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by

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1. Introduction

Let X and E be topological spaces. Let Y be a superspace of X. We say that X is E-embedded in Y provided that every continuous function $f : X \to E$ admits a continuous extension $f^* : Y \to E$. We shall be concerned with R-embedding, where R is the space of the reals. All spaces will be assumed to be Hausdorff completely regular. For some spaces X, the fact that X is R-embedded in Y can be decided by examining the extension of only one function $f : X \to R$ (this property of X is formulated precisely in the next section). This paper contains some partial results concerning the characterization of such spaces.

We shall now formulate a few statements of purely technical character.

A function $f : X \to R$ is said to be *absolutely extendable* provided that for every superspace Y of X, f admits a continuous extension $f^* : Y \to R$. We say that f *has vanishing oscillation outside compact subsets of* X provided that for every $\varepsilon > 0$ there exists a compact subset C of X such that

$$\omega(f, X \setminus C) = \sup \{|f(p)-f(q)| : p, q \in X \setminus C\} < \varepsilon.$$

1.1 Proposition. *A function $f : X \to R$ is absolutely extendable if and only if f can be continuously extended over every compactification of X.*

1.2 Proposition. *A function $f : X \to R$ has vanishing oscillation outside compact subsets of X if and only if f is the limit of a uniformly convergent sequence $f_1, f_2, \cdots$ of functions on X each of which is constant outside a compact subset of X.*

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PROOF. The sufficiency of the condition is obvious. To prove the necessity consider the functions $\alpha_{sn} : R \to R$, $s \in R$, $n = 1, 2, \cdots$, defined by

$$\alpha_{sn}(t) = s \qquad \text{for} \quad |t-s| \leqq \frac{1}{n};$$

$$\alpha_{sn}(t) = t - \frac{1}{n} \quad \text{for} \quad t \geqq s + \frac{1}{n};$$

$$\alpha_{sn}(t) = t + \frac{1}{n} \quad \text{for} \quad t \leqq s - \frac{1}{n}.$$

Each α_{sn} is continuous and $|\alpha_{sn}(t)-t| \leqq 1/n$ for every $t \in R$.

Assume that $f : X \to R$ has vanishing oscillation outside compact subsets of X. For every positive integer n find a compact set $C_n \subset X$ such that $\omega(f, X \setminus C_n) \leqq 1/n$. Let s be a value of f on $X \setminus C_n$ (if $X \setminus C_n$ is empty, then there is nothing to prove) and let f_n be the composition $\alpha_{sn} \circ f$. f_n is constant outside C_n and the sequence $f_1, f_2, \cdots$ is uniformly convergent to f on X.

1.3 PROPOSITION. *A function $f : X \to R$ is absolutely extendable if and only if f has vanishing oscillation outside compact subsets of X.*

PROOF. The necessity of the condition is obvious; the sufficiency follows from 1.1 and 1.2.

2. The property (P_1). R-compact spaces

2.1 DEFINITION. We say that a space X *has the property* (P_1) provided that there exists a continuous function $f : X \to R$ such that for every superspace Y of X, X is R-embedded in Y if and only if the function f admits the continuous extension $f^* : Y \to R$.

A function f with the above property will be called (for a lack of a better term) a *proper function* on X.

In this section we shall give a characterization of R-compact spaces having property (P_1). A space is R-compact iff it is homeomorphic to a closed subspace of some topological power R^m of R. Intuitively speaking, an R-compact space is a space which is either compact or admits a large number of continuous unbounded functions (precisely: X is R-compact iff for every $p_0 \in \beta X \setminus X$ there is a continuous function $f : X \to R$ which is unbounded on every neighbourhood of p_0). In the next section we shall state some partial results concerning the property (P_1) in arbitrary space.

A function $f : X \to R$ is said to be *bounded only on compact* *subsets of* X provided that for every subset A of X, if f is bounded on A, then $\bar{A}$ ($=$ the closure of A in X) is compact.

2.2 Proposition. *A space* X *admits a continuous function* $f : X \to R$ *that is bounded only on compact subsets of* X *if and only if* X *is locally compact and Lindelöf.*

Proof is obvious [2].

2.3 Proposition. *Let* $f : X \to R$ *be a continuous function which is bounded only on compact subsets of* X. *Let* Y *be a superspace of* X. X *is* R*-embedded in* Y *if and only if the function* f *admits a continuous extension* $f^* : Y \to R$.

Proof. The necessity of the condition is obvious; we shall prove the sufficiency. Let f^* be the continuous extension of f with $f^* : Y \to R$. Let g be an arbitrary continuous function with $g : X \to R$. It is clear that if a continuous function $\alpha : R \to R$ tends sufficiently fast to $+\infty$ as $|t| \to +\infty$, then the function $g(p)/\alpha(f(p))$ has vanishing oscillation outside compact subsets of X. (It suffices to take a continuous function $\alpha : R \to R$ such that

$$\alpha(t) \geqq n \cdot \sup \{|g(p)| : p \in C_{n+1}\} + 1$$

for $|t| \geqq n$, where $C_n = \{p \in X : |f(p)| \leqq n\}$.) By 1.3, $g(p)/\alpha(f(p))$ is absolutely extendable. Let g^* be a continuous extension of $g(p)/\alpha(f(p))$ with $g^* : Y \to R$. It is clear that $g^*(p) \cdot \alpha(f^*(p))$ is a continuous extension of g over Y.

2.4 Corollary. *A locally compact Lindelöf subspace* X *of* Y *is* R*-embedded in* Y *if and only if there exists a continuous function* $g : Y \to R$ *such that* $g|X$ *is bounded only on compact subsets of* X.

2.5 Theorem. *Let* X *be an* R*-compact space.* X *has property* (P_1) *if and only if* X *is locally compact and Lindelöf. Furthermore, a continuous function* $f : X \to R$ *is a proper function if and only if* f *is bounded only on compact subsets of* X.

Proof. Assume that $f : X \to R$ is a continuous function and assume that there is a set A such that f is bounded on A and $\bar{A}^X$ is not compact. Then $\bar{A}^{\beta X} \setminus X \neq \emptyset$; let $p_0 \in \bar{A}^{\beta X} \setminus X$. f can be

[2] Recall that for a locally compact space X the following conditions are equivalent.

(a) *X is Lindelöf*;

(b) *X is σ-compact* (i.e., *X is the union of countably many compact subsets*);

(c) $X = \bigcup_n C_n$, *where C_n are compact and* $C_n \subset \operatorname{Int} C_{n+1}$;

(d) *the ideal point ∞ in the one-point compactification $\iota X = X \cup \{\infty\}$ of X satisfies the first axion of countability.*

extended over $X \cup \{p_0\}$; in fact, one can modify f outside a neighborhood of p_0 so that it becomes bounded on the whole of X. But X is not R-embedded in $X \cup \{p_0\}$. Thus f is not a proper function on X. The rest of the theorem follows now from Propositions 2.2 and 2.3.

3. Proper functions for arbitrary spaces

From the results of the previous section it is easy to obtain a characterization of proper functions for arbitrary spaces. We have to recall a few known facts and definitions.

An *extension* of X is any superspace εX of X such that X is dense in εX. The *canonical map* of extension $\varepsilon_1 X$ into an extension $\varepsilon_2 X$ is a continuous function $\varphi : \varepsilon_1 X \to \varepsilon_2 X$ which is the identity on X. The canonical map (if it exists) is unique. We write $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$ provided that there exists a canonical map of $\varepsilon_1 X$ onto $\varepsilon_2 X$ which is a homeomorphism. (For further information see [4], Chi I.) βX admits a canonical map onto any compactification of X.

The *Q-closure* of a subset P of a space X is the set of all points $q \in X$ such that for every continuous function $f : X \to R$, if $f(p) > 0$ for every $p \in P$, then $f(p) > 0$. P is said to be *Q-closed* in X provided that it is equal to its Q-closure in X. The Q-closure of any subset of an R-compact space is again R-compact. If $c_1 X$ and $c_2 X$ are compactifications of X, A is a Q-closed subset of $c_2 X$ containing X, and φ is a canonical map of $c_1 X$ onto $c_2 X$, then φ maps the Q-closure of X in $c_1 X$ into A.

$\beta_R X$ is an R-compact extension of X such that X is R-embedded in $\beta_R X$. ($\beta_R X$ is also called the *Nachbin completion* of X.) $\beta_R X$ coincides with the Q-closure of X in βX. X is R-compact iff $X = \beta_R X$; equivalently, X is R-compact iff X is Q-closed in βX.

Every continuous function $f : X \to R$ can be continuously extended over βX if we allow $\pm\infty$ to be values of this extension. We shall denote this extension by f^β (note that f^β is a function into the two-point compactification of R). It is easy to see that f is bounded only on compact subsets of X iff $f^\beta(p) = \pm\infty$ for every $p \in \beta X \setminus X$.

3.1 Theorem. *A continuous function $f : X \to R$ is proper if and only if it satisfies the following two conditions*:

(a) *f^β is one-to-one on $\beta_R X \setminus X$;*

(b) *$f^\beta(p) = \pm\infty$ for every $p \in \beta X \setminus \beta_R X$.*

PROOF. Only the sufficiency requires a proof. Assume that Y is a superspace of X such that f admits a continuous extension $g : Y \to R$. We can assume that Y is R-compact (if not, replace Y by $\beta_R Y$). Let $\tilde{X}$ be the Q-closure of X in Y; $\tilde{X}$ is R-compact, hence $\tilde{X}$ is Q-closed in $\beta\tilde{X}$. Let $g_0 = g|\tilde{X}$ (= the restriction of g to $\tilde{X}$); let φ be the canonical map of βX onto $\beta\tilde{X}$. The quality

$$f^{\beta}(p) = g_0^{\beta}(\varphi(p)) \tag{1}$$

holds for every $p \in X$; hence, by continuity, (1) holds for every $p \in \beta X$. Since g_0^{β} is finite on $\tilde{X}$, we infer from (1) and (b) that $\varphi^{-1}[\tilde{X}] \subset \beta_R X$. Since $\tilde{X}$ is Q-closed in $\beta\tilde{X}$, the reverse inclusion also holds. Consequently, $\beta_R X = \varphi^{-1}[\tilde{X}]$. It follows that $\varphi_0 = \varphi|\beta_R X$ is a closed map (the restriction of a closed map to a full counter-image is again closed). From (1) and (a) we infer that φ_0 is one-to-one. Thus φ_0 is a homeomorphism; consequently, $\beta_R X =_{\text{ext}} \tilde{X}$. Hence X is R-embedded in $\tilde{X}$. But from (1) and (b) we infer that $g_0^{\beta}(q) = \pm\infty$ for every $q \in \beta\tilde{X}\setminus\tilde{X}$; hence g_0 is bounded only on compact subsets of $\tilde{X}$. Since $\tilde{X}$ is R-compact, we infer from Theorem 2.5 that g_0 is a proper function for $\tilde{X}$. Consequently, $\tilde{X}$ is R-embedded in Y. Thus X is R-embedded in Y. The theorem is shown.

We did not find any interesting characterization of arbitrary spaces with property (P_1). The following partial results follow directly from Theorem 3.1.

3.2 PROPOSITION. *If X has (P_1), then $\beta_R X$ is locally compact and Lindelöf.*

Recall that a space X is called *extremal* (in the sense of Fréchet, see [1]) provided that every continuous function $f : X \to R$ is bounded. Such spaces are also called *pseudocompact* or *quasicompact*. X is extremal iff $\beta_R X =_{\text{ext}} \beta X$.

3.3. THEOREM. *Let X be an extremal space. A continuous function $f : X \to R$ is proper if and only if f^{β} is one-to-one on $\beta X\setminus X$.*

3.4. THEOREM. *Let X be an extremal locally compact space. X has (P_1) if and only if the Čech outgrowth of X, $\beta X\setminus X$, is homeomorphic to a subspace of the closed interval $I = [0, 1]$.*

3.5. THEOREM. *Every space with a countable Čech outgrowth has (P_1).*

PROOF. If card $(\beta X\setminus X) < 2^{\mathfrak{c}}$, then X is extremal. On the other hand, for every countable subset of an arbitrary space there exists a continuous real function on the space which is one-to-one on this subset (see [4], Theorem 1).

On the basis of Theorem 3.5 it is easy to give examples showing that an extremal space with property (P_1) need not to be locally compact and its Čech outgrowth need not to be homeomorphic to a subspace of I.

We say that a space X has property (P_1^*) provided that there exists continuous function $f: X \to I$ (I is the closed interval $[0, 1]$) such that for every superspace Y of X, X is I-embedded in Y iff f admits a continuous extension $g: Y \to I$. A function f with this property will be called a $*$-proper function. It can be shown that a continuous function $f: X \to I$ is $*$-proper iff f^{β} is one-to-one on $\beta X \setminus X$. It follows that a space X does not have property (P_1^*) unless X is extremal. But for such spaces properties (P_1) and (P_1^*) coincide.

We conclude with two questions.

It follows from Theorem 3.4 that for a locally compact extremal space X property (P_1) depends only on the topological type of the Čech outgrowth of X. Is this true for arbitrary extremal spaces?

Does $\beta X \setminus X$ being homeomorphic to a subspace of I imply that X has (P_1)?

4. Generalizations of properties (P_1) and (P_1^*)

The purpose of this section is to state some questions concerning generalizations of properties (P_1) and (P_1^*) to higher cardinalities.

Let $\mathfrak{m}$ be an arbitrary cardinal; we shall say that *a space X has property* $(P_{\mathfrak{m}})$ provided that there exists a class $\mathfrak{F}$ of continouous real-valued function on X such that card $\mathfrak{F} \leqq \mathfrak{m}$ and for every superspace Y of X, X is R-embedded in Y iff each function in $\mathfrak{F}$ can be extended to a continuous real-valued function on Y. Such a class $\mathfrak{F}$ will be called a proper class on X. Property $(P_{\mathfrak{m}}^*)$ and $*$-proper classes are defined in an analogous way. It is clear that $(P_{\mathfrak{m}})$ implies $(P_{\mathfrak{n}})$ and $(P_{\mathfrak{m}}^*)$ implies $(P_{\mathfrak{n}}^*)$ for $\mathfrak{n} > \mathfrak{m}$; furthermore, every space has property $(P_{\mathfrak{m}})$ (as well as $(P_{\mathfrak{m}}^*)$) for a sufficiently large $\mathfrak{m}$. (Properties (P_0) and (P_0^*) are equivalent; each of them asserts that X is R-embedded in each of its superspaces; such spaces coincide with those having exactly one compactification.) It can be easily shown that $\mathfrak{F}$ is a $*$-proper class on X iff the continuous extensions of functions in $\mathfrak{F}$ over βX separate points of $\beta X \setminus X$ (compare with Theorem 3.1 and the remarks at the end of § 3). This, in turn, implies that X cannot have property $(P^*_{\aleph_0})$ unless X is extremal; consequently, for $\mathfrak{m} \leqq \aleph_0$, $(P^*_{\mathfrak{m}})$ implies $(P_{\mathfrak{m}})$. Clearly, there are non-extremal spaces having property $(P^*_{2^{\aleph_0}})$; I do not know if it can be shown without the continuum hypothesis that $2^{\aleph_0}$ is the first such cardinal. It can also be shown that a locally compact extremal space has $(P_{\mathfrak{m}})$ iff its Čech outgrowth is homeomorphic to a subspace of the Tihonov cube $I^{\mathfrak{m}}$ (compare with 3.4). It follows that for locally compact extremal spaces properties $(P_{\mathfrak{m}})$ and $(P_{\mathfrak{n}})$ are not equivalent for any two distinct cardinals $\mathfrak{m}$ and $\mathfrak{n}$.[3] On the other hand, for R-compact spaces, properties $(P_1), (P_2), \cdots, (P_n), \cdots, n < \aleph_0$, are equivalent; this can be demonstrated by showing that $\mathfrak{F} = \{f_1, \cdots, f_n\}$ is a proper class for X iff $f = \max \{|f_1|, \cdots, |f_n|\}$ is a proper function for X.

Property $(P_{\mathfrak{m}})$ for R-compact spaces is somewhat related to the concept of R-defect introduced in [3]. (An *R-non-extendable* class for X is a class $\mathfrak{F}$ of continuous real-valued functions on X such that for every extension εX of X with $\varepsilon X \neq X$, at least one of the functions in $\mathfrak{F}$ does not admit a continuous real-valued extension over εX. The *R-defect* of X [in symbols: $\text{def}_R X$] is the smallest cardinal $\mathfrak{m}$ such that X admits an R-non-extendable class of cardinality $\mathfrak{m}$. For further information see [3] and [5].) It can be easily shown that

4.1. *If an R-compact space has property* $(P_{\mathfrak{m}})$, *then* $\text{def}_R X \leqq \mathfrak{m}$. *In fact, a proper class on X is an R-non-extendable class for X.*

From 2.5 and from 5.9 in [5] we infer that for $\mathfrak{m} = 1$ the above implication can be reversed.

4.2. *Let X be R-compact. X has* (P_1) *if and only if* $\text{def}_R X \leqq 1$.

The converse of 4.1 fails for infinite $\mathfrak{m}$. We have the following.

4.3. *Let* $\mathfrak{m}$ *be an infinite cardinal of the form* $\mathfrak{m} = 2^{\mathfrak{n}}$ *and let X be a space with weight* $X \leqq \mathfrak{m}$. *X has* $(P_{\mathfrak{m}})$ *if and only if* card $C(X, R) \leqq \mathfrak{m}$.

Proof. The "if" part is obvious. To prove the converse it suffices to show that for every class $\mathfrak{F}$ of continuous real-valued functions on X with card $\mathfrak{F} \leqq m$ there is a superspace Y of X such that Y has only $\mathfrak{m}$ continuous real-valued functions and each function in

[3] In [2] Glicksberg proves that if $X \times Y$ is extremal, then $\beta(X \times Y =_{\text{ext}} \beta X \times \beta Y$. On the other hand, if X is compact and Y is extremal, then $X \times Y$ is also extremal. Consequently, for every compact space X we have $\beta X^* \setminus X^* =_{\text{top}} X$, where $X^* = X \times S(\Omega)$ and $S(\Omega)$ is the space of all ordinals $< \Omega$.

$\mathfrak{F}$ admits a continuous extension over Y. We can assume that $\mathfrak{F}$ is an R-separating class for X. The parametric map h of X corresponding to F (see Theorem 2.1 in [5]) is a homeomorphism of X into $R^{\mathfrak{m}}$. It suffices to take as Y a superspace of X that is homeomorphic to $R^{\mathfrak{m}}$ by an extension of the homeomorphism h. $R^{\mathfrak{m}}$ has only $\mathfrak{m}$ continuous real-valued functions; indeed, $R^{\mathfrak{m}}$ has a dense subset of cardinality $\mathfrak{n}$.

It follows from 4.2 that if $\mathfrak{m} = 2^{\mathfrak{n}}$ is infinite, then the discrete space $X_{\mathfrak{m}}$ of cardinality $\mathfrak{m}$ does not have $(P_{\mathfrak{m}})$. On the other hand, $\mathrm{def}_R X_{\mathfrak{m}} \leqq \mathfrak{m}$ for "almost all" infinite cardinals; in particular, for all $\mathfrak{m} = 2^{\mathfrak{n}}$, where $\mathfrak{n}$ is Ulam non-measurable.

Consequently, the converse of 4.1 fails for all such cardinals. I do not know if 4.3 holds for infinite cardinals that are not of the form $2^{\mathfrak{n}}$ as well as if 4.2 fails for such cardinals. It appears that the answer to this question depends upon the assumed rules of exponentation of cardinals. In particular, I do not know if 4.2 holds for the cardinal $\aleph_0$. Let Q be the space of irrational numbers; we have $\mathrm{def}_R Q = \aleph_0$; does Q have $(P_{\aleph_0})$? (Note that $\mathrm{def}_R P > \aleph_0$, where P is the space of rational numbers; therefore P does not have $(P_{\aleph_0})$.) We have $\mathrm{def}_R R^{\aleph_0} = \aleph_0$; does $R^{\aleph_0}$ have $(P_{\aleph_0})$?

One can discuss the above problems in a more general context. The property analogous to $(P_{\mathfrak{m}})$ but referring to functions with values in a space E will be denoted by $P_{\mathfrak{m}}(E)$; in the formulation of this property all spaces are assumed to be E-completely regular. F. Marin has pointed out to us that 4.1 holds true in this general context: if and E-compact space X has property $P_{\mathfrak{m}}(E)$, then $\mathrm{def}_E X \leqq \mathfrak{m}$. The study of property $P_{\mathfrak{m}}(E)$ for E-extremal spaces is more difficult. (An E-extremal space is an E-completely regular space X with the property: for every continuous function $f : X \to E$, $f[X]$ is compact.)

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