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Representations of infinite weak product groups

by

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In this paper we consider certain unitary representations of groups G of the form $\coprod_{i\in I} G_i$. Here I is an arbitrary indexing set, and, for each $i \in I$, G_i is a group with identity $e_i \cdot \coprod_{i\in I} G_i$ denotes the subgroup of the group $\prod_{i\in I} G_i$ consisting of all $(g_i)_{i\in I}$ with $g_i = e_i$ for all but a finite number of $i \in I$. Suppose, for each $i \in I$, that $U_i : g_i \to U_i(g_i)$ is a unitary representation of G_i on the W^*-algebra $\mathscr{A}_i$: by this we mean that $g_i \to U_i(g_i)$ is a homomorphism of G_i into the group of unitary operators of $\mathscr{A}_i$ *and that* $\mathscr{A}_i$ *is generated by* (is the smallest W^*-subalgebra of $\mathscr{A}$ containing)

$$\{U_i(g_i) : g_i \in G_i\}.$$

Suppose that, for each $i \in I$, μ_i is a normal state of $\mathscr{A}_i$ with $\mu_i(1) = 1$. Define the unitary representation $U = \otimes_{i\in I}(U_i, \mu_i)$ of G on the W^*-algebra $\mathscr{A} = \otimes_{i\in I}(\mathscr{A}_i, \mu_i)$ by:

$$U((g_i)) = \otimes_{i\in I} U_i(g_i) \text{ for all } (g_i) \in G. \tag{0.1}$$

(For the definition and properties of $\otimes(\mathscr{A}_i, \mu_i)$, see § 1, below).

We remark that, if the G_i are topological groups, and if G is provided with the topology induced by the product topology on $\prod G_i$, then $\otimes_{i\in I}(U_i, \mu_i)$ is weakly continuous whenever each U_i is weakly continuous.

If each U_i is a factor representation (i.e. if $\mathscr{A}_i$ is a factor), then U is a factor representation [1, p. 178]. If each U_i is faithful (i.e. is an isomorphism) then U is faithful.

We say that a unitary representation U of G on $\mathscr{A}$ is *equivalent* (quasi-equivalent is the more usual term) to a unitary representation V of G on $\mathscr{B}$ when there exists a *-isomorphism Φ of $\mathscr{A}$ onto $\mathscr{B}$ such that:

$$\Phi(U(g)) = V(g) \text{ for all } g \in G.$$

It is easy to see that if $\otimes(U_i, \mu_i)$ and $\otimes(U_i, \nu_i)$ are equivalent then

$$\lim_{i\in I} ||\mu_i - \nu_i|| = 0$$

(see theorem 2.3, below). A sufficient condition for the equivalence of $\otimes\ (U_i,\ \mu_i)$ and $\otimes\ (U_i,\ \nu_i)$ is

$$\sum_{i\in I} [d(\mu_i,\ \nu_i)]^2 < \infty. \tag{0.2}$$

Here d is a metric on the set of normal states of a W^*-algebra $\mathscr{B}$: d is defined essentially by

$$d(\mu,\ \nu) = \inf \{||x-y||),$$

the infimum being taken over all vectors x and y inducing μ and ν relative to a representation of $\mathscr{B}$ as a von Neumann algebra (see [2]). Using the genralized Kakutani product theorem of [2], we easily prove that, whenever each $\mathscr{A}_i$ is semi-finite, (0,2) is necessary and sufficient for the equivalence of $\otimes(U_i,\ \mu_i)$ and $\otimes\ (U_i,\ \nu_i)$ (see theorem 2.2, below).

We can construct other unitary representations of $G = \prod G_i$ by using the above tensor product construction twice (see § 3, below). Whether or not two such double product representations are equivalent can be determined, under certain conditions, by the results of § 2 mentioned above. It is not difficult to see (theorem 3.6 and corollary 3.10) that, provided an infinite number of the $\mathscr{A}_i$'s are non-trivial, roughly speaking almost all of these double product representations are *non-direct-product* representations: we call a representation U of $G = \prod G_i$ on $\mathscr{A}$ non-direct-product if there exists no non-zero normal state ω of $\mathscr{A}$ such that

$$\omega(U((g_i))) = \prod \omega(U(g_i)) \ \text{ for all } \ (g_i) \in G \tag{0.3}$$

(Here each G_i has been identified in the natural way with a subgroup of G). (A normal state ω satisfying (0.3) is called a product state for U. That $\otimes\ \mu_i$ is a product state for $\otimes\ (U_i.\ \mu_i)$ is clear).

All our results hold equally well for representations which are not necessarily unitary.

Other authors have obtained some of these results for representations of certain product groups arising in quantum field theory: namely, the groups associated with representations of the canonical commutation relations (e.g. [6]) and the anticommutation relations (e.g. [5]) for an infinite number of degrees of freedom.

1. Tensor products of W^*-algebras

This section is a summary of definitions and results which we need in the sequel. With the exception of lemma 1.2, which is rather obvious, everything here is from [1], [2], [3] or [7].

Suppose that $(\mathscr{A}_i)_{i\in I}$ is a family of W^*-algebras. We call a W^*-algebra $\mathscr{A}$, together with a family $(\alpha_i)_{i\in I}$ with each α_i an isomorphism of $\mathscr{A}_i$ into $\mathscr{A}$ (and $\alpha_i(1) = 1$), a *product* for $(\mathscr{A}_i)_{i\in I}$ when the following conditions hold:

(1.1) $\qquad \alpha_i(\mathscr{A}_i)$ commutes with $\alpha_j(\mathscr{A}_j)$ for all $i, j \in I$

with $i \neq j$.

(1.2) $\qquad \mathscr{A}$ is generated by $\{\alpha_i(\mathscr{A}_i) : i \in I\}$.

We say that two products $(\mathscr{A}, (\alpha_i))$ and $(\mathscr{B}, (\beta_i))$ for $(\mathscr{A}_i)$ are *product isomorphic* if there exists an isomorphism Φ of $\mathscr{A}$ onto $\mathscr{B}$ such that $\Phi \circ \alpha_i = \beta_i$ for all $i \in I$. By a *product state* of $(\mathscr{A}, (\alpha_i))$ we mean a normal state ν of $\mathscr{A}$ for which there exist states ν_i of $\mathscr{A}_i$ such that:

(1.3) $$\nu\Big(\prod_{i\in I} \alpha_i(A_i)\Big) = \prod_{i\in I} \nu_i(A_i) \text{ for all families } (A_i)$$

with each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in I$ (a.a. $i \in I$ will mean *all but a finite number of* $i \in I$). We write $\nu = \otimes\, \nu_i$ if (1.3) holds. If a normal state ν of $\mathscr{A}$ satisfies $\nu(1) = 1$, then ν is a product state of $(\mathscr{A}, (\alpha_i))$ if and only if the following condition is satisfied (see [3]):

(1.4) $$\nu\Big(\prod_{i\in I} B_i\Big) = \prod_{i\in I} \nu(B_i) \text{ for all families } (B_i)$$

with each $B_i \in \alpha_i(\mathscr{A}_i)$ and $B_i = 1$ for a.a. $i \in I$. Condition (1.4) means precisely that the family $(\alpha_i(\mathscr{A}_i))$ is independent with respect to ν.

Suppose that, for each $i \in I$, μ_i is a normal state of $\mathscr{A}_i$ with $\mu_i(1) = 1$. We call a product $(\mathscr{A}, (\alpha_i))$ for $(\mathscr{A}_i)$ a (μ_i)-*tensor product* for $(\mathscr{A}_i)$ if the following condition holds:

(1.5) For every family (ν_i) with each ν_i a normal state of $\mathscr{A}_i$ and $\nu_i = \mu_i$ for a.a. $i \in I$, the product state $\otimes\, \nu_i$ exists on $(\mathscr{A}, (\alpha_i))$. Furthermore if $A \in \mathscr{A}^+$ and $(\otimes\, \nu_i)(A) = 0$ for all such families (ν_i), then $A = 0$.

The fact that a (μ_i)-tensor product $(\mathscr{A}, (\alpha_i))$ exists and is unique up to product isomorphism is proved in [3]. We write

$$\mathscr{A} = \bigotimes_{i\in I} (\mathscr{A}_i, \mu_i),$$

and, if no confusion can arise, we let

$$\bigotimes_{i\in I} A_i = \prod_{i\in I} \alpha_i(A_i)$$

for every family $(A_i)_{i\in I}$ with each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in I$.

Suppose that $\mathscr{A}_i$ is a von Neumann algebra on the Hilbert space H_i, and that $x_i \in H_i$ induces μ_i:

$$\mu_i(A_i) = (A_i x_i | x_i) \text{ for all } A_i \in \mathscr{A}_i.$$

Then the von Neumann algebra $\mathscr{A} = \otimes_{i\in I} (\mathscr{A}_i, x_i)$ on the Hilbert space $\otimes_{i\in I} (H_i, x_i)$, together with the natural injections a_i of $\mathscr{A}_i$ into $\mathscr{A}$, forms a (μ_i)-tensor product for $(\mathscr{A}_i)_{i\in I}$ (see [1], [3], [7]).

The following lemma is proved in [2]. Part (I) is almost obvious from the definitions; part (II) is the generalized Kakutani theorem of [2].

Lemma 1.1. *Suppose that $(\mathscr{A}_i)_{i\in I}$ is a family of W^*-algebras and that, for each $i \in I$, μ_i and ν_i are normal states of $\mathscr{A}_i$ with $\mu_i(1) = \nu_i(1) = 1$. Let the conditions* (1.6), (1.7) *and* (1.8) *be as follows*:

(1.6) $$\sum_{i\in I} [d(\mu_i, \nu_i)]^2 < \infty.$$

(1.7) $$\bigotimes_{i\in I} (\mathscr{A}_i, \mu_i) \text{ is product isomorphic to } \bigotimes_{i\in I} (\mathscr{A}_i, \nu_i).$$

(1.8) $$\bigotimes_{i\in I} \nu_i \text{ exists on } \bigotimes_{i\in I} (\mathscr{A}_i, \mu_i).$$

Then: (I). (1.6) implies (1.7) and (1.7) implies (1.8).

(II). Provided each $\mathscr{A}_i$ is *semi-finite*, the three conditions are equivalent.

Proof. See [2].

We now proceed to find a condition on $d(\mu_i, \nu_i)$ which is implied by (1.8) whether or not the $\mathscr{A}_i$ are semi-finite. First we need the following lemma.

Lemma 1.2. *Suppose that ω is a normal state of $\mathscr{A} = \otimes_{i\in I}(\mathscr{A}_i, \mu_i)$ and that $\omega(1) = 1$. For each $i \in I$, define the normal state ω_i of $\mathscr{A}_i$ by*:

$$\omega_i(A_i) = \omega(\alpha_i(A_i)) \text{ for all } i \in I.$$

Then:

$$\lim_{i\in I} d(\omega_i, \mu_i) = 0$$

In other words, given $\varepsilon > 0$, there exists a finite subset F_ε of I such that:

$$d(\omega_i, \mu_i) < \varepsilon \text{ for all } i \in I - F_\varepsilon.$$

Proof. We may assume without loss of generality that each $\mathscr{A}_i$

is a von Neumann algebra on H_i, that $x_i \in H_i$ induces μ_i, and that $\mathscr{A} = \otimes (\mathscr{A}_i, x_i)$. We may also assume that every normal state of $\mathscr{A}$ is a vector state: for one of the $\mathscr{A}_i$ may be replaced by $\mathscr{A}_i \otimes \mathscr{C}_{H'}$ with $\mathscr{C}_{H'}$ the scalar operators on an infinite dimensional Hilbert space; then the new $\mathscr{A}$ is isomorphic to $[\otimes(\mathscr{A}_i, x_i)] \otimes \mathscr{C}_{H'}$, for which it is evident that every normal state is a vector state.

Proceeding under these assumptions, we let ω be induced by the vector z in $H = \otimes (H_i, x_i)$. Select an orthonormal basis $(\varphi_i^\lambda)_{\lambda \in \Lambda(i)}$ for each H_i with $0 \in \Lambda(i)$ and $\varphi_i^0 = x_i$. Define Λ to be

$$\{(\lambda(i)) \in \prod_{i \in I} \Lambda(i) : \lambda(i) = 0 \text{ for a.a. } i \in I\}.$$

For each $\lambda = (\lambda(i)) \in \Lambda$, let $\varphi^\lambda = \otimes_{i \in I} \varphi_i^{\lambda(i)}$. Then [1] $(\varphi^\lambda)_{\lambda \in \Lambda}$ is an orthonormal basis for H. Let

$$z = \sum_{\lambda \in \Lambda} C_\lambda \varphi^\lambda.$$

Given $\varepsilon > 0$, there exists a finite subset Λ' of Λ such that $||z' - z|| < \varepsilon/2$ if

$$z = \sum_{\lambda \in \Lambda'} C_\lambda \varphi^\lambda.$$

Let F be

$$\{i \in I : \lambda(i) \neq 0 \text{ for some } (\lambda(i)) \in \Lambda'\}.$$

Then F is finite, and, if ω' is the state of $\mathscr{A}$ induced by z' and ω_i' is defined by

$$\omega_i'(A_i) = \omega'(\alpha_i(A_i)) \text{ for all } A_i \in \mathscr{A}_i,$$

then, by direct calculation:

$$\omega_i' = ||z||^2 \mu_i \text{ for all } i \in I - F.$$

Thus, for all $i \in I - F$:

$$\begin{aligned} d(\omega_i, \mu_i) &\leqq d(\omega_i, \omega_i') + d(\omega_i', \mu_i) \\ &\leqq ||z - z'|| + (1 - ||z'||) \\ &\leqq 2||z - z'|| < \varepsilon. \end{aligned}$$

COROLLARY 1.3. *Under the assumptions of lemma* 1.1 (*but without the assumption that each* $\mathscr{A}_i$ *is semi-finite*), *condition* (1.8) *implies condition* (1.9):

$$\text{(1.9)} \qquad \lim_{i \in I} d(\mu_i, \nu_i) = 0.$$

LEMMA 1.4. *Suppose that* $\mathscr{A}$ *is a von Neumann algebra on* H, *and* *that states μ and ν of $\mathscr{A}$ with $\mu(1)$, $\nu(1) \leqq 1$ are induced by vectors x and y in H. Then:*

$$||\mu-\nu|| \leqq 2||x-y||.$$

PROOF. By direct calculation.

COROLLARY 1.5. *For all normal states μ and ν of a W^*-algebra $\mathscr{A}$ with $\mu(1)$, $\nu(1) \leqq 1$,*

$$||\mu-\nu|| \leqq 2d(\mu, \nu).$$

REMARK. The inequality

$$[d(\mu, \nu)]^2 \leqq ||\mu-\nu||$$

under the same conditions as those of corollary 1.5 is proved in [2]. These inequalities together show that, on the set of normal states μ of $\mathscr{A}$ with $\mu(1) \leqq 1$, d and $|| \; ||$ induce the same topology.

2. Tensor-product representations

Suppose that $G = \prod_{i \in I} G_i$ and that U_i is a unitary representation of G_i on $\mathscr{A}_i$ for each $i \in I$. If $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)_{i \in I}$, then we may define a unitary representation U of G on $\mathscr{A}$ by:

$$U((g_i)) = \prod_{i \in I} \alpha_i(U_i(g_i)) \text{ for all } (g_i) \in G. \tag{2.1}$$

We call U the $(\mathscr{A}, (\alpha_i))$ product of (U_i). If $\mathscr{A} = \otimes_{i \in I} (\mathscr{A}_i, \mu_i)$ and α_i is the natural injection of $\mathscr{A}_i$ into $\mathscr{A}$, the $(\mathscr{A}, (\alpha_i))$ product U is the tensor-product representation $\otimes_{i \in I} (U_i, \mu_i)$ defined by (0.1). We obtain immediately the following general lemma.

LEMMA 2.1. *Suppose that $(\mathscr{A}, (\alpha_i))$ and $(\mathscr{B}, (\beta_i))$ are products for $(\mathscr{A}_i)_{i \in I}$, and suppose that U (respectively V) is the $(\mathscr{A}, (\alpha_i))$ (respectively $(\mathscr{B}, (\beta_i))$) product of (U_i). Then U is equivalent to V if and only if the product $(\mathscr{A}, (\alpha_i))$ is product isomorphic to the product $(\mathscr{B}, (\beta_i))$.*

PROOF. If Φ is an isomorphism of $\mathscr{A}$ onto $\mathscr{B}$ with $\Phi \circ \alpha_i = \beta_i$ for each $i \in I$, then for all $g = (g_i) \in G$:

$$\Phi(U(g)) = \Phi(\prod \alpha_i(U_i(g_i))) = \prod \beta_i(U_i(g_i)) = V(g)$$

That proves the sufficiency of the condition.

Suppose, on the other hand, that U is equivalent to V, so that there exists an isomorphism Φ of $\mathscr{A}$ onto $\mathscr{B}$ such that

$$\Phi(U(g)) = V(g) \text{ for all } g \in G.$$

In particular, then, for each $i \in I$:

$$\Phi(\alpha_i(U_i(g_i))) = \beta_i(U_i(g_i)) \text{ for all } g_i \in G_i. \tag{2.2}$$

Now $\{U_i(g_i) : g_i \in G_i\}$ generates $\mathscr{A}_i$. Hence, by the linearity and ultraweak continuity of $\Phi \circ \alpha_i$ and β_i, we can conclude from (2.2) that, for each $i \in I$:

$$\Phi(\alpha_i(A_i)) = \beta_i(A_i) \text{ for all } A_i \in \mathscr{A}_i.$$

That proves that Φ is a product isomorphism of $(\mathscr{A}, (\alpha_i))$ onto $(\mathscr{B}, (\beta_i))$.

THEOREM 2.2. *Suppose that each $\mathscr{A}_i$ is* semi-finite. *Then $\otimes_{i\in I} (U_i, \mu_i)$ is equivalent to $\otimes_{i\in I} (U_i, \nu_i)$ if and only if*

$$\sum_{i\in I} [d(\mu_i, \nu_i)]^2 < \infty. \tag{2.3}$$

PROOF. Use lemma 2.1 and lemma 1.1. (II).

THEOREM 2.3. *With no conditions on the $\mathscr{A}_i$, (2.3) is a sufficient condition for $\otimes_{i\in I}(U_i, u_i)$ and $\otimes_{i\in I}(U_i, \nu_i)$ to be equivalent, and (2.4) is a necessary condition*:

$$\lim_{i\in I} d(\mu_i, \nu_i) = 0 \tag{2.4}$$

PROOF. Use lemmas 2.1, 1.1. (I) and 1.3.

3. Non-direct-product representations

Suppose, as in § 2, that $G = \coprod_{i\in I} G_i$, and that, for each $i \in I$, U_i is a unitary representation of G_i on $\mathscr{A}_i$ and μ_i is a normal state of $\mathscr{A}_i$ with $\mu_i(1) = 1$. Let us suppose, further, that $(I(k))_{k\in K}$ is a disjoint family of subsets of I whose union is I. For each $k \in K$, let $\mathscr{B}_k$ be $\otimes_{i\in I(k)} (\mathscr{A}_i, \mu_i)$ with β_i the natural injection of $\mathscr{A}_i$ into $\mathscr{B}_k$. Suppose that ν_k is a normal state of $\mathscr{B}_k$ with $\nu_k(1) = 1$. Let $\mathscr{A}$ be $\otimes_{k\in K}(\mathscr{B}_k, \nu_k)$ with γ_k the natural injection of $\mathscr{B}_k$ into $\mathscr{A}$. Let $\alpha_i = \gamma_k \circ \beta_i$ for all $i \in I(k)$ and for all $k \in K$. Then $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)_{i\in I}$. Let U be the $(\mathscr{A}, (\alpha_i))$ product of $(U_i)_{i\in I}$, in the notation of § 2. Then U is a unitary representation of G on $\mathscr{A}$. Because $G = \coprod_{i\in I} G_i$ is naturally isomorphic to $\prod_{k\in K} \coprod_{i\in I(k)} G_i$, we may identify U with the double tensor-product representation

$$\bigotimes_{k\in K} [\bigotimes_{i\in I(k)} (U_i, \mu_i), \nu_k].$$

It is evident that the results of § 2 will give necessary and sufficient conditions for two such representations defined by the same partition $(I(k))_{k\in K}$ to be equivalent. In particular, if each $I(k)$ is finite, and if each $\mathscr{A}_i$ is semi-finite, a necessary and sufficient condition for

$$\bigotimes_{k\in K}\Big(\bigotimes_{i\in I(k)} U_i, \nu_k\Big)$$

to be equivalent to

$$\bigotimes_{k\in K}\Big(\bigotimes_{i\in I(k)} U_i, \nu'_k\Big)$$

is that

$$\sum_{k\in K}[d(\nu_k, \nu'_k)]^2 < \infty$$

(theorem 2.2 and [4, I, § 6, proposition 14]). For two such double tensor-product representations defined using *different* partitions of I, a crude condition for non-equivalence can be obtained by using lemma 1.2.

DEFINITION 3.1. Suppose that U is a unitary representation of $G = \prod_{i\in I} G_i$ on $\mathscr{A}$ and that, for each $i \in I$, U_i is defined by

$$U_i(g_i) = U(\lambda_i(g_i)) \text{ for all } g_i \in G_i,$$

where λ_i is the natural injection of G_i into G. Then by a *product state* for U, we mean a non-zero normal state ν of $\mathscr{A}$ such that

$$\nu(U((g_i))) = \prod_{i\in I}\nu(U_i(g_i)) \text{ for all } (g_i) \in G. \tag{3.1}$$

U will be called a *direct-product* (respectively *non-direct-product*) representation of $G = \prod G_i$ if the set of product states ν is faithful on $\mathscr{A}$ (respectively if 0 is the only normal state of $\mathscr{A}$ satisfying (3.1)).

U will be called a *tensor-product* representation of $G = \prod G_i$ if U is equivalent to a representation of the form $\otimes_{i\in I}(U_i, \mu_i)$.

REMARK. Evidently a tensor-product representation is a direct-product representation, but the converse fails (see the remark following lemma 3.2).

LEMMA 3.2. *Under the assumptions of definition* 3.1, *let $\mathscr{A}_i$ be the W^*-subalgebra of $\mathscr{A}$ generated by $\{U_i(g_i) : g_i \in G\}$, and let α_i be the inclusion mapping of $\mathscr{A}_i$ into $\mathscr{A}$. Then $(\mathscr{A}, (\alpha_i))$ is a product for $(\mathscr{A}_i)$ and U is the $(\mathscr{A}, (\alpha_i))$ product of (U_i). Furthermore, a normal state ν of $\mathscr{A}$ is a product state for U if and only if ν is a product state of $(\mathscr{A}, (\alpha_i))$ and $\nu(1) = 1$.*

PROOF. The first assertian is obvious. A product state ν of $(\mathscr{A}, (\alpha_i))$ with $\nu(1) = 1$ satisfies (1.4), and (3.1) follows immediately. To prove the converse, suppose that ν is a non-zero normal state of $\mathscr{A}$ satisfying (3.1). Evidently $\nu(1) = 1$. Letting ν_i be the restriction of ν to $\mathscr{A}_i$, we can write (3.1) as follows:

$$(3.2) \qquad \nu[\prod_{i \in I} \alpha_i(U_i(g_i))] = \prod_{i \in I} \nu_i(U_i(g_i)) \text{ for all } (g_i) \in G.$$

Now $\{U_i(g_i) : g_i \in G_i\}$ generates $\mathscr{A}_i$: therefore the linear span of this group is ultraweakly dense in $\mathscr{A}_i$. Since both sides of (3.2) are linear and ultraweakly continuous in $U_i(g_i)$, we obtain:

$$(3.3) \qquad \nu(\prod_{i \in I} \alpha_i(A_i)) = \prod_{i \in I} \nu_i(A_i) \text{ for all families } (A_i)$$

with each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in I$. (3.3) means that ν is a product state of $(\mathscr{A}, (\alpha_i))$.

Remark. Suppose that U is a factor representation. Then ([3] and lemma 3.2), the existence of one non-zero product state $\nu = \otimes \nu_i$ for U implies that U is equivalent to the tensor-product representation $\otimes (U_i, \nu_i)$. If, however, U is not a factor representation, then U may be a direct-product representation and fail to be a tensor-product representation (see [3, example 7.3]). For double tensor-product representations, however, this phenomenon cannot occur (see lemma 3.7).

Definition 3.3. Suppose that $(\mathscr{B}, (\beta_i))$ is a product for $(\mathscr{A})_{i \in J}$ and that ω is a normal state of $\mathscr{B}$ with $\omega(1) = 1$. Let Σ_P denote the set of product states μ of $(\mathscr{B}, (\beta_i))$ with $\mu(1) = 1$. Then define:

$$d(\omega, \Sigma_P) = \inf \{d(\omega, \mu) : \mu \in \Sigma_P\}$$

Lemma 3.4. *Assume the conditions of definition* 3.3, *and let* $J = \{1, 2\}$. *Then*

$$6[d(\omega, \Sigma_P)] \geqq \sup \{|\omega(E)\omega(F) - \omega(EF)| : E \in \beta_1(\mathscr{A}_1), \\ F \in \beta_2(\mathscr{A}_2) \text{ and } ||E||, ||F|| \leqq 1\}$$

Proof. For all $E \in \beta_1(\mathscr{A}_1)$ and $F \in \beta_2(\mathscr{A}_2)$ with $||E||, ||F|| \leqq 1$, and for all $\mu \in \Sigma_P$:

$$|\omega(EF) - \omega(E)\omega(F)| \leqq |\omega(EF) - \mu(EF)| + |\mu(E)\mu(F) - \omega(E)\omega(F)| \\ \leqq ||\omega - \mu|| + 2||\omega - \mu|| = 3||\omega - \mu|| \leqq 6d(\omega, \mu).$$

The last inequality is by corollary 1.5.

Remark. Clearly it would be easy to rephrase lemma 3.4 for arbitrary indexing sets J.

Lemma 3.5. *Let* $(\mathscr{B}, (\beta_i))$ *be a product for* $(\mathscr{A}_i)_{i \in J}$ *and let* ω *be a normal state of* $\mathscr{B}$. *Then* $d(\omega, \Sigma_P) = 0$ *implies* $\omega \in \Sigma_P$.

PROOF. $d(\omega, \Sigma_P) = 0$ means that ω is in the norm closure of Σ_P by corollary 1.5. Hence we need to show that Σ_P is norm closed.

For each family $(A_i)_{i\in K}$ with each $A_i \in \mathscr{A}_i$ and $A_i = 1$ for a.a. $i \in K$,

$$\nu \to \nu(\prod \alpha_i(A_i)) - \prod \nu(\alpha_i(A_i))$$

is evidently a complex-valued function on the set Σ of normal states of $\mathscr{B}$, continuous in the norm topology on Σ. Since Σ_P is that subset of Σ consisting of those ν on which all such functions vanish (i.e. those ν for which (1.4) holds), Σ_P is norm closed.

THEOREM 3.6. *Let U be $\otimes_{k\in K}[\otimes_{i\in I(k)}(U_i, \mu_i), \nu_k]$ with the assumptions and notations of the first paragraph of § 3. Let Σ_k denote Σ_P for the product $(\mathscr{B}_k, (\beta_i))$ of $(\mathscr{A}_i)_{i\in I(k)}$ (i.e. for $\otimes_{i\in I(k)}(\mathscr{A}_i, \mu_i)$ with the natural injections). Then:*

(I). *U is a non-direct-product representation of $\prod G_i$ whenever the following condition fails:*

$$\lim_{k\in K} d(\nu_k, \Sigma_k) = 0. \tag{3.4}$$

(II). *Suppose that each $\mathscr{B}_k$ is semi-finite. Then U is non-direct-product representation whenever*

$$\sum_{k\in K} [d(\nu_k, \Sigma_k)]^2 = \infty. \tag{3.5}$$

PROOF. We will show that the existence of a non-zero product state ω for U implies (3.4) and, whenever each $\mathscr{B}_k$ is semi-finite, contradicts (3.5).

Suppose then that ω is a non-zero product state for U. Then (lemma 3.2) $\omega(1) = 1$ and ω is a product state of

$$(\mathscr{A}, (\alpha_i)) : \omega = \bigotimes_{i\in I} \omega_i$$

where $\omega_i = \omega \circ \alpha_i$. Evidently ω is a product state of $(\mathscr{A}, (\mathscr{B}_k))$. In fact, $\omega = \otimes_{k\in K}\omega^k$ where $\omega^k = \otimes_{i\in I(k)}\omega_i$ for the product $(\mathscr{B}_k, (\alpha_i)_{i\in I(k)})$, and certainly, then, $\omega^k \in \Sigma_k$. Now $(\mathscr{A}, (\mathscr{B}_k))$ is the tensor product $\otimes_{k\in K}(\mathscr{B}_k, \nu_k)$ with the natural injections. Hence, by lemma 1.3,

$$\lim_{k\in K} d(\nu_k, \omega^k) = 0. \tag{3.6}$$

If each $\mathscr{B}_k$ is semi-finite, lemma 1.1.(II) demonstrates the stronger result:

$$\sum_{k\in K} [d(\nu_k, \omega^k)]^2 < \infty. \tag{3.7}$$

Recalling that $\omega^k \in \Sigma_k$, we see that (3.6) implies (3.4) and that (3.7) contradicts (3.5). That finishes the proof.

To identify the tensor-product representations U, the following lemma is useful.

LEMMA 3.7. *Suppose that* $\mathscr{A}$ *is* $\otimes_{k\in K}[\otimes_{i\in I(k)}(\mathscr{A}_i, \mu_i), \nu_k]$ and *the* α_i *are as before. Suppose that a non-zero product state* $\omega = \otimes_{i\in I}\omega_i$ *exists for* $(\mathscr{A}, (\alpha_i))$. *Then* $(\mathscr{A}, (\alpha_i))$ *is product isomorphic to* $\otimes_{i\in I}(\mathscr{A}_i, \omega_i)$ *with the natural injections.*

PROOF. We will need the following result [3, corollary 4.4 and lemma 5.1]:

(*) Suppose that $(\mathscr{A}, (\alpha_i))$ is a (μ_i)-tensor product for $(\mathscr{A}_i)$ and that $\nu = \otimes \nu_i$ is a product state of $(\mathscr{A}, (\alpha_i))$. Then $(\mathscr{A}, (\alpha_i))$ is a (ν_i)-tensor product for $(\mathscr{A}_i)$.

For each $k \in K$, $\omega^k = \gamma_k \circ \omega$ is a product state $\otimes_{i\in I(k)}\omega_i$ for $(\mathscr{B}_k, (\beta_i)_{i\in I(k)})$. Now $\mathscr{B}_k = \otimes_{i\in I(k)}(\mathscr{A}_i, \mu_i)$ and the β_i are the natural injections; therefore, by (*), $\mathscr{B}_k$ may be identified with $\otimes_{i\in I(k)}(\mathscr{A}_i, \omega_i)$ and the β_i with the natural injections.

Similarly, since $\omega = \otimes_{k\in K}\omega^k$ for $(\mathscr{A}, (\gamma_k))$, $\mathscr{A}$ may be identified with

$$\bigotimes_{k\in K}[\bigotimes_{i\in I}(\mathscr{A}_i, \omega_i), \omega^k]$$

and the γ_k with the natural injections. By a standard associativity argument (see [3] or [7]), the product $(\mathscr{A}, (\alpha_i))$ or $(\mathscr{A}, (\gamma_k \circ \beta_i))$ is product isomorphic to $\otimes_{i\in I}(\mathscr{A}_i, \omega_i)$ with the natural injections.

THEOREM 3.8. *With* U *as in theorem* 3.6, *suppose that*

$$\sum_{k\in K}[d(\nu_k, \Sigma_k)]^2 < \infty. \tag{3.8}$$

Then U *is a tensor-product representation.*

PROOF. Suppose that (3.8) holds. Then (using lemma 3.5 for uncountable K), we can find states $\omega^k \in \Sigma_k$ such that

$$\sum_{k\in K}[d(\nu_k, \omega^k)]^2 < \infty.$$

This implies (lemma 1.1. (I)) that $\omega = \otimes_{k\in K}\omega_k$ exists as a product state of $\mathscr{A} = \otimes_{k\in K}(\mathscr{B}_k, \nu_k)$. For each $k \in K$, we have $\omega^k \in \Sigma_k$ so that $\omega^k = \otimes_{i\in I(k)}\omega_i$ on $\mathscr{B}_k = \otimes_{i\in I(k)}(\mathscr{A}_i, \mu_i)$. It is now evident, that with respect to the product $(\mathscr{A}, (\alpha_i))$ for $(\mathscr{A}_i)_{i\in I}$, ω has the form $\otimes_{i\in I}\omega_i$. Now lemma 3.7 shows that $(\mathscr{A}, (\alpha_i))$ is product isomorphic with $\otimes_{i\in I}(\mathscr{A}_i, \omega_i)$, and lemma 2.1 shows that U is equivalent to $\otimes_{i\in I}(U_i, \omega_i)$.

Lemma 3.9. *Suppose that $\mathscr{B} = \mathscr{A}_1 \otimes \mathscr{A}_2$ and that, for $i = 1$ or 2. μ_i and ν_i are normal states $\mathscr{A}_i$ with orthogonal supports and with $\mu_i(1) = \nu_i(1) = 1$. Let $0 \leqq \alpha \leqq 1$ and let ω be the normal state $\alpha(\mu_1 \otimes \mu_2)+(1-\alpha)(\nu_1 \otimes \nu_2)$ of $\mathscr{B}$. Then:*

$$d(\omega, \Sigma_P) \geqq (\alpha-\alpha^2)/6. \tag{3.9}$$

Proof. Let E_i be the support of μ_i. Then $\nu_i(E_i) = 0$. We have:

$$\omega\big((\alpha_1(E_1))(\alpha_2(1-E_2)\big) = 0$$
$$\omega(\alpha_1(E_1)) = \alpha \quad \text{and} \quad \omega(\alpha_2(1-E_2)) = 1-\alpha$$

Therefore (3.9) follows from lemma 3.4.

Corollary 3.10. *Suppose that $\mathscr{A}_i$ fails to reduce to the scalars for an infinite number of $i \in I$. Then certain double tensor-products of $(U)_{i\in I}$ are non-direct-product representations of $G = \prod_{i\in I} G_i$.*

Proof. Use lemma 3.9 and theorem 3.6. (I).

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