tilde{z}(k) = G \sim z(k)$. e denotes the identity element of G.

THEOREM 3. *Let G be a LCA group, h a nonconstant trigonometric polynomial on G, $\|h\|_\infty \leqq 1$.*

1. *$\mathscr{L}_1(h)$ is a radical algebra if any one of the following conditions is satisfied:*
   i) *G is connected*
   ii) $z(h) = \emptyset$
   iii) $e \in z(h)$
   iv) *G is infinite, $e \in \tilde{z}(h)$ and $\tilde{z}(h)$ contains no nontrivial subgroups.*
2. *If G is disconnected then the following are equivalent:*
   i) *$\mathscr{L}_1(h)$ is not a radical algebra.*
   ii) *There exists a unique open and closed subgroup $K \subset \tilde{z}(h)$ such that*
      a) $h(t) = h(e)$, $t \in K$.
      b) *If $t, s \in \tilde{z}(h) \sim K$ then $ts \notin K$.*

Moreover, if $\mathscr{L}_1(h)$ is not a radical algebra then the multiplicative linear functionals on $\mathscr{L}_1(h)$ are precisely those continuous linear functionals of the form

$$F_\gamma(f) = \int_G f(t)\alpha_\gamma(t)\,dm(t), \qquad f \in \mathscr{L}_1(h),$$

where

$$\alpha_\gamma(t) = \begin{cases} (h(e))^2(t, \gamma), & t \in K \\ 0, & t \notin K \end{cases}$$

and $\gamma \in \hat{K}$. The correspondence $\gamma \langle - \rangle \alpha_\gamma$ is bijective.

REMARKS. a) In view of this theorem we shall now restrict our attention to disconnected groups. If h is a trigonometric polynomial on such a group, then any subgroup K of $\tilde{z}(h)$ which satisfies the restrictions in 2 ii) of Theorem 3 will be called the *solution group* for $\mathscr{L}_1(h)$. Clearly the solution group is unique.

b) Some concrete examples of solutions for (**) can be found in (6).

4. A decomposition theorem

Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. It is evident that $L_1(K)$, the Banach space of functions absolutely integrable with respect to Haar measure on K, can be considered as a closed linear subspace of $\mathscr{L}_1(h)$. Thus for any $f \in \mathscr{L}_1(h)$ we see that $f_2 = \chi_K f \in L_1(K) \subset \mathscr{L}_1(h)$, where χ_K denotes the characteristic function of K. We set $f_1 = f - f_2$. Clearly f_1 can be considered as an element of $L_1(G \sim K)$, the Banach space of functions absolutely integrable with respect to the restriction to $G \sim K$ of Haar measure on G. This space can also in an obvious manner be considered as a subspace of $\mathscr{L}_1(h)$. Moreover Theorem 3 shows that f_1 is an element of the radical of $\mathscr{L}_1(h)$, that is, a member of the intersection of the kernels of the multiplicative linear functionals on $\mathscr{L}_1(h)$. Thus we see that $\mathscr{L}_1(h)$ is a sum of its radical and $L_1(K)$. The next theorem shows that somewhat more can be said about this decomposition. We shall denote the radical of $\mathscr{L}_1(h)$, which is a closed ideal, by R.

THEOREM 4. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$, and suppose $\mathscr{L}_1(h)$ is not a radical algebra. If K is the solution group for $\mathscr{L}_1(h)$ then*:

i) $R = \{f | f \in \mathscr{L}_1(h), f = 0$ *a. e. on* $K\}$.

ii) $\mathscr{L}_1(h) = R \oplus L_1(K)$.

iii) $L_1(K)$ *is an associative subalgebra of* $\mathscr{L}_1(h)$.

iv) *There exists a homeomorphic algebra isomorphism of* $\mathscr{L}_1(h)/R$ *onto* $L_1(K)$.

PROOF. i) and ii) are easily verified. Suppose $f_2, g_2 \in L_1(K)$. Then

$$f_2 \circ g_2(s) = h(e) \int_K f_2(st^{-1})h(st^{-1})g_2(t)dm(t), \quad s \in G,$$

as $K \subset \{t | h(t) = h(e)\}$ and $g_2 = 0$ a.e. on $G \sim K$. Since K is a group it follows that if $s \notin K$ then $st^{-1} \notin K$ for all $t \in K$. This, combined with the fact that $f_2 = 0$ a.e. on $G \sim K$, reveals at once that

$$f_2 \circ g_2(s) = \begin{cases} (h(e))^2 f_2 * g_2(s), & s \in K \\ 0, & s \notin K. \end{cases}$$

The validity of iii) is now evident.

Some routine computations together with the results in i)–iii) show that the mapping $\beta : \mathscr{L}_1(h)/R \to L_1(K)$ defined by $\beta(f_2+R) = (h(e))^2 f_2$ satisfies the requirements of part iv).

REMARKS. a) Parts ii)–iv) of the theorem show that the Wedderburn first principal structure theorem is valid for nonradical $\mathscr{L}_1(h)$ (8, p. 59).

b) Moreover the theorem also provides examples of Banach algebras where the Wedderburn theorem holds but the sufficient conditions utilized by Feldman (2) to insure the validity of this theorem may not be satisfied. In particular it is evident that neither R nor $\mathscr{L}_1(h)/R$ need be finite dimensional (2, p. 776).

c) In the sequel for any $f \in \mathscr{L}_1(h)$ we shall always write $f = f_1 + f_2$ where $f_1 \in R$ and $f_2 \in L_1(K)$ is the decomposition given by the preceding theorem.

It is clear that the decomposition theorem should be a powerful tool in the study of the algebras $\mathscr{L}_1(h)$. We apply it first in the next section to the question of associativity.

5. The question of associativity

As indicated previously the algebras $\mathscr{L}_1(h)$ need not satisfy the associative law of multiplication. In this section we shall examine this problem more closely. To begin we shall prove a lemma which will also be useful in the investigation of the ideal structure of $\mathscr{L}_1(h)$.

LEMMA 1. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. Then the following are equivalent*:

i) $K = \tilde{z}(h)$.
ii) $f \circ g = 0, \quad f, g \in R$.
iii) $f \circ g = 0, \quad f \in L_1(K), \; g \in R$.
iv) $f \circ g = 0, \quad f \in \mathscr{L}_1(h), \; g \in R$.

PROOF. If $K = \tilde{z}(h)$ and $g \in R$ then by Theorem 4 i) we conclude that $gh = 0$ a.e. It is then immediate that i) implies ii), iii) and iv). Clearly iv) implies ii) and iii).

On the other hand suppose that $\tilde{z}(h) \sim K \neq \emptyset$. Then since $\tilde{z}(h) \sim K$ is open we may choose an open set E contained in $\tilde{z}(h) \sim K$ such that $0 < m(E) < +\infty$. Let $f = \chi_E \bar{h}$, where the bar denotes complex conjugation. $f \in R$ by Theorem 4 i) and $f \circ f = \chi_E |h|^2 * \chi_E |h|^2$. Since E has positive measure it is evident that $f \circ f \neq 0$ a.e., and we conclude that ii) implies i). A similar type of argument shows that iii) implies i).

Combining all of these implications we see that i) – iv) are equivalent.

An immediate consequence of the lemma is the observation that $f \circ g = (h(e))^2 f_2 * g_2$, $f, g \in \mathscr{L}_1(h)$, whenever $K = \tilde{z}(h)$. The proof of the next theorem is then apparent.

THEOREM 5. *Let G be a disconnected LCA group, h a nonconstant tri-*

gonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. If $K = \tilde{z}(h)$, then $\mathscr{L}_1(h)$ is a commutative (associative) Banach algebra.

It is not clear whether the converse of this theorem is valid. Some results in the converse direction are given after the following lemma.

LEMMA 2. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose that $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. If $K \neq \tilde{z}(h)$, then there exists $s, t, \in \tilde{z}(h) \sim K$ such that $st \notin \tilde{z}(h)$.*

PROOF. Suppose the conclusion of the lemma is not valid. Then since K is the solution group for $\mathscr{L}_1(h)$ we must have $st \in \tilde{z}(h) \sim K$ for all s, t in $\tilde{z}(h) \sim K$. In particular if $t_0 \in \tilde{z}(h) \sim K$ then $t_0^k \in \tilde{z}(h) \sim K$, $k = 1, 2, 3, \cdots$. When G is finite this clearly leads to a contradiction since K is a group, whereas if G is infinite then a simple argument reveals that all the positive integral powers of t_0 are distinct. Furthermore by Proposition 5 ii) in (6) we see that $t_0^{-k} \in G \sim \tilde{z}(h)$, $k = 1, 2, 3, \cdots$. Thus the function h restricted to the infinite discrete group generated by t_0 has the property that $h(t_0^k) \neq 0$, $k = 0, 1, 2, \cdots$, while $h(t_0^{-k}) = 0$, $k = 1, 2, 3, \cdots$. Such a property is however incompatible with the almost periodicity of h, and hence leads to a contradiction. This latter assertion can be proved by an argument used several tines in (6). (See for example the proof of Proposition 2 in (6)).

THEOREM 6. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. If $\tilde{z}(h)$ is a closed set and $K \neq \tilde{z}(h)$ then $\mathscr{L}_1(h)$ is not associative.*

PROOF. By the preceding lemma there exists $s, t \in \tilde{z}(h) \sim K$ such that $st \notin \tilde{z}(h)$. Since $\tilde{z}(h)$ is closed and $\tilde{z}(h) \sim K$ is open we can choose neighbourhoods U and V of the identity in G with finite measure such that a) $stU \cap \tilde{z}(h) = \emptyset$, b) $sV \cup tV \subset \tilde{z}(h) \sim K$ and c) $stV^2 \subset stU$. Moreover we may assume without loss of generality that there exists a $\delta > 0$ such that $|h(u)| \geqq \delta$, $u \in sV \cup tV$. Let $g = \chi_{sV}/h$, $k = \chi_{tV}/h$. Clearly $g, k \in \mathscr{L}_1(h)$ and $g \circ k = \chi_{sV} * \chi_{tV}$. Thus $g \circ k = 0$ a.e. on $G \sim sVtV \supset G \sim stU$ which implies that $(g \circ k)h = 0$ a.e. as $stU \cap \tilde{z}(h) = \emptyset$. Hence for any $f \in \mathscr{L}_1(h)$ we have $f \circ (g \circ k) = fh * (g \circ k)h = 0$.

On the other hand since K is open we can choose an open neighbourhood W of the identity in G which is contained in K, has finite measure and is such that $sVW \subset \tilde{z}(h) \sim K$. If $f = \chi_W$ then $(f \circ g) \circ k = h(e)((\chi_W * \chi_{sV})h) * \chi_{tV}$, which is clearly a continuous function on G which is not identically zero. Therefore $\mathscr{L}_1(h)$ is not associative.

COROLLARY 1. *Let G be a discrete LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. Then the following are equivalent*:

i) $K = \tilde{z}(h)$.
ii) $\mathscr{L}_1(h)$ *is associative.*

REMARKS. a) For finite groups it is possible that $K \neq \tilde{z}(h)$ whereas for infinite discrete groups it is not known whether this can occur or not (6).

b) The question of associativity can also be connected with the existence of an involution on $\mathscr{L}_1(h)$. To be precise, one can show whenever $h(e)$ is either real or pure imaginary that $K = \tilde{z}(h)$ if and only if the mapping $f \to f^*$ is an involution on $\mathscr{L}_1(h)$. As usual $f^*(t) = \overline{f(t^{-1})}$.

6. The ideal structure of $\mathscr{L}_1(h)$

As to be expected the ideal structure of $\mathscr{L}_1(h)$ is closely related to that of the group algebra $L_1(K)$ where K is the solution group for $\mathscr{L}_1(h)$. Nevertheless there are several differences which set the algebras $\mathscr{L}_1(h)$ apart from the usual group algebras. When $K = \tilde{z}(h)$ a fairly complete description of the ideals in $\mathscr{L}_1(h)$ can be given, whereas if $K \neq \tilde{z}(h)$ then the situation appears to be a good deal more complicated and no really satisfactory results are yet available.

Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. Then it is apparent that if I is a closed ideal in $\mathscr{L}_1(h)$ then $I = I_1 \oplus I_2$ where $I_1 \subset R$ is a closed ideal in $\mathscr{L}_1(h)$ and $I_2 \subset L_1(K)$ is a closed ideal in the group algebra $L_1(K)$. However I_2 need not be an ideal in $\mathscr{L}_1(h)$, nor is the sum of ideals I_1 and I_2 satisfying the previous conditions necessarily an ideal in $\mathscr{L}_1(h)$. As is easily seen, in order for I_2 to be an ideal in $\mathscr{L}_1(h)$ it is necessary and sufficient that $R \circ I_2 = 0$. Thus, for example, if $K \neq \tilde{z}(h)$, then by Lemma 1 we see that $L_1(K)$ is not an ideal in $\mathscr{L}_1(h)$. The simplest way to avoid such difficulties is to concentrate ones attention on the case where $K = \tilde{z}(h)$. It is then easily seen that the closed ideals I in $\mathscr{L}_1(h)$ are precisely the closed subspaces of $\mathscr{L}_1(h)$ of the form $I_1 \oplus I_2$ where $I_1 \in R$ and $I_2 \subset L_1(K)$ satisfy the above mentioned conditions.

It is equally easy, using the observation that $f_2 \circ g_2 = (h(e))^2 f_2 * g_2$, to verify in all cases where $\mathscr{L}_1(h)$ is not a radical algebra that $R \oplus I_2$ is a closed proper regular ideal in $\mathscr{L}_1(h)$ whenever I_2 is such an ideal in the group algebra $L_1(K)$. This however does not give a complete picture of the regular ideals even in the case where $K = \tilde{z}(h)$. For example the radical itself may be a regular ideal in $\mathscr{L}_1(h)$.

THEOREM 7. *Let G be a disconnected LCA group, h a trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$. Then the following are equivalent:*

i) *K is discrete.*
ii) *R is a regular ideal in $\mathscr{L}_1(h)$.*

PROOF. If K is discrete then as is well known (9, p. 6) the group algebra $L_1(K)$ possesses an identity δ. Setting $e_2 = \delta/(h(e))^2 \in L_1(K)$ we see at once that $e_2 \circ f_2 - f_2 = 0, f_2 \in L_1(K)$. Thus for any $f \in \mathscr{L}_1(h)$ we have $e_2 \circ f - f = e_2 \circ f_1 - f_1 \in R$, that is, R is regular.

Conversely, if R is regular then there exists some $e = e_1 + e_2 \in \mathscr{L}_1(h)$, $e_2 \neq 0$, such that $e \circ f - f = e \circ f_1 + e_1 \circ f_2 - f_1 + e_2 \circ f_2 - f_2 \in R, f \in \mathscr{L}_1(h)$. Since R is an ideal it follows at once that $e_2 \circ f_2 - f_2 \in R \cap L_1(K)$, and hence $e_2 \circ f_2 - f_2 = (h(e))^2 e_2 * f_2 - f_2 = 0, f_2 \in L_1(K)$, by Theorem 4 ii). Thus $\delta = (h(e))^2 e_2$ is an identity for $L_1(K)$ and K is discrete (9, p. 30).

An argument similar to the one just given and Lemma 1 establish the following corollary.

COROLLARY 2. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and K is the solution group for $\mathscr{L}_1(h)$.*

i) *K is nondiscrete if and only if R contains no ideals of $\mathscr{L}_1(h)$ which are regular in $\mathscr{L}_1(h)$.*

ii) *If K is discrete and $K = \tilde{z}(h)$, then if $I_1 \subset R$ is a regular ideal in $\mathscr{L}_1(h)$ then $I_1 = R$.*

This theorem and corollary combined with the foregoing discussion allow us to establish a complete description of the ideals in $\mathscr{L}_1(h)$ when $K = \tilde{z}(h)$.

THEOREM 8. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra, and the solution group $K = \tilde{z}(h)$.*

i) *If K is discrete then I is a closed proper regular ideal in $\mathscr{L}_1(h)$ if and only if $I = R$ or $I = R \oplus I_2$ where I_2 is a closed proper regular ideal in the group algebra $L_1(K)$.*
ii) *If K is nondiscrete then I is a closed proper regular ideal in $\mathscr{L}_1(h)$ if and only if $I = R \oplus I_2$ where I_2 is a closed proper regular ideal in the group algebra $L_1(K)$.*
iii) *I is a maximal regular ideal in $\mathscr{L}_1(h)$ if and only if $I = R \oplus I_2$ where I_2 is a maximal regular ideal in the group algebra $L_1(K)$.*

iv) *I is a closed ideal in $\mathscr{L}_1(h)$ such that $kh(I) = I$ if and only if $I = R \oplus I_2$ where I_2 is a closed ideal in the group algebra $L_1(K)$ such that $kh(I_2) = I_2$.*

PROOF. In view of the preceding results it is evident that to establish i) and ii) it is sufficient to show that if $I = I_1 \oplus I_2$ is a regular ideal in $\mathscr{L}_1(h)$ where I_2 is a proper regular ideal in $L_1(K)$ then $I_1 = R$. But in this case if $e \in \mathscr{L}_1(h) \sim I$ is an identity modulo I we conclude from Lemma 1 that $e \circ f_1 - f_1 = -f_1 \in I_1 \oplus I_2$, $f_1 \in R$. That is, $R \subset I_1$, which proves the assertion.

iii) and iv) are easily verified using Theorems 3 and 4.

REMARKS. a) As indicated previously the situation when $K \neq \tilde{z}(h)$ seems a good deal more complicated. One can for example show that $I = I_1 \oplus L_1(K)$ is a regular ideal in $\mathscr{L}_1(h)$ if and only if $I_1 \subset R$ is an ideal in $\mathscr{L}_1(h)$ which is a regular ideal in the subalgebra R such that $R \circ L_1(K) \subset I_1$.

b) For group algebras it is well known (9, p. 157) that the collection of closed ideals is identical with the collection of closed linear subspaces invariant under translation by the group. Such a result is no longer valid for the algebras $\mathscr{L}_1(h)$. One can, however, show that every closed linear subspace which is invariant under translation by the solution group K is a closed ideal provided that $K = \tilde{z}(h)$. The converse is, however, not valid.

As the last subject of this section we shall discuss some results concerning convex ideals. In (1) Aubert studied the existence and character of convex and absolutely convex ideals in the real Banach algebra $L_1^r(G)$ consisting of the real functions in the group algebra $L_1(G)$. If we set $\mathscr{L}_1^r(h) = \{f | f \in \mathscr{L}_1(h), f \text{ real}\}$ then it is easily seen when $\mathscr{L}_1(h)$ is not a radical algebra, the solution group $K = \tilde{z}(h)$ and $(h(e))^2$ is real that $\mathscr{L}_1^r(h)$ is a real Banach algebra. With the same definitions of convexity and almost the same proofs as in (1) it is not difficult to see that the only convex maximal regular ideal I^r in $\mathscr{L}_1^r(h)$ is $I^r = \{f | f \in \mathscr{L}_1^r(h), \hat{f}(e) = 0\}$ where $\hat{f}(\gamma) = (h(e))^2 \int_K f(t)(t^{-1}, \gamma) dm(t)$, $\gamma \in \hat{K}$. And furthermore that the intersection of a family of maximal regular ideals in $\mathscr{L}_1^r(h)$ is convex if and only if it is contained in $\{f | f \in \mathscr{L}_1^r(h), \hat{f}(e) = 0\}$. Thus the results for convex ideals in $\mathscr{L}_1^r(h)$ are completely analogous to those for $L_1^r(G)$ (1, p. 183 and 186).

On the other hand $\mathscr{L}_1^r(h)$ may contain proper regular absolutely convex ideals and always contains closed proper absolutely convex ideals while both sorts of ideals fail to exist in $L_1^r(G)$(1, p. 183). For example $R^r = R \cap \mathscr{L}_1^r(h)$ and $L_1^r(K) = L_1(K) \cap \mathscr{L}_1^r(h)$ are closed proper absolutely convex ideals. More precise information is contained in the following theorem.

THEOREM 9. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $\|h\|_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra, the solution group $K = \tilde{z}(h)$ and $(h(e))^2$ is real.*

i) *If K is discrete, then I^r is a proper regular absolutely convex ideal in $\mathscr{L}_1^r(h)$ if and only if $I^r = R^r$.*

ii) *If K is nondiscrete, then there exists no proper regular absolutely convex ideals in $\mathscr{L}_1^r(h)$.*

iii) *If G is discrete, then $I^r \subset R^r$ is a closed proper absolutely convex ideal in $\mathscr{L}_1^r(h)$ if and only if there exists a subset $E \subset G \sim K$ such that $I^r = \{f \mid f \in \mathscr{L}_1^r(h), f = 0$ a.e. on $E \cup K\}$.*

PROOF. If $I^r = R^r$ then by Theorem 4 i) and 7 it is evident that I^r is a proper regular absolutely convex ideal when K is discrete. Conversely suppose that I^r is a proper regular absolutely convex ideal in $\mathscr{L}_1^r(h)$. Then, as is easily seen, $I = I^r \oplus iI^r$ is a proper regular ideal in $\mathscr{L}_1(h)$. Thus by Theorem 8 i) and ii) we see that either $I = R$ or $I = R \oplus I_2$ when K is discrete or $I = R \oplus I_2$ when K is nondiscrete, where I_2 is a proper regular ideal in the group algebra $L_1(K)$. If $I = R \oplus I_2$ then $I^r = I \cap \mathscr{L}_1^r(h) = R^r \oplus I_2^r$ implies that I_2^r is a proper regular absolutely convex ideal in $L_1^r(K)$ since I^r is such an ideal in $\mathscr{L}_1^r(h)$. But this contradicts the fact that no such ideals exist in $L_1^r(K)$ (1, p. 183). Consequently $I^r = R^r$ when K is discrete, proving i); and when K is nondiscrete we must conclude that no proper regular absolutely convex ideals exist in $\mathscr{L}_1^r(h)$, thereby proving ii).

Since by Lemma 1 and Theorem 4 i) every closed linear subspace of $L_1^r(G \sim K)$ is a proper closed ideal in $L_1^r(h)$ it is sufficient in establishing iii) to show that each closed absolutely convex ideal I^r has the desired form. If $I^r = R^r$ then iii) holds with $E = \emptyset$. While if $I^r \neq R^r$ and no such E exists then for each $t \in G \sim K$ there is some $f \in I^r$ such that $f(r) > 1$. The absolute convexity of I^r then shows that $\chi_{\{t\}} \in I^r$, $t \in G \sim K$, and hence $I^r = L_1^r(G \sim K) = R^r$ as I^r is closed, contrary to assumption.

REMARKS. In passing we mention two further structural results without proof. When h is nonconstant then $\mathscr{L}_1(h)$ never possesses an approximate identity. Thus the algebras $\mathscr{L}_1(h)$ cannot be group algebras in the general sense considered in (3). Furthermore, when $\mathscr{L}_1(h)$ is not radical and $K = \tilde{z}(h)$ then $\mathscr{L}_1(h) \circ \mathscr{L}_1(h)$ is always a proper subset of $\mathscr{L}_1(h)$, that is, factorization is not generally valid as in group algebras.

7. The multipliers for $\mathscr{L}_1(h)$

A multiplier for $\mathscr{L}_1(h)$ is a bounded linear operator $S : \mathscr{L}_1(h) \to \mathscr{L}_1(h)$ such that $Sf \circ g = f \circ Sg, f, g \in \mathscr{L}_1(h)$. It is evident that the multipliers form an operator norm closed linear subspace $M(\mathscr{L}_1(h))$ of the Banach algebra $E(\mathscr{L}_1(h))$ of all bounded linear operators on $\mathscr{L}_1(h)$. When $\mathscr{L}_1(h)$ is not a radical algebra and the solution group $K = \tilde{z}(h)$ then it is quite easy to exhibit several different kinds of elements in $M(\mathscr{L}_1(h))$. For example translation by an element of K, the projections P_1 and P_2 of $\mathscr{L}_1(h)$ onto R and $L_1(K)$ respectively, the mapping $Sf = (h(e))^2 f_2 * \mu$, $f \in \mathscr{L}_1(h)$, where μ is a bounded regular Borel measure on K, and the mapping $Sf = S_1 f_1$, $f \in \mathscr{L}_1(h)$, where S_1 is any bounded linear operator on $L_1(G \sim K)$, are all instances of multipliers for $\mathscr{L}_1(h)$.

The latter three examples actually provide us with a complete description of $M(\mathscr{L}_1(h))$. This and other facts about the multipliers are contained in the next theorem. The proof of the theorem, which relies on Theorem 4 and Lemma 1, is relatively straight forward and will be omitted. $E(L_1(G \sim K))$ will denote the Banach algebra of all bounded linear operators on the Banach space $L_1(G \sim K)$ and $M(L_1(K))$ the commutative Banach algebra of multipliers for the group algebra $L_1(K)$. This latter algebra can be identified with the Banach algebra under convolution of all bounded regular Borel measures on K (9, p. 75).

THEOREM 10. *Let G be a disconnected LCA group, h a nonconstant trigonometric polynomial on G, $||h||_\infty \leqq 1$. Suppose $\mathscr{L}_1(h)$ is not a radical algebra and the solution group $K = \tilde{z}(h)$. Then the following are equivalent*:

i) $S \in M(\mathscr{L}_1(h))$.
ii) *There exists a unique* $S_1 \in E(L_1(G \sim K))$, $S_2 \in M(L_1(K))$ *such that* $Sf = S_1 f_1 + S_2 f_2$, $f \in \mathscr{L}_1(h)$.
iii) $S \in E(\mathscr{L}_1(h)$ *is such that*:
   a) $S : R \to R$, $S : L_1(K) \to L_1(K)$.
   b) $SP_2 T_s = T_s SP_2$, $s \in K$.

Moreover $M(\mathscr{L}_1(h))$ is a closed subalgebra of $E(\mathscr{L}_1(h))$ and the correspondence determined by the relationship in ii) *defines an isometric algebra isomorphism of $M(\mathscr{L}_1(h))$ onto $E(L_1(G \sim K)) \oplus M(L_1(K))$.*

REMARKS. a) One cannot use the more general definition of multiplier as given in (11) since $\mathscr{L}_1(h)$ is not without order.

b) In the cases considered one should note several differences between the multipliers for the group algebra $L_1(G)$ and those for $\mathscr{L}_1(h)$. First, $M(\mathscr{L}_1(h))$ is clearly not commutative while $M(L_1(G))$ is (11, p. 1133). Secondly, translations by elements of G are multipliers for $L_1(G)$ while only the translations by elements of the solution group K are multipliers for $\mathscr{L}_1(h)$. Finally, the multipliers for $L_1(G)$ can be characterized as the bounded linear operators which commute with translation by G (9, p. 74). This characterization fails to hold for the elements of $M[\mathscr{L}_1(h)]$. The most that can be said is contained in Theorem 10 iii).

8. Almost periodic functions

With the exception of the second remark following Corollary 1 all of the preceding results remain valid if the trigonometric polynomial h is replaced by a bounded continuous almost periodic function. The proofs remain the same. If one does this, however, then the measures $d\mu = h\, dm$ need not be almost invariant. Moreover a characterization of such measures μ in terms of relatively simple intrinsic properties of μ which are abstractions of well known properties of Haar measure does not seem to exist. Some results in the direction of such a description are given in (7). Consequently we have restricted our attention in the body of the paper to trigonometric polynomials.

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(Oblatum 20–VI–69)

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