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by

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1. Introduction

Let R be the reals and βR the Stone-Čech compactification of R. If $p \in R$, let T^p be the homeomorphism of βR such that $T^p : R \to R$ is translation by p. Let $\pi : R \times \beta R \to \beta R$ be defined by $\pi(p, w) = T^p w$. If $f \in C(\beta R)$, then it is elementary to show that $f \circ \pi \in C(R \times \beta R)$ iff $f|R$ is uniformly continuous. It is even known [3, theorem 2] that separate continuity implies joint continuity.

In this paper we are concerned with the Baire measurability of $\pi : R \times \beta R \to \beta R$. Using known results from semigroup theory, it can be shown that if $f \in C(\beta R)$, then $(p, w) \to f(T^p w)$ is measurable iff $f|R$ is uniformly continuous, so that $\pi : R \times \beta R \to \beta R$ is non-measurable. This result is presumably known, but for completeness we sketch a proof in section 2. In section 3, we construct a function f continuous on βR such that for a large class of T^p-invariant probability Baire measures on βR, the map $p \to f \circ T^p$ from R to $L^1(m)$ is discontinuous (in the L^1-topology), and from this fact we use the theorem of section 2 to conclude that if K is the support in βR of the measure m (i.e., the smallest closed set such that $m(K) = 1$), then the map $\pi : R \times K \to K$ (K is clearly T^p-invariant for each $p \in R$) is non-Baire measurable. We know that the map $\pi : R \times \beta R \to \beta R$ is non-measurable, and one may wonder whether this would not directly imply the non-measurability of $\pi : R \times K \to K$. But this seems unlikely to be easy because it is known [2, 2.9] that the support of an invariant probability measure in βN is an 'extremely' non-dense subset of βN, and the same is likely to be true of the set K as a subset of βR.

2. Measurability implies continuity

THEOREM. *Let $\{T^p : p \in R\}$ be a group of homeomorphisms of a compact T^2 space X onto itself, and assume $C(X)$ is separable in* *the sup-norm* $\|\ \|$. *If the map* $(p, w) \to T^p w$ *is Baire measurable on* $R \times X \to X$, *then it is continuous, and for each* $f \in C(X)$, *the map* $p \to f \circ T^p$ *is strongly continuous, i.e.,* $\lim_{n\to\infty} p_n = p$ *implies* $\lim_{n\to\infty} \|f \circ T^{p_n} - f \circ T^p\| = 0$.

Sketch of Proof. First, to show that if $f \in C(X)$, then the map $p \to f \circ T^p$ from the reals to the Banach space $C(X)$ is a measurable map, we show that it is separably valued and weakly measurable [4, pp. 130—131]. We have assumed separability. If m is a Baire measure on X, measurability of $(p, w) \to f(T^p w)$ implies measurability of $p \to \int_X f \circ T^p dm$ [1, p. 148], and this is just weak measurability.

Now since each T^p is a homeomorphism, $\|f \circ T^p\| = \|f\|$ for all p, so the map $p \to \|f \circ T^p\|$ is Lebesgue integrable on any interval $[a, b]$, and the Bochner integral $\int_a^b f \circ T^p dp$ is defined [4, p. 133]. That the map $p \to f \circ T^p$ is strongly continuous follows as in [4, pp. 233—234]. Since this is true for all $f \in C(X)$, an elementary argument gives continuity of $(p, w) \to T^p w$.

Corollary. *If* $f \in C(\beta R)$, *the map* $(p, w) \to f(T^p w)$ *is measurable iff* $f|R$ *is uniformly continuous, and the map* $\pi : R \times \beta R \to \beta R$ *defined above is non-measurable.*

Proof. Measurability of $(p, w) \to f(T^p w)$ implies the condition that if $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} \|f \circ T^{p_n} - f \circ T^p\| = 0$, and this is just uniform continuity of fR. Since not every $g \in C(\beta R)$ has this property, the theorem gives the non-measurability of $(p, w) \to T^p w$.

3. Construction of the non-measurable function *f*

(i) We begin by defining $A \subset R$ as follows: let

$$B_n = \bigcup_{k=0}^{2^n-1} [2n + 5k2^{-(n+2)}, 2n + (5k+1)2^{-(n+2)}]$$

and $A = \bigcup_{n=0}^{\infty} B_n$. We now enumerate some properties of A.

(ii) For all n, $B_n \subset [2n, 2n+3 \cdot 2^{-1}]$. Since

$$\max B_n = 2n + (5(2^n-1)+1)2^{-(n+2)}$$

we must show that

$$3 \cdot 2^{-1} \geqq (5(2^n-1)+1)2^{-(n+2)} = 5 \cdot 4^{-1} - 2^{-n}.$$

But $3 \cdot 2^{-1} < 5 \cdot 4^{-1} - 2^{-n} < 5 \cdot 4^{-1}$ implies $12 < 10$.

(iii) If $m < n$, then $B_n \cap (B_n+2^{-m})$ is finite. For

$$B_n+2^{-m} = \bigcup_{K=0}^{2^n-1} [2n+(5k+2^{n-m+2})2^{-(n+2)}, \\ 2n+(5k+2^{n-m+2}+1)2^{-(n+2)}].$$

Now

$$[2n+5j2^{-(n+2)}, 2n+(5j+1)2^{-(n+2)}]$$

and

$$[2n+(5k+2^{n-m+2})2^{-(n+2)}, 2n+(5k+2^{n-m+2}+1)2^{-(n+2)}]$$

are intervals of length $2^{-(n+2)}$ whose end points are integral multiples of $2^{-(n+2)}$. They can't coincide, because 5 divides $5j$ and 5 doesn't divide $5k+2^{n-m+2}$. Hence they meet in at most one point, and B_n meets B_n+2^{-m} in a finite set.

(iv) If $m \geqq 2$, then for all n, $(B_n+2^{-m}) \cap B_{n+1}$ is null. For since $2^{-m} \leqq 4^{-1}$ and $B_n \subset [2n, 2n+3 \cdot 2^{-1}]$, we have

$$(Bn+2^{-m}) \cap B_{n+1} \subset [2n, 2n+7 \cdot 4^{-1}] \\ \cap [2(n+1), 2(n+1)+3 \cdot 2^{-1}] = \emptyset.$$

(v) $\lim_{T\to\infty} T^{-1} \int_0^T \chi_A(p)dp = \frac{1}{8}$.

For each B_n is the union of 2^n disjoint intervals of length $2^{-(n+2)}$, measure $(B_n) = 2^n \cdot 2^{-(n+2)} = 4^{-1}$. Hence if n is the largest number such that $2n+2 \leqq T < 2n+4$, we have

$$T^{-1} \int_0^T \chi_A(p)dp = T^{-1} \sum_{j=0}^{n} \text{measure } (B_i) \\ +T^{-1} \int_{2n+2}^T \chi_A(p)dp \\ = T^{-1}(n+1)4^{-1}+T^{-1} \int_{2n}^T \chi_A(p)dp,$$

and since n is the largest integer such that $2n+2 \leqq T$, we have $T^{-1}(n+1)4^{-1} = T^{-1}(2n+2)8^{-1}$ goes to 8^{-1} as $n \to \infty$. The remainder obviously goes to zero.

We now define $f \in C(R)$ to be any function such that: (a) support $(f) \subset A$, (b) $0 \leqq f \leqq 1$, (c) if $A_n = \{x \in B_n : f(x) = 1\}$, then measure $(A_n) \geqq 8^{-1}$ (In (v) above we showed that measure $(B_n) = 4^{-1}$.). It is easy to see that

$$(*) \qquad \liminf_{T\to\infty} T^{-1} \int_0^T f^2(p)dp \geqq \tfrac{1}{16} > 0.$$

Let m be any Baire measure on βR which is the weak $-*$ limit of the functionals $g \to T^{-1} \int_0^T g(p)dp$. (This is the class referred to in the introduction.) Then m is a Baire probability measure invariant under $\{T^p, p \in R\}$. We now wish to show that the map $p \to f \circ T^p \in L^1(m)$ is discontinuous (with respect to the $L^1(m)$ norm $\|\ \|_1$).

For $n = 2, 3, \cdots$ we define $p^n = -2^{-n}$.

(v) $\int f^2 dm \geqq \frac{1}{16}$, and for $n = 2, 3, \cdots$ we have

$$\int (f \circ T^{p_n}) f\, dm = 0.$$

Hence

$$\tfrac{1}{16} \leqq \left| \int f^2 dm - \int f(f \circ T^{p_n}) dm \right| \leqq \|f\| \int |f - f \circ T^{p_n}|\, dm.$$

Proof. The first assertion follows from (*). For the second, if $m \geqq 2$ is given, then $n > m$ implies $f|[2n, 2n+2]$ is supported by B_n, while $f \circ T^{p_m}|[2n, 2n+2]$ is supported by $B_n + 2^{-m}$. Since by (iii) $B_n \cap (B_n + 2^{-m})$ is finite, $n > m$ implies $(f \circ T^{p_m})f|[n, \infty)$ is a Lebesgue-null function, so

$$\lim_{T \to \infty} T^{-1} \int_0^T (f \circ T^{p_m})(p) f(p)\, dp = 0.$$

This proves (v).

Now let K be the support of m. K is a T^p-invariant set because m is a T^p-invariant measure. We assume that the map $\pi : R \times K \to K$ is measurable, and derive a contradiction to (v).

We'll show that the hypotheses of the theorem of section 2 are satisfied. K is a compact T^2 space, and $\{T^p : p \in R\}$ a group of mappings of K. Since K is compact in βR, every element in $C(K)$ may be extended to an element of $C(\beta R)$, so that the restriction map from $C(\beta R)$ to $C(K)$ is continuous. Since $C(\beta R)$ is separable, so must $C(K)$ be separable. By the theorem, since π is assumed to be Baire measurable, $\lim_{p \to 0} \|f \circ T^p - f\| = 0$, where $\|\ \|$ is the sup-norm on $C(K)$. Letting $p_m = -2^{-m}$ as in (v), and recalling that the f we have defined is non-negative, we get (using v)

$$\begin{aligned}
\tfrac{1}{16} &\leqq \int_K f^2 dm - \int_K f(f \circ T^{p_n}) dm \\
&= \left| \int_K f(f - f \circ T^{p_n}) dm \right| \\
&\leqq \|f\|\, \|f - f \circ T^{p_n}\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

a contradiction.

4. Final remark

The result obtained is for a rather restricted class of invariant means on βR, namely those which can be computed as limits of averages over R itself. If $w \in \beta R - R$, and

$$K = \text{closure } \{T^p w : p \in R\},$$

then it may be that there are invariant means supported by K which are computable as averages along the orbit $\{T^p w : p \in R\}$ of w, and a construction analagous to ours carried out. Of course, one would have to know something about $\{T^p w : p \in R\}$ as a subset of βR.

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