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# Asymptotic expansions in renewal theory

by

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## 1. Introduction and statement of the results

Let $\mu$ be a probability measure defined on the Borel sets of $(-\infty, \infty)$ with $\int |x|\mu\{dx\} < \infty$ and $\mu_1 = \int x\mu\{dx\} > 0$. Then the renewal measure $\nu$ belonging to $\mu$, defined by $\nu = \sum_0^\infty \mu^{n*}$, assigns finite measure to bounded Borel sets.

In this paper our aim is to get approximations of $\nu\{x+E\}$, $E$ some Borel set, for $x \to -\infty$ if $\mu\{(-\infty, x)\}$ decreases exponentially, and for $x \to \infty$ if $\mu\{(x, \infty)\}$ has this property. Woık on this has been done in Stone [1] and [2]. Results are obtained for $\mu$ lattice and for the case that some $\mu^{m*}$ is non-singular (we call $\mu$ lattice with span $d$ if $\mu$ is concentrated on $\{nd : -\infty < n < \infty\}$ but not on $\{nd' : -\infty < n < \infty\}$ for any $d' > d$, and we call $\mu^{m*}$ non-singular if it contains an absolutely continuous component).

Let $g(s)$ be the moment generating function of $\mu$, defined by $g(s) = \int e^{sx}\mu\{dx\}$, the domain being all complex numbers for which the integral exists absolutely. As far as defined let $A(s_0)$ denote the residue of $1/(1-g(s))$ at $s = s_0$.

THEOREM 1. *Let $\mu$ be lattice with span* 1.

a) *If $g(s)$ exists for some $s$ with* $\operatorname{Re} s = -R < 0$, *then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = -r$, the set*

$$S = \{s_0 : g(s_0) = 1, -r < \operatorname{Re} s_0 < 0, -\pi < \operatorname{Im} s_0 \leqq \pi\}$$

*is finite, $A(s_0)$ exists for $s_0 \in S$ and for integer $k \to -\infty$*

$$\nu\{k\} = \sum_{s_0 \in S} A(s_0)e^{-s_0 k}+o(e^{rk}) \tag{1.1}$$

$$\nu\{(-\infty, k]\} = \sum_{s_0 \in S} (1-e^{s_0})^{-1}A(s_0)e^{-s_0 k}+o(e^{rk}). \tag{1.2}$$

b) *If $g(s)$ exists for some $s$ with* $\operatorname{Re} s = R > 0$, *then for any $r \in (0, R]$ with $g(s) \neq 1$ on* $\operatorname{Re} s = r$, *the set*

$$S' = \{s_0 : g(s_0) = 1, 0 < \operatorname{Re} s_0 < r, -\pi < \operatorname{Im} s_0 \leqq \pi\}$$

*is finite, $A(s_0)$ exists for $s_0 \in S'$ and for integer $k \to \infty$*

$$\nu\{k\} = \mu_1^{-1} - \sum_{s_0 \in S} A(s_0)e^{-s_0 k} + o(e^{-rk}). \tag{1.3}$$

*Moreover, if $\mu_2 = \int x^2 \mu\{dx\} < \infty$ then*

$$\begin{aligned}\nu\{(-\infty, k)\} \\ = k/\mu_1 + \tfrac{1}{2}(\mu_2/\mu_1)^2 + \sum_{s_0 \in S'} (1-e^{-s_0})^{-1} A(s_0)e^{-s_0 k} + o(e^{-rk}).\end{aligned} \tag{1.4}$$

Under mild conditions $S$ is not empty and contains even one real point which provides the leading term. This does not hold for the set $S'$.

THEOREM 2. *Let $\mu$ be lattice with span 1, $\mu\{(-\infty, 0)\} > 0$ and let $\boldsymbol{I}$ be the interior of the interval $I$ of real points $s < 0$ for which $g(s)$ exists. Suppose $I$ is not empty.*

a) *If $\boldsymbol{I} = I$ or if there exists some $s$ with $g(s) = 1$ and $\operatorname{Re} s \in \boldsymbol{I}$, or even $\operatorname{Re} s \in I$ and $\operatorname{Im} s \neq 2\pi k$, $k = 0, \pm 1, \cdots$, then there exists exactly one real $s_0 \in \boldsymbol{I}$ with $g(s_0) = 1$. Moreover, $g'(s_0) < 0$ and for some $r > -s_0$*

$$\nu\{k\} = -e^{-s_0 k}/g'(s_0) + o(e^{rk}), \qquad k \to -\infty \tag{1.5}$$

$$\nu\{(-\infty, k]\} = -e^{-s_0 k}/\{g'(s_0)(1-e^{s_0}) + o(e^{rk}), \quad k \to -\infty. \tag{1.6}$$

b) *If $\boldsymbol{I} \neq I$ and there does not exist such an $s_0 \in \boldsymbol{I}$ then for any $-r \in \boldsymbol{I}$*

$$\nu\{k\} = o(e^{rk}), \quad k \to -\infty$$
$$\nu\{(-\infty, k]\} = o(e^{rk}), \quad k \to -\infty.$$

*Moreover, if even there does not exist such an $s_0 \in I$ then these order relations hold for $r = R$, where $-R$ is the (finite) left boundary of $I$.*

The corresponding theorems for $\mu$ non-lattice are:

THEOREM 3. *Let $\mu^{m*}$ be non-singular.*

a) *If $g(s)$ exists for some $s$ with $\operatorname{Re} s = -R < 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = -r$, for which the singular part $\zeta$ of $\mu^{m*}$ satisfies*

$$\int_{-\infty}^{0} e^{-rx} \zeta\{dx\} + \int_{0}^{\infty} (1+x)\zeta\{dx\} < 1, \tag{1.7}$$

*the set*

$$S = \{s_0 : g(s_0) = 1, -r < \operatorname{Re} s_0 < 0\}$$

*is finite, $A(s_0)$ exists for $s_0 \in S$ and for $x \to -\infty$*

$$(1.8)\qquad \nu\{x+E\} = \sum_{s_0\in S} A(s_0)e^{-s_0 x}\int_E e^{-s_0 t}dt+o(e^{rx})$$

*for every Borel set E bounded from above. In particular, for* $x \to -\infty$

$$(1.9)\qquad \nu\{(-\infty, x)\} = -\sum_{s_0\in S} s_0^{-1}A(s_0)e^{-s_0 x}+o(e^{rx}).$$

b) *If* $g(s)$ *exists for some* $s$ *with* $\operatorname{Re} s = R > 0$, *then for any* $r \in (0, R]$ *with* $g(s) \neq 1$ *on* $\operatorname{Re} s = r$, *for which the singular part* $\zeta$ *of* $\mu^{m*}$ *satisfies*

$$(1.10)\qquad \int_{-\infty}^{\infty} (1-x)\,\zeta\{dx\}+\int_0^{\infty} e^{rx}\zeta\{dx\} < 1,$$

*the set*

$$S' = \{s_0 : g(s_0) = 1,\ 0 < \operatorname{Re} s_0 < r\}$$

*is finite,* $A(s_0)$ *exists for* $s_0 \in S'$ *and for* $x \to \infty$

$$(1.11)\qquad \nu\{x+E\} = |E|/\mu_1-\sum_{s_0\in S'} A(s_0)e^{-s_0 x}\int_E e^{-s_0 t}dt+o(e^{-rx}),$$

*for every Borel set E bounded from below of finite length* $|E|$. *Moreover, if* $\mu_2 = \int x^2\mu\{dx\} < \infty$ *then*

$$(1.12)\qquad \nu\{(-\infty, x)\} = x/\mu_1+\tfrac{1}{2}(\mu_2/\mu_1)^2+\sum_{s_0\in S'} s_0^{-1}A(s_0)e^{-s_0 x}+o(e^{-rx}).$$

THEOREM 4. *Let* $\mu^{m*}$ *be non-singular,* $\mu\{(-\infty, 0)\} > 0$, *let the singular part of* $\mu^{m*}$ *be restricted to* $(-\infty, 0]$, *let* $\boldsymbol{I}$ *be the interior of the interval* $I$ *of real points* $s < 0$ *for which* $g(s)$ *exists and let E be a Borel set bounded from above. Suppose I is not empty.*

a) *If* $\boldsymbol{I} = I$ *or if there exists some* $s$ *with* $g(s) = 1$ *and* $\operatorname{Re} s \in \boldsymbol{I}$, *or even* $\operatorname{Re} s \in I$ *and* $\operatorname{Im} s \neq 0$, *then there exists exactly one real* $s_0 \in I$ *with* $g(s_0) = 1$. *Moreover,* $g'(s_0) < 0$ *and for some* $r > -s_0$

$$(1.13)\qquad \nu\{x+E\} = -e^{-s_0 x}\int_E e^{-s_0 t}dt/g'(s_0)+o(e^{rx}),\quad x\to-\infty.$$

*In particular*

$$(1.14)\qquad \nu\{(-\infty, x)\} = e^{-s_0 x}/e_0 g'(s_0)+o(e^{rx}),\quad x\to-\infty$$

b) *If* $\boldsymbol{I} \neq I$ *and there does not exist such an* $s_0 \in \boldsymbol{I}$ *then for any* $-r \in \boldsymbol{I}$

$$\nu\{x+E\} = o(e^{rx}),\quad x\to-\infty$$
$$\nu\{(-\infty, x)\} = o(e^{rx}),\quad x\to-\infty.$$

*Moreover, if even there does not exist such an* $s_0 \in I$ *then these order relations hold for* $r = R$, *where* $-R$ *is the (finite) left boundary of* $I$.

## 2. Proof of the theorems

PROOF OF THEOREM 1a). $g(s)$ is analytic for $\operatorname{Re} s \in (-R, 0)$, continuous for $\operatorname{Re} s \in [-R, 0]$, $g(i\theta) \neq 1$ for $|\theta| \in (0, 2\pi)$ and

(2.1) $$g(s) = 1+\mu_1 s+o(|s|), \text{ for } |s| \to 0 \text{ and } \operatorname{Re} s \leqq 0.$$

Therefore, for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = -r$ and $\varepsilon > 0$ sufficiently small the function $1/(1-g(s))$ is continuous on $\Gamma$, and analytic within $\Gamma$ with the exception of a finite number of poles. Here $\Gamma$ is the contour in the complex $s$-plane shown in fig. 1. If for one or more $s_0$ with $\operatorname{Re} s_0 \in (-r, 0)$ it occurs that

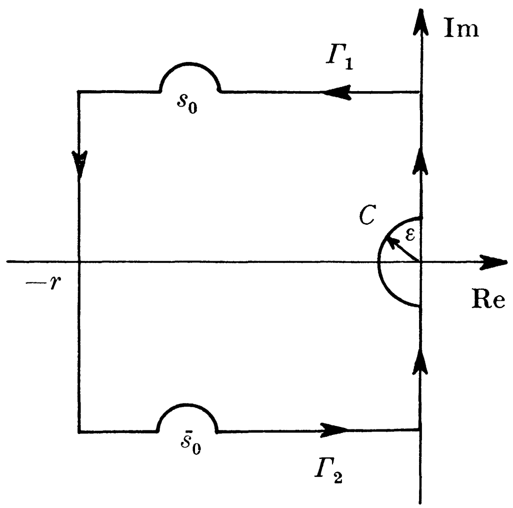


Figure 1

$g(s_0) = 1$ with $\operatorname{Im} s_0 = \pi$ and also $g(\bar{s}_0) = 1$ with $\operatorname{Im} \bar{s}_0 = -\pi$ then the parts $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ are slightly deformed as indicated.

Setting
$$\psi(s) = \{1-g(s)\}^{-1}+\{\mu_1(1-e^s)\}^{-1}$$
we get with the Cauchy residue theorem

(2.2) $$\frac{1}{2\pi i}\int_\Gamma e^{-sk}\Psi(s)ds = \sum_{s_0 \in S} A(s_0)e^{-s_0 k}.$$

According to Stone [3], (20), for $k < 0$ we have

(2.3)
$$\nu\{k\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\{e^{-ik\theta}\Psi(i\theta)\}d\theta = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\substack{\varepsilon \leqq |\operatorname{Im} s| \leqq \pi \\ \operatorname{Re} s=0}} e^{-sk}\Psi(s)ds.$$

With the Riemann-Lebesgue lemma

$$(2.4)\qquad \frac{1}{2\pi i}\int_{-r-i\pi}^{-r+i\pi} e^{-sk}\Psi(s)ds = \frac{1}{2\pi}e^{rk}\int_{-\pi}^{\pi} e^{-i\theta k}\Psi(i\theta-r)d\theta = o(e^{rk}),\quad k\to-\infty.$$

Since $g(s+2\pi i) = g(s)$, the contributions of $\Gamma_1$ and $\Gamma_2$ to the integral in (2.2) cancel out. With (2.1) we see that the contribution of $C$ to the integral in (2.2) tends to zero for $\varepsilon\to 0$. So (1.1) follows from (2.2)—(2.4) and (1.2) follows from (1.1).

b) The proof of (1.3) is similar to that of (1.1). Use Stone [1], (20), for $k \geqq 0$. With (1.3)

$$\nu\{k, N) = \frac{N-k}{\mu_1} + \sum_{s_0\in S'}(1-e^{-s_0})^{-1}A(s_0)e^{-s_0 k}+o(e^{-rk})+o(e^{-rN}),\quad k\to\infty,\ N\to\infty$$

and, as is well-known,

$$\lim_{N\to\infty}\left[\nu\{(-\infty, N)\}-\frac{N}{\mu_1}\right] = \tfrac{1}{2}(\mu_2/\mu_1)^2$$

we get (1.4).

Lemma A. *Let $I$ and $\boldsymbol{I}$ be defined as in theorem* 2. *Suppose $I$ is not empty and $\mu\{(-\infty, 0)\} > 0$. If $\boldsymbol{I} = I$ or if $g(s_1) = 1$ for some $s_1$ with* $\operatorname{Re} s_1 \in I$ *then there exists exactly one real $s_0\in I$ with $g(s_0) = 1$. We have $g'(s_0) < 0$.*

Proof. Let

$$(2.5)\qquad \begin{aligned} g_1(s) &= \int_{[0,\infty)} (e^{sx}-1)\mu\{-dx\}, & -s\in I\\ g_2(s) &= \int_{(0,\infty)} (1-e^{-sx})\mu\{dx\}, & -s\in I. \end{aligned}$$

Since $g_1(0) = g_2(0) = 0$, $0 < g_1'(0^+) < g_2'(0^+)$ and $g_1$ is convex and $g_2$ concave, there is at most one $s_0$ with

$$0 = g_1(-s_0)-g_2(-s_0) = g(s_0)-1,$$

and then $g'(s_0) < 0$. If $g(s_1) = 1$ with $\operatorname{Re} s_1\in I$ then $g(\operatorname{Re} s_1) \geqq 1$. But $g'(0^-) > 0$ and so there exists $s_0\in I$ with $g(s_0) = 1$. Finally, if $\boldsymbol{I} = I$ i.e. $I$ is open to the left, then $g_1(-s)\to\infty$ if $s$ tends to the left boundary of $I$. This also assures that there is $s_0\in I$ with $g(s_0) = 1$.

PROOF OF THEOREM 2.

a) According to lemma A the set $S$ in theorem 1 contains exactly one real $s_0 \in I$ with $g'(s_0) < 0$ and $s_0 \geqq \operatorname{Re} s_1$ for any $s_1 \in S$. But $\mu$ has span 1 and so $s_0 > \operatorname{Re} s_1$ and $s_0 \in I$. With $A(s_0) = -1/g'(s_0)$ and theorem 1 we see that (1.5) holds for some $r > -s_0$. (1.6) follows from (1.5).

b) This part follows immediately from theorem 1.

In the following for any signed measure $\psi$ let $|\psi|$ denote its variation. We call $\psi$ finite if the measure $|\psi|$ is finite.

LEMMA B. *Let $\mu^{m*}$ be non-singular, $\zeta$ the singular part of $\mu^{m*}$, and let $K(x)$ and $L_s(x)$, $s \in T$ with $T$ an arbitrary index-set, be non-negative Borel functions in $x$ so that*

(2.6) *for every fixed finite interval $I$* .

$$\int_{\{x+I\}} K(y-x)\mu\{dy\} \text{ is bounded in } x,\ -\infty < x < \infty$$

$$\lim_{\varepsilon\to 0}\int_{\{x+I\}} |K(y+\varepsilon)-K(y)|(\mu^{m*}-\zeta)\{dy\} = 0,\ -\infty > x > \infty$$

(2.7) $\int K(x)\mu\{dx\} < \infty$

(2.8) $L_s(x) \leqq K(x), \qquad -\infty < x < \infty, \quad s \in T$

(2.9) $L_s(x+y) \leqq L_s(x)L_s(y), \quad -\infty < x,\ y < \infty,\ s \in T$

(2.10) $\sup_{s\in T} \int L_s(x)\zeta\{dx\} < 1.$

*Then for any $\varepsilon > 0$ there exist an integer $n_0 \geqq 1$, a measure $\varphi$ with infinitely often differentiable density with compact support, and a signed measure $\varphi'$ such that*

(2.11) $\mu^{n_0*} = \varphi+\varphi'$,

(2.12) $|\varphi'|\{(-\infty,\ \infty)\} < \varepsilon$,

(2.13) $1-\varepsilon \leqq \varphi\{(-\infty,\ \infty)\} \leqq 1$,

(2.14) $\sup_{s\in T} \int L_s(x)\varphi\{dx\} < \infty$,

(2.15) $\sup_{s\in T} \int L_s(x)|\varphi'|\{dx\} < \varepsilon.$

*Moreover, for $\varepsilon < 1$ the renewal measure*

$$\nu = \sum_0^\infty \mu^{k*}$$

*can be written as*

$$(2.16) \qquad \nu = \nu' + \nu''$$

*with*

$$\nu'' = (\mu^{0*} + \cdots + \mu^{(n_0-1)}) * \sum_0^\infty \varphi'^{k*}$$

$$\nu' = \varphi * \nu'' * \sum_0^\infty \mu^{kn_0*}.$$

*Here* $\nu''$ *is a finite signed measure with*

$$(2.17) \qquad \sup_{s\in T} \int L_s(x)|\nu''|\{dx\} < \infty.$$

PROOF. With $\zeta\{(-\infty, \infty)\} < 1$, (2.9) and (2.10) it follows that for $n$ sufficiently large

$$(2.18) \qquad \zeta^{n*}\{(-\infty, \infty)\} < \frac{\varepsilon}{4},$$

$$(2.19) \qquad \sup_{s\in T} \int L_s(x)\zeta^{n*}\{dx\} < \frac{\varepsilon}{4}\cdot$$

Setting $\xi = \mu^{m*} - \zeta$ and $n_0 = nm$ we get

$$(2.20) \qquad \mu^{n_0*} = \zeta^{n*} + \sum_{k=1}^{n} \binom{n}{k} \cdot \xi^{k*} * \zeta^{(n-k)*}.$$

The second term on the right hand side of (2.20) is absolutely continuous. Let $h(x)$ be its density. With (2.7), (2.8) and (2.9) for $A > 0$

$$\sup_{s\in T} \int_{|x|\geqq A} L_s(x)\mu^{n_0*}\{dx\} \leqq n_0 \left[\int K(x)\mu\{dx\}\right]^{n_0-1} \cdot \int_{|x|\geqq A/n_0} K(x)\mu\{dx\}$$

and so with (2.7) and (2.20) for $A$ sufficiently large

$$(2.21) \qquad \int_{|x|\geqq A} h(x)dx < \frac{\varepsilon}{4}$$

$$(2.22) \qquad \sup_{s\in T} \int_{|x|\geqq A} L_s(x)h(x)dx < \frac{\varepsilon}{4}\cdot$$

Set

$$q_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\tfrac{1}{2}\sigma^{-2}x^2\}, \quad \sigma > 0$$

$$h_\sigma(x) = \int q_\sigma(x-t)h(t)dt$$

and let, for $\delta > 0$, $\theta(x)$ be some infinitely often differentiable function with

$$\begin{aligned} \theta(x) &= 1, & |x| &\leqq A-\delta \\ 0 \leqq \theta(x) &\leqq 1, & A-\delta \leqq |x| &\leqq A \\ \theta(x) &= 0, & |x| &\geqq A. \end{aligned}$$

With (2.6),

$$\int_{|x|\leqq A} K(x)\mu^{n_0 *}\{dx\} < \infty. \tag{2.33}$$

So with (2.20)

$$\int_{|x|\leqq A} K(x)h(x)dx < \infty$$

and therefore for $\delta$ sufficiently small, again with (2.6)

$$\int_{-A}^{A} |h(x)-h_\sigma(x)|dx < \frac{\varepsilon}{4} \tag{2.24}$$

$$\int_{-A}^{A} K(x)|h(x)-h_\sigma(x)|dx < \frac{\varepsilon}{4}. \tag{2.25}$$

Finally, for $\delta$ sufficiently small

$$\int_{A-\delta\leqq|x|\leqq A} (1-\theta(x))h_\sigma(x)dx < \frac{\varepsilon}{4} \tag{2.26}$$

$$\int_{A-\delta\leqq|x|\leqq A} K(x)(1-\theta(x))h_\sigma(x)dx < \frac{\varepsilon}{4}. \tag{2.27}$$

Let $\varphi$ be the measure with density

$$\begin{aligned} p_\varphi(x) &= \theta(x)h_\sigma(x), & |x| &\leqq A \\ &= 0 & |x| &> A \end{aligned}$$

and $\varphi'$ the sum of the measure $\zeta^{n*}$ and the signed measure with density $h(x)-p_\varphi(x)$. Then (2.11) holds, $\varphi$ and $\varphi'$ are finite with $\varphi\{(-\infty, \infty)\} \leqq 1$, and $p_\varphi$ is infinitely often differentiable with compact support $[-A, A]$.

With (2.18), (2.21), (2.24), (2.26)

$$\begin{aligned} |\varphi'|\{(-\infty, \infty)\} &\leqq \zeta^{n*}\{(-\infty, \infty)\} + \int_{|x|\geqq A} h(x)dx \\ &+ \int_{-A}^{A} |h(x)-h_\sigma(x)|dx + \int_{A-\delta\leqq|x|\leqq A} (1-\theta(x))h_\sigma(x)dx < \varepsilon, \end{aligned}$$

which proves (2.12). With (2.11) this gives (2.13). From (2.8), (2.20) and (2.23) we get (2.14). With (2.19), (2.8), (2.22), (2.25), (2.27)

$$\sup_{s\in T}\int L_s(|\varphi'|\{dx\} \leqq \sup_{s\in T}\int L_s(x)\zeta^{n*}\{dx\}+\sup_{s\in T}\int_{|x|\geqq A} L_s(x)h(x)dx$$
$$+\int_{-A}^{A} K(x)\,|h(x)-h_\sigma(x)|\,dx+\int_{A-\delta\leqq|x|\leqq A} K(x)(1-\theta(x))h_\sigma(x)dx < \varepsilon$$

which proves (2.15).

Moreover, if $\varepsilon < 1$ then from (2.12) it follows that $\nu''$ is a finite signed measure. So $\nu-\nu''$ is defined, and with (2.11),

$$\begin{aligned}\nu-\nu'' &= (\mu^{0*}+\cdots+\mu^{(n_0-1)*}) * \sum_{k=1}^{\infty}(\mu^{kn_0*}-\varphi'^{k*})\\ &= (\mu^{0*}+\cdots+\mu^{(n_0-1)*}) * \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\varphi^{*}\mu^{jn_0*} * \varphi'^{(k-1-j)*}\\ &= (\mu^{0*}+\cdots+\mu^{(n_0-1)*}) * \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\varphi * \mu^{jn_0*} * \varphi'^{(k-1-j)*}\\ &= \varphi * \nu'' * \sum_{0}^{\infty}\mu^{kn_0*},\end{aligned}$$

which proves (2.16). Note that the summations with respect to $j$ and $k$ may be interchanged since $\nu''$ is finite.

Finally, (2.17) follows with (2.7), (2.9) and (2.15).

PROOF OF THEOREM 3.

a) Let $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = -r$. We apply Lemma B for $T = [-r, 0]$,

$$\begin{aligned}L_s(x) &= e^{sx}, && x < 0\\ &= 1+x, && x \geqq 0\\ K(x) &= L_{-r}(x), && -\infty < x < \infty\end{aligned}$$

and keep the same notations. Denoting the moment generating function of any finite measure or finite signed measure $\psi$ different from $\mu$ by $\psi_1$, we get that $\int|x||\nu''|\{dx\}$, $\varphi_1(-r)$ and $|\nu''|_1(-r)$ are finite.

In (1.8) and (1.9) we may replace $\nu$ by $\nu'$ since

$$e^{-rx}|\nu''|\{(-\infty, x)\} \leqq \int_{(-\infty,x)} e^{-ry}|\nu''|\{dy\} \to 0 \text{ if } x \to -\infty.$$

Note that $g(s)$ and $g^{n_0}(s)$ are analytic for $\operatorname{Re} s \in (-R, 0)$, continuous for $\operatorname{Re} s \in [-R, 0]$, that $g^{n_0}(i\theta) \neq 1$ for $\theta \neq 0$ and that

(2.28) $\quad g^{n_0}(s) = 1+n_0\mu_1 s+o(|s|)$, for $|s| \to 0$ and $\operatorname{Re} s \leqq 0$.

Since $\varphi_1(s)$ tends to zero if $|\text{Im } s| \to \infty$, uniformly in $\text{Re } s \in [-R, 0]$ and $|\varphi'|_1(-r)$ can be made arbitrary small

$$(2.29)\quad |g^{n_0}(s)| \leqq C < 1,\ \text{Re } s \in [-R, 0], \text{ for } |\text{Im } s| \text{ sufficiently large.}$$

Therefore, for $N$ sufficiently large and $\varepsilon$ sufficiently small the function $1/(1-g^{n_0}(s))$ is continuous on $\Gamma$ and analytic within $\Gamma$ with the exception of a finite number of poles. Here $\Gamma$ is the contour in the proof of theorem with $\pi$ replaced by $N$.

If $\chi = n_0^{-1} \cdot \varphi * \nu''$, then $\chi$ is a finite signed measure with $\chi\{(-\infty, \infty)\} = 1$, and $\chi_1(s)$ is continuous on $\Gamma$ and analytic within $\Gamma$. Setting

$$\Psi(s) = \chi_1(s)[\{1-g^{n_0}(s)\}^{-1}+(n_0\mu_1 s)^{-1}]$$

we get with the Cauchy residue theorem

$$(2.30)\qquad \frac{1}{2\pi i}\int_\Gamma e^{-sx}\Psi(s)ds = \sum_{s_0 \in Z} B(s_0)\chi_1(s_0)e^{-s_0 x}.$$

Here $B(s_0)$ is the residue of $1/(1-g^{n_0}(s))$ at $s = s_0$ and $Z$ is defined by

$$Z = \{s_0 : g^{n_0}(s) = 1,\ -r < \text{Re } s_0 < 0\}.$$

But $\chi_1(s_0) = 1$ if $g(s_0) = 1$ and $\chi_1(s_0) = 0$ if $g^{n_0}(s_0) = 1$ and $g(s_0) \neq 1$. If $s_0 \in S \subset Z$ then $B(s_0) = n_0^{-1}A(s_0)$. So we get

$$(2.31)\qquad \sum_{s_0 \in Z} B(s_0)\chi_1(s_0)e^{-s_0 x} = n_0^{-1} \sum_{s_0 \in S} A(s_0)e^{-s_0 x}$$

Let $p(x)$ be the density of $\nu'$. In the same way as in the proof of Stone [2], Theorem, it follows that

$$\int |\chi(i\theta+s)|d\theta < \infty,\ s \in [-R, 0].$$

and

$$(2.32)\qquad \begin{aligned} &p(x)-\mu_1^{-1} \cdot \chi\{(-\infty, x)\} \\ &\frac{n_0}{2\pi}\int \text{Re } \{e^{-ix\theta}\Psi(i\theta)\}d\theta = \lim_{\varepsilon \to 0} \frac{n_0}{2\pi i}\int_{\substack{|\text{Im } s| \geqq \varepsilon \\ \text{Re } s=0}} e^{-sx}\Psi(s)ds.\end{aligned}$$

It follows easily that

$$(2.33)\qquad \chi\{(-\infty, x)\} = o(e^{rx}),\quad x \to -\infty.$$

With (2.29) and the Riemann-Lebesgue lemma

$$(2.34)\qquad \lim_{N\to\infty}\frac{n_0}{2\pi i}\int_{-r-iN}^{-r+iN} e^{-sx}\Psi(s)ds = \frac{n_0}{2\pi}e^{rx}\int e^{-i\theta x}\Psi(i\theta-r)d\theta = o(e^{rx}),\quad x\to-\infty.$$

The contributions of $\Gamma_1$ and $\Gamma_2$ to the integral of (2.30) tend to zero for $N\to\infty$. This follows with (2.29) and the fact that $\chi_1(s)$ tends to zero for $|\mathrm{Im}\, s|\to\infty$, uniformly in $\mathrm{Re}\, s\in[-R, 0]$. With (2.28) we see that the contribution of $C$ to the integral in (2.30) tends to zero for $\varepsilon\to 0$. Therefore, from (2.30)—(2.34)

$$(2.35)\qquad p(x) = \sum_{s_0\in S} A(s_0)e^{-s_0x}+o(e^{rx}),\quad x\to-\infty$$

and (1.8), (1.9) follow from (2.35).

b) Compare the corresponding part of the proof of theorem 1.

Proof of theorem 4. Compare the proof of theorem 2. Use Lemma A and theorem 3. Since $g(s) < 1$ for real $s\in(s_0, 0)$ and $g(s_0) = 1$ the condition (1.7) is fulfilled for some $r > -s_0$.

## 3. Final remarks

Remark 1. Let $\mu$ be lattice or some $\mu^{m*}$ be non-singular. Suppose $\mu\{(-\infty, 0)\} > 0$ and let $g(s)$ exist for some $s < 0$. Then there exists always a finite real number $r < 0$ such that $\int e^{sx}\nu\{dx\}$ converges for $s\in(r, 0)$ and diverges for $s\in(-\infty, r)$.

This follows from theorem 2 and $\nu\{k\}$ bounded, and from theorem 4, (2.35) and $p(x)$ bounded.

Remark 2.

a) Suppose $g(s)$ exists for $\mathrm{Re}\, s\leqq 0$. If

$$(3.1)\qquad \liminf_{r\to\infty} e^{rk_0}\int_0^{\pi}|g(i\theta-r)-1|^{-1}d\theta = 0$$

then the sum in (1.1) converges for $r\to\infty$ and equals $\nu\{k\}$, $k\leqq k_0 < 0$. This follows from the fact that the left side of (2.4) tends to zero for $r\to\infty$, uniformly in $k\leqq k_0$. Note that the sum remains a finite one and (3.1) holds if the number of lattice-points of $\mu$ in $(-\infty, 0)$ is finite.

b) Suppose $g(s)$ exists for $\mathrm{Re}\, s\geqq 0$. Similarly, if

$$(3.2)\qquad \liminf_{r\to\infty} e^{-rk_0}\int_0^{\pi}|g(i\theta+r)-1|^{-1}d\theta = 0$$

then the sum in (1.3) converges for $r \to \infty$ and equals $\nu\{k\}-\mu_1^{-1}$, $k \geqq k_0 \geqq 0$. Note that the sum remains a finite one and (3.2) holds if the number of lattice-points of $\mu$ in $(0, \infty)$ is finite.

*Postscript.* Further investigations have led to the stronger result that theorem 3 continues to hold if (1.7) and (1.10) are replaced by $\zeta_1(-r) < 1$ and $\zeta_1(r) < 1$. The condition in theorem 4 that $\zeta\{(0, \infty)\} = 0$ can be dropped. We refer to van der Genugten [4].

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