COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 22, nº 2 (1970), p. 175-180

<http://www.numdam.org/item?id=CM_1970__22_2_175_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 2, 1970, pag. 175-180
Wolters-Noordhoff Publishing
Printed in the Netherlands

THE LUSTERNIK-SCHNIRELMAN CATEGORIES OF A PRODUCT SPACE

by

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1. Introduction

For Cartesian products of connected C.W. complexes, the following formula concerning the weak Lusternik-Schnirelman category is known (see for example [1]):

$$\operatorname{cat}(X \times Y) \leqq \operatorname{cat}(X) + \operatorname{cat}(Y).$$

In § 3 we shall prove the same formula for the strong Lusternik-Schnirelman category:

$$\operatorname{Cat}(X \times Y) \leqq \operatorname{Cat}(X) + \operatorname{Cat}(Y)$$

under the assumption that X and Y have the homotopy type of a connected C.W. complex. We shall refer to these formulars as the weak resp. strong product theorem. In § 4 the methods § 3 are extended in order to obtain the

MIXED PRODUCT THEOREM: If X and Y have the homotopy type of a connected C.W. complex, then

$$\operatorname{Cat}(X \times Y) \leqq \max.(\operatorname{Cat}(\mathrm{X}), 1) + \operatorname{cat}(Y).$$

This theorem has the following corollary (take $Y =$ one point): If X has the homotopy type of a C.W. complex then

$$\operatorname{Cat}(X) \leqq \operatorname{cat}(X) + 1.$$

[Note that from the definitions it follows that $\operatorname{cat}(X) \leqq \operatorname{Cat}(X)$].

Discussions with T. Ganea helped me to simplify the proof and to give the theorem its present generality; after these discussions T. Ganea found an independent and shorter proof for the corollary using only homotopy-categorical notions; this proof is given in § 5.

* Research partially supported by the Netherlands Organisation for the Advancement of Pure Research (Z.W.O.).

2. Definitions

All spaces will have the homotopy type of a C.W. complex, or equivalently, of a polyhedron. We shall use $X \overset{h}{\sim} Y$ for: X is homotopy equivalent with Y.

If K is a simplicial complex, then the corresponding polyhedron is also denoted by K; if we have a polyhedron K with a specific triangulation t, we use the notation (K, t).

DEFINITION 1. If X is a connected space, the *weak Lusternik-Schnirelman category cat* (X) is the smallest number k, such that there is a covering of X with $k+1$ open sets $\{U_i\}_{i=0}^{k}$, each of which is contractible in X.

The *strong Lusternik-Schnirelman category Cat* (X) is the smallest number n, such that there is a C.W. complex or, equivalently, a polyhedron $Y \overset{h}{\sim} X$ and a covering $\{V_i\}_{i=0}^{n}$ with subcomplexes of Y, each of which is self-contractible.

[Sometimes, for example in [1], cat( ) and Cat( ) are defined so that they are one larger than according to our definition; our definition is used for example in [2]).

REMARK. It is a standard fact [1] that cat( ) is a homotopy invariant. If the space X is a C.W. complex, or even a simplicial complex, one can also require, without changing the notion of cat( ), that the sets U_i in definition 1 are, with respect to some subdivision, closed subcomplexes which are contractible in X. From this it easily follows that cat( ) $\leqq$ Cat( ).

Finally we have to say something about the topology of a product space. Let K_1 and K_2 be simplicial complexes with a given ordering of the vertices. A product $K_1 \times K_2$ of the simplicial complexes K_1 and K_2 can be obtained by taking as the set of vertices of $K_1 \times K_2$ the Cartesian product of the sets of vertices and as n-simplices subsets $(a_0, b_0), \cdots, (a_n, b_n)$ with

(i) $a_i \leqq a_{i+1}, b_i \leqq b_{i+1}$, but $(a_i, b_i) \neq (a_{i+1}, b_{i+1})$,

(ii) some simplex of K_1 containes $a_0, \cdots, a_n$; some simplex of K_2 contains $b_0, \cdots, b_n$.

The simplicial complex $K_1 \times K_2$ depends on the ordering of the vertices of K_1 and K_2. The corresponding polyhedron $K_1 \times K_2$ however does not depend on these orderings. This polyhedron will be called $K_1 \times_S K_2$; the product of the polyhedra K_1 and K_2, with the Cartesian product topology will be called $K_1 \times_C K_2$. It is not difficult to see that there is a natural $1-1$ continuous map $t : K_1 \times_S K_2 \rightarrow K_1 \times_C K_2$ which is in general not a homeomorphism.

According to Milnor [3] however, $K_1 \times_C K_2$ has the homotopy type of a

C.W. complex. From this and the fact that t induces isomorphisms between the homotopy groups of $K_1 \times_S K_2$ and $K_1 \times_C K_2$, it follows that t is a homotopy equivalence. In the following "$\times$" will always mean "$\times_S$"; by the above remark however, all final statements, containing only homotopy invariants, hold for the cartesian product as well.

3. The strong product theorem

LEMMA 2. *If* $\mathrm{Cat}(X) \leqq k$, *then, for each number* n *there is a polyhedron* $K \overset{h}{\sim} X$ *and a set of* $n+1$ *coverings* $\{\{U_{i,j}\}_{i=0}^{k}\}_{j=0}^{n}$ *of* K *with the following properties*:

1. *There is a triangulation* t *of* K, *such that all* $U_{i,j}$*'s are closed subcomplexes of* (K, t).
2. $U_{i,j}$ *is contractible (over itself) for all* i, j.
3. $\bigcup_{i=0}^{k} U_{i,j} = K$ *for each* j.
4. $U_{i,j} \bigcap U_{i',j'} = \emptyset$ *if* $i+j = i'+j'$.

PROOF. We shall prove the lemma by induction on k (for fixed n). For $k = 0$ the lemma is trivial (take $U_{0,j} = X$). For arbitrary k we use the fact [2] that if $\mathrm{Cat}(X) = k$, then there is a map $\alpha : L \to M$ between polyhedra such that

(i) $\mathrm{Cat}(M) = k-1$.

(ii) X has the homotopy type of the mapping cone of α.

If $h : M \to M'$ is some homotopy equivalence, then, for $h\alpha : L \to M'$ clearly also (i) and (ii) hold. By induction we may assume that for given M, with $\mathrm{Cat}(M) = k-1$, there is a polyhedron $M' \overset{h}{\sim} M$ and a set of n coverings $\{\{V_{i,j}\}_{i=0}^{k-1}\}_{j=0}^{n}$ of M' satisfying the properties 1. $\cdots$ 4. in Lemma 2. Because a homotopic change of $h\alpha$ does not change the homotopy type of the mapping cone, we may assume that $h\alpha$ is simplicial (with respect to triangulations t and t' of L and M' such that the sets $V_{i,j}$ are subcomplexes with respect to t') and consequently that the mapping cone of $h\alpha$ is a polyhedron.

Now take K = mapping cone of $h\alpha$ =

$$M' \cup L \times [0, 1]/((l, 0) = h\alpha(l) \forall l \in L; (l, 1) = (l', 1)\, l, l' \in L)$$

For any subset W of M', $W[t]$ is the following subset of K:

$$W[t] = W \cup \{(l, \bar{t}) | h\alpha(l) \in W \quad \text{and} \quad \bar{t} \leqq t\}.$$

Note that $W[t] \overset{h}{\sim} W$.

The coverings of K can now be defined by

$$U_{i,j} = V_{i,j}\left[\frac{1}{j+2}\right] \text{ for } i = 0, \cdots, k-1 \text{ and } j = 0, \cdots, n.$$

$$U_{k,j} = \left\{(l, t) | l \in L, t \in \left[\frac{1}{j+2}, 1\right]\right\} \subset k$$

The properties $1 \cdots 4$. follow directly.

PROOF OF THE STRONG PRODUCT THEOREM.

We have to prove that $\text{Cat}(X \times Y) \leqq \text{Cat}(X) + \text{Cat}(Y)$ in case X and Y have the homotopy type of a C.W. complex. Let $\text{Cat}(X) = k$ and $\text{Cat}(Y) = n$. There is a polyhedron $L \overset{h}{\sim} Y$ and a covering $\{V_j\}_{j=0}^n$ of L with closed contractible subcomplexes. By lemma 2 there is a polyhedron $K \overset{h}{\sim} X$ and a set of $n+1$ coverings $\{\{U_{i,j}\}_{i=0}^k\}_{j=0}^n$ of K satisfying the properties $1. \cdots 4$. of lemma 2.

Now we take the following covering $\{W_s\}_{s=0}^{k+n}$ of $K \times L$:

$$W_s = \bigcup_{i+j=s} U_{i,j} \times V_j.$$

$U_{i,j} \times V_j$ is contractible. Because $U_{i,j} \cap U_{i',j'} = \emptyset$ for $i+j = i'+j'$ we have $(U_{i,j} \times V_j) \cap (U_{i',j'} \times V_{j'}) = \emptyset$ if $i+j = i'+j'$, so W_s is the union a finite number of contractible components. In order to obtain the required covering we make each of the W_s connected, by connecting the components with arcs in $K \times L$.

4. The proof of the mixed product theorem

As observed in § 2 it suffices to prove that

$$\text{Cat}(K \times L) \leqq \max(\text{Cat}(K), 1) + \text{cat}(L)$$

for K and L connected polyhedra.

Let $k = \max(\text{Cat}(K), 1)$ and $n = \text{cat}(L)$. There is a covering $\{V_j\}_{j=0}^n$ of (L, t) with closed subcomplexes, each contractible in L. We may assume that (K, t') admits a set of $n+1$ coverings $\{\{U_{i,j}\}_{i=0}^k\}_{j=0}^n$ with closed subcomplexes satisfying properties 1, 2, 3 and 4 of lemma 2; because $k \geqq 1$, we may assume that none of the $U_{i,j}$'s is the whole of K.

As in the proof of the strong product theorem we consider the covering of $K \times L$ with subcomplexes of the form $U_{i,j} \times V_j$. These subcomplexes are in general not self-contractible; they have the homotopy type of V_j. Self-contractible subsets can be obtained by attaching a cone over V_j to $U_{i,j} \times V_j$. To get space enough for this attachement we multiply $L \times K$ and each $U_{i,j} \times V_j$ with a sufficiently large contractible polyhedron. We now give the details.

By assumption V_j is contractible in L, so there are maps $\varphi'_j : C(V_j) \to L$; $C(V_j) = V_j \times [0, 1]/V_j \times 1$ and $\varphi'_j(v, 0) = v$. There is a triangulation t_j of $C(V_j)$ and a $\varphi_j \overset{h}{\sim} \varphi'_j$ such that φ_j is a simplicial map $(C(V_j), t_j) \to (L, t)$ and $\varphi_j(v, 0) = v$.

Δ^J is the contractible simplicial complex whose set of vertics is J, and whose set of q-simplices is the set of all subsets of $q+1$ elements of J. The set J is taken so large that each $(C(V_j), t_j)$ can be embedded simplicially in Δ^J; let $\sigma_i : C(V_j) \to \Delta^J$ be such embeddings.

The maps $\kappa_i : (C(V_j), t_j) \to [0, 1]$ are defined as follows: Each vertex of the base is mapped to 0, all the other vertices are mapped to 1; κ_i is linear on each simplex.

Finally we construct a sequence of embeddings $\alpha_{i,j} : [0, 1] \to K$ such that

(a) $\alpha_{i,j}[0, 1] \cap U_{i,j} = \alpha_{i,j}(0)$,

(b) $(U_{i,j} \cup \operatorname{Im} \alpha_{i,j}) \cap (U_{i',j'} \cup \operatorname{Im} \alpha_{i',j'}) = \emptyset$ of $i+j = i'+j'$,

(c) there is a subdivision t'' of (K, t') such that for all (i, j), $\alpha_{i,j}$ is a linear map onto one 1-simplex.

Note that the $\alpha_{i,j}$'s can only be constructed if none of the $U_{i,j}$'s equals K.

After all these preparations we can attach a cone over V_j to $U_{i,j} \times V_j \times \Delta^J$ by taking

$$W_{i,j} = (U_{i,j} \times V_j \times \Delta^J) \cup \{(\alpha_{i,j}\kappa_j(u, t), \varphi_j(u, t), \sigma_j(u, t)) | (u, t) \in C(V_j)\}.$$

These $W_{i,j}$ have the following properties:

1. $W_{i,j}$ is a closed subcomplex of $K \times L \times \Delta^J$; this follows from the fact that $\alpha_{i,j}\kappa_j, \varphi_j$ and σ_j are simplicial with respect to the same triangulation t_j of $C(V_j)$.

2. $W_{i,j}$ is self-contractible.

3. $W_{i,j} \cap W_{i',j'} = \emptyset$ if $i+j = i'+j'$; this can be seen in the projection onto K.

4. $\bigcup_{i,j} W_{i,j} = K \times L \times \Delta^J$.

As in the proof of the strong product theorem, this gives $\operatorname{Cat}(K \times L \times \Delta^J) \leqq k+n$. Because $K \times L \overset{h}{\sim} K \times L \times \Delta^J$ we are done.

5. Ganea's proof of the corollary: Cat $\leqq$ cat+1

Let X be a space with cat $(X) = k$ having the homotopy type of a C.W. complex. Then there is a polyhedron $K \overset{h}{\sim} X$ and a covering $\{U_i\}_{i=0}^k$ of K with closed subcomplexes, each contractible in K. Let K' be the polyhedron obtained by attaching a cone over each of the U_i's. Clearly

$\text{Cat}(K') \leqq k$ (K' is the union of $k+1$ cones; each cone is self-contractible).

We first show that K' is homotopy equivalent with $K \vee (\Sigma U_0) \vee \cdots \vee (\Sigma U_k)$, the one-point union of K, $(\Sigma U_0), \cdots, (\Sigma U_k)$ (ΣU_i is the suspension of U_i).

Let K'_0 be the space obtained from K by attaching a cone over U_0. K'_0 is the mapping cone of the natural inclusion map $i_0 : U_0 \to K$. Changing the map i_0 within its homotopy class does not change the homotopy type of the mapping cone of i_0. Because U_0 is contractible in K, $i_0 : U_0 \to K$ is homotopic with a constant map $\tilde{\imath}_0$ (i.e. $\text{Im}(\tilde{\imath}_0) =$ one point). The mapping cone of $\tilde{\imath}_0$ is clearly $K \vee (\Sigma U_0)$, so $K'_0 \overset{h}{\sim} K \vee (\Sigma U_0)$. The same argument, applied to each of the U_i's, gives $K' \overset{h}{\sim} K \vee (\Sigma U_0) \vee \cdots \vee (\Sigma U_k)$.

Let us call $Z = (\Sigma U_0) \vee \cdots \vee (\Sigma U_k)$. We now have:

$$\text{Cat}(K \vee Z) \leqq k \text{ because } K \vee Z \overset{h}{\sim} K' \text{ and } \text{Cat}(K') \leqq k;$$

$K \overset{h}{\sim}$ ($K \vee Z$ with a cone attached over Z).

From Ganea [2] it then follows that $\text{Cat}(K) \leqq k+1$, and consequently $\text{Cat}(X) \leqq k+1$.

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(Oblatum 13–X–69)

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