COMPOSITIO MATHEMATICA

H. I. BROWN

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Compositio Mathematica, tome 22, nº 1 (1970), p. 23-28
<http://www.numdam.org/item?id=CM_1970__22_1_23_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 1, 1970, pag. 23-28
Wolters-Noordhoff Publishing
Printed in the Netherlands

A note on type P methods of summation

by

H. I. Brown [1]

Let A denote a conservative matrix summability method and let c denote the set of convergent sequences. (For the notation and terminology used in this note see, for example, [8].) The summability field of A is denoted by c_A. If c is dense in c_A then A is called perfect, while if the zero sequence is the only member of l (the set of absolutely convergent series) which acts as a left annihilator of A then A is said to be of type M. If A is reversible (i.e., one-to-one and onto c) then the type M property is necessary but not sufficient in order that A be perfect. On the other hand, the two properties are equivalent for *coregular* reversible matrices [1, 3, 4, 5]. In an attempt to establish an analogous result for *conull* reversible matrices, Jürimaë [2] introduced the concept of a type P matrix (defined below). However, he incorrectly assumed that for reversible matrices type P is equivalent to perfectness. (The identity matrix serves as a suitable counterexample.) In this note we first point out that there is a kind of perfectness that agrees with the type P property for reversible matrices; namely, that the null sequences be dense in c_A. It then follows that for *conull* reversible matrices, type P is equivalent to perfectness. We then consider the class of multiplicative matrices. (A conservative matrix is multiplicative if and only if each of its columns is a null sequence.) It is known that a member A of this class is of type M if and only if the closure of the null sequences is a maximal linear subspace of c_A [6; p. 342]. We show that if a reversible member of this class is of type P then the image of the null sequences is dense in the image of the null summability field (defined below), and the adjoint and transpose matrix have the same kernels (in the sense given below).

Let c'_A represent the dual space of c_A. If $f \in c'_A$ then there exist sequences t and β in l and a constant α such that for each $x \in c_A$, [2]

[1] Research supported in part by NSF Grant GP-8199.

[2] Unless otherwise specified the indices of summation run from 0 to ∞. Moreover, all quantities with negative indices are taken to be equal to zero.

$$f(x) = \alpha \cdot \lim_A x + \sum t_n (Ax)_n + \sum \beta_n x_n,$$

where $\lim_A x = \lim (Ax)_n$. If A is reversible then $\beta_n = 0$ for each n. (See, for example, [7; p. 230].) A representation of c'_A that is more suitable for our purposes can be obtained by applying Abel's summation by parts formula. This is done in [2] and we summarize it in the following Proposition. V denotes the set of absolutely convergent sequences, that is, $w \in V$ if and only if

$$\sum |w_n - w_{n+1}| < \infty.$$

PROPOSITION 1. *If $f \in c'_A$, then there exist sequences $\beta \in l$ and $w \in V$ such that for all $x \in c_A$,*

$$f(x) = \sum w_n (y_n - y_{n-1}) + \sum \beta_n x_n, \tag{1}$$

where $y_n = \sum a_{nk} x_k$. If A is reversible then $\beta_n = 0$ for each n.

The matrix A is then said to be of type P if the conditions $w \in V$ and $\sum w_n (a_{nk} - a_{n-1,k}) = 0$ for each k imply that $w_n = 0$ for each n. A straightforward computation shows that the identity matrix is not of type P. The following example shows that matrices which are of type P need not be perfect. Define $A = (a_{nk})$ by setting $a_{n0} = a_{n,n+1} = -a_{n+1,n+1} = 1$ for $n = 0, 1, 2, \cdots$, and $a_{nk} = 0$ otherwise. Letting $w \in V$ and considering the equations $\sum w_n (a_{nk} - a_{n-1,k}) = 0$ for $k = 0, 1, 2, \cdots$, we see that $w_0 = 0$ and $w_n = n w_1$ for $n = 1, 2, 3, \cdots$. Since $w \in V$ it follows that $w_n = 0$ for every n and so A is of type P. However, A is not perfect because

$$f(x) = \lim (x_n - x_{n-1})$$

is a non-trivial continuous linear functional on c_A which is zero on c. Hence, c is not dense in c_A.

PROPOSITION 2. *Matrices having the type P property are always of type M.*

PROOF. Let A be of type P and let $t \in l$ with $tA = 0$. Setting $w_n = \sum_{i=n}^{\infty} t_i$ and $s_n = \sum_{i=0}^{n} t_i$ we see that $w \in V$ and

$$\begin{aligned}\sum_{n=0}^{m} w_n (a_{nk} - a_{n-1,k}) &= \sum_{n=0}^{m} (\sum t_i - s_{n-1})(a_{nk} - a_{n-1,k}) \\ &= s_m a_{mk} - \sum_{n=0}^{m-1} s_n (a_{n+1,k} - a_{nk}) + a_{mk} \sum_{i=m+1}^{\infty} t_i \\ &= \sum_{n=0}^{m} t_n a_{nk} + a_{mk} \sum_{i=m+1}^{\infty} t_i.\end{aligned}$$

Taking the limit (as $m \to \infty$) we obtain the equations

$$\sum w_n(a_{nk}-a_{n-1,k}) = 0$$

for each k and so $w_n = 0$ for every n. This means that $t_n = 0$ for every n and so A is of type M.

If E is a subset of c_A then by $\bar{E}$ we shall mean the closure of E in the c_A-topology. Let c_0 denote the set of null sequences and let e^k denote the sequence having a 1 in the k^{th} coordinate and zeros elsewhere.

THEOREM 3. *If A is reversible, then A is of type P if and only if $\bar{c}_0 = c_A$.*

PROOF. By using the representation (1) with $\beta_n = 0$ for every n it is easily seen that when A is of type P then the only member of c'_A which vanishes on c_0 is the zero functional.

Conversely, suppose $\bar{c}_0 = c_A$ and let $w \in V$ with

$$\sum w_n(a_{nk}-a_{n-1,k}) = 0 \text{ for each } k.$$

Then the functional

$$f(x) = \sum_n w_n \sum_k (a_{nk}-a_{n-1,k})x_k$$

defines a member of c'_A which vanishes on c_0 (because $f(e^k) = 0$ for each k) and so it must vanish identically on c_A. Since A is onto c we may choose x^k in c_A so that $Ax^k = e^k$ for $k = 0, 1, 2, \cdots$. Then $f(x^k) = 0$ and so $w_k = w_{k+1}$ for $k = 0, 1, 2, \cdots$, that is, $w = (\gamma)$ a constant sequence. But then for each x in c_A we have

$$0 = \sum_n \gamma \sum_k (a_{nk}-a_{n-1,k})x_k = \gamma \cdot \lim_A x.$$

Since A is reversible this means that $\gamma = 0$; hence, A is of type P.

COROLLARY 4. *If A is a conull reversible matrix, then A is perfect if and only if A is of type P.*

PROOF. This follows from Theorem 3 since for conull matrices, $\bar{c}_0 \supseteqq c$.

We have already noted that the identity matrix is not of type P and so a multiplicative type M matrix need not be of type P. However, a multiplicative type M matrix is almost of type P in the following sense.

PROPOSITION 5. *If A is multiplicative and of type M then the conditions $w \in V$ and $\sum w_n(a_{nk}-a_{n-1,k}) = 0$ for each k imply that w is a constant sequence.*

PROOF. Let A and w satisfy the hypothesis and set $t_n = w_n - w_{n+1}$ and $s_n = \sum_{i=1}^{n} t_i$. Then $t \in l$ and for each $k = 0, 1, 2, \cdots$, we have

$$\begin{aligned}\sum_{n=0}^{m} t_n a_{nk} &= s_m a_{mk} - \sum_{n=0}^{m-1} s_n(a_{n+1,k} - a_{nk}) \\ &= \sum_{n=0}^{m} (s_m - s_{n-1})(a_{nk} - a_{n-1,k}) \\ &= \sum_{n=0}^{m} w_n(a_{nk} - a_{n-1,k}) - w_{m+1} a_{mk}.\end{aligned}$$

Since A is multiplicative, $\lim_m a_{mk} = 0$ for each k; hence, taking the limit (as $m \to \infty$) we see that $tA = 0$. Since A is of type M it follows that $t = 0$. Thus, w must be a constant sequence.

In contrast with this, we remark here that if A is not multiplicative then the only constant sequence w in V such that $\sum w_n(a_{nk} - a_{n-1,k}) = 0$ for every k is the zero sequence. (Indeed, if $w = (\gamma)$, then $\sum \gamma(a_{nk} - a_{n-1,k}) = \gamma \cdot \lim_n a_{nk}$ for each k.)

For most of what follows we shall require only that A preserve null sequences, that is, $A : c_0 \to c_0$. (The class of all such matrices contains, among others, the multiplicative conservative matrices.) Given such a matrix, its null summability field is defined to be the set $c_A^0 = \{x : Ax \in c_0\}$. Then $A : c_A^0 \to c_0$ is a linear continuous operator and so its adjoint $A' : c_0' \to c_A^{0\prime}$ is given by the equation $\langle A't, x\rangle = \langle t, Ax\rangle$ for each t in l and each x in c_A^0 (where we have identified, as usual, c_0' with l). Let A^t represent the transpose matrix of A and, for an operator T, let ker T denote its kernel.

LEMMA 6. *If $A : c_0 \to c_0$, then* $\ker A' \subseteqq l \cap \ker A^t$.

PROOF. If $t \in \ker A'$, then $\langle t, Ax\rangle = \langle A't, x\rangle = 0$ for each $x \in c_A^0$. In particular, this is true for each e^k. Thus,

$$0 = \langle t, Ae^k\rangle = \sum t_n a_{nk} \text{ for } k = 0, 1, 2, \cdots,$$

and so $t \in l \cap \ker A^t$.

THEOREM 7. *If $A : c_0 \to c_0$, then* $\ker A' = l \cap \ker A^t$ *if and only if $A(c_0)$ is dense in $A(c_A^0)$, where the closure is taken in the norm topology of c_0.*

PROOF. To prove the necessity let $f \in c_0'$ with $f = 0$ on $A(c_0)$. Then [7; p. 91], $f(y) = \sum t_n y_n$ for some t in l and all y in c_0. Since $e^k \in c_0$ we see that $0 = f(Ae^k)$ for $k = 0, 1, 2, \cdots$ and so $t \in l \cap \ker A^t = \ker A'$. It follows that $\langle A't, x\rangle = 0$ for each x in c_A^0 and hence that $0 = \langle t, Ax\rangle = f(Ax)$ for each x in c_A^0, which shows that $A(c_0)$ is dense in $A(c_A^0)$.

Conversely, let $t \in l \cap \ker A^t$. Then $\langle t, y\rangle$ defines a continuous linear functional on c_0 which vanishes on $A(c_0)$. (Indeed, if $x \in c_0$ then

$$\langle t, Ax\rangle = \sum_n t_n \sum_k a_{nk}x_k = \sum_k (\sum_n t_n a_{nk})x_k = 0$$

where the interchange of the order of summation is justified by the absolute convergence of the series.) Hence $\langle t, y\rangle$ vanishes on $A(c_A^0)$ which implies that $\langle A't, x\rangle = 0$ for each x in c_A^0. Thus, $t \in \ker A'$. This, combined with Lemma 6, proves the Theorem.

THEOREM 8. *Let $A : c_0 \to c_0$. If A is reversible then $A(c_0)$ is dense in $A(c_A^0)$ if and only if c_0 is dense in c_A^0.*

PROOF. The sufficiency is obvious (without the assumption that A be reversible) since A is continuous.

Conversely, let $f \in c_A^{0\prime}$. Since A is reversible, a development similar to that found in [7; p. 230] shows that we may write $f(x) = \sum_n t_n \sum_k a_{nk}x_k$ for some t in l and all x in c_A^0. Now suppose that f vanishes on c_0 and let x be any member of c_A^0 which is not in c_0. (If $c_A^0 = c_0$ then we are done.) By hypothesis, to each $\varepsilon > 0$ there corresponds a sequence u in c_0 such that

$$\sup_n |\sum_k a_{nk}(u_k - x_k)| < \varepsilon.$$

Since $f(u) = 0$ we may write $f(x) = f(x-u)$ and so

$$\begin{aligned} |f(x)| &= |\sum_n \sum_k t_n a_{nk}(x_k - u_k)| \\ &< \varepsilon \cdot \sum_n |t_n|. \end{aligned}$$

Therefore, $f = 0$ on c_A^0.

THEOREM 9. *Let A be a reversible multiplicative matrix. Then the following statements are equivalent:*

(1) $\bar{c}_0 = c_A^0$.

(2) $\ker A' = l \cap \ker A^t$.

(3) *$A(c_0)$ is dense in $A(c_A^0)$.*

If, in addition, A is of type P then A satisfies each of these conditions.

PROOF. The equivalence of the three statements follows from Theorems 7 and 8. The last sentence is a consequence of Theorem 3.

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(Oblatum 24–II–69)

Department of Mathematics
SUNY at Albany
Albany, New York 12203, USA