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The diffeomorphic excision of closed local compacta from infinite-dimensional Hilbert manifolds

by

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If M is an infinite-dimensional manifold, which open, dense submanifolds of it are homeomorphic or diffeomorphic to it by functions arbitrarily close to the identity? R. D. Anderson, David W. Henderson, and the author together have shown [2] that if M is a metrizable manifold modelled on a separable, infinite-dimensional Fréchet space, then each open, dense submanifold N of M with the property that for each open set U of M, U and $U \cap N$ have the same homotopy type is homeomorphic to M by a homeomorphism which may be required to be the identity on any closed subset of M lying in N and may be limited by any open cover of M. (A function f from a subset X of M into M is said to be limited by the open cover G of M if the collection $\{\{x, f(x)\} | x \in X\}$ refines G.) Such submanifolds include the complements of all closed, locally compact subsets of M, but the method of proof used cannot readily be adapted to give diffeomorphisms when M is a differentiable manifold, as it involves homeomorphisms between Fréchet spaces which are not diffeomorphic. The principal tools used in [2] may be traced conceptually from the proof due to V. L. Klee, Jr., [5], that a separable, infinite-dimensional Hilbert space is homeomorphic to the complement of each of its compacta. In 1966, Cz. Bessaga [3] produced a differentiable version of Klee's theorem in the special case of a single point, so it seemed natural to the author to try proving a differentiable version of [2] for complements of closed, locally compact subsets of differentiable manifolds on separable, infinite-dimensional Hilbert spaces (real). The analogy is complete, as the following statement of the theorem of this paper shows: *If M is a metrizable C^p-manifold* ($1 \leqq p \leqq \infty$) *modelled on separable, infinite-dimensional Hilbert spaces, X is a closed, locally compact subset of M, U is an open subset of M containing X, and G is an open cover of M, then there is a C^p-diffeomorphism of M onto $M \setminus X$ which is the identity off U and is limited*

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by G. The proof is elementary in the sense that it requires only the Inverse Function Theorem, differentiable partitions of unity, and Bessaga's result, which requires nothing more sophisticated in its proof.

After completing most of the work on this paper, the author was apprised by R. D. Anderson that (a) by minor modifications of [1], it is possible to show that a number of topological linear spaces which possess Schauder bases are C^∞-diffeomorphic to the complements of each of their compacta and (b) Peter Renz of the University of Washington has recently obtained by a different method the result that a metrizable manifold modelled on a separable, infinite-dimensional Hilbert space is diffeomorphic to the complement of each of its closed local compacta. The author has also learned from David Henderson that quite recently D. Burghelea, N. Kuiper, and N. Moulis have proven results implying that each two open subsets of a separable, infinite-demensional Hilbert space which have the same homotopy type are C^∞-diffeomorphic.

Throughout this paper, H will denote a separable, infinite-dimensional (real) Hilbert space, and differentiability will be taken in the sense of Fréchet. The term "manifold" will denote a manifold without boundary. The term "C^∞-partition of unity on H" is taken to mean a collection S of C^∞-functions s from H into $[0,1]$ and a locally finite open cover $\{U_s\}$ of H such that

$$\overline{s^{-1}((0,\infty))} \subset U_s \quad \text{and} \quad \sum_{s\in S} s(x) = 1$$

for each x in H; S is said to be subordinate to an open cover G of H if $\{\overline{s^{-1}((0,\infty))} \mid s \in S\}$ refines G. Given any open cover G of H, there is a C^∞-partition of unity subordinate to G (see [6]; p. 30).

The proof of Theorem 1 is broken into a sequence of 8 lemmas, several of which are not new but are included for purposes of completeness and reference.

LEMMA 1. *If X is a closed, locally compact subset of H, there is a complete, orthonormal basis $\{e_n\}_{n=1}^\infty$ for H with the property that if H_1 is the closed, linear span of $\{e_{2n-1}\}_{n=1}^\infty$ and P_1 is the (orthogonal) projection of H onto H_1, then P_1 is a homeomorphism on each compact subset of X.*

PROOF: Because X is a separable, locally compact, metric space, it is possible to find a collection $\{X_i\}_{i=1}^\infty$ of compacta of X for which each X_i is contained in the interior (relative to X) of its successor and X is the union of the X_i's. Let $\{z_n\}_{n=1}^{\infty}$ be a complete, orthonormal basis for H, and for each i and n let

$$a_{i,n} = \sup \{|(x, z_n)|\, |x \in X_i\}.$$

By the compactness of the X_i's, $\{a_{i,n}\}_{n=1}^{\infty}$ converges to zero for each i. Let $\{n(i)\}_{i=1}^{\infty}$ be a subsequence of the positive integers such that for each i, $a_{i,n(i)} \leqq 1/i$, and observe that if $j \leqq i$, then $a_{j,n(i)} \leqq a_{i,n(i)}$. Let $\{A_k\}_{k=1}^{\infty}$ be an infinite collection of pairwise disjoint infinite subsets of the positive integers such that if each A_k is indexed by the positive integers in the natural order and is denoted by $\{m(k, p)\}_{p=1}^{\infty}$, then for each k and p, $m(k, p) \geqq 2^{k+p}$. Let, for each k, $x_k = \sum_{p=1}^{\infty} 2^{-p/2} z_{n(m(k,p))}$. Since for each k, the point $y_k = 2^{-\frac{1}{2}} z_{n(m(k,1))} - z_{n(m(k,2))}$ is orthogonal to each x_j and the y_k's are all orthogonal, there is a complete, orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for H such that for each n, $e_{2n} = x_n$. Such a basis will suffice, for if x and y are in X and $P_1(x) = P_1(y)$, then $x-y$ is in the closed, linear span of the x_k's. Thus,

$$x-y = \sum_{k=1}^{\infty} (x-y, x_k) = \sum_{k=1}^{\infty} \Big(\sum_{p=1}^{\infty} (x-y, z_{n(m(k,p))}) z_{n(m(k,p))} \Big).$$

Hence,

$$(x-y, z_{n(m(k',p))}) = \Big(\sum_{k=1}^{\infty} (x-y, x_k) x_k, z_{n(m(k',p))} \Big),$$

but since

$$(x_k, z_{n(m(k',p))}) = 0 \text{ unless } k = k',$$

it is true that

$$(x-y, z_{n(m(k',p))}) = (x-y, x_{k'})(x_{k'}, z_{n(m(k',p))}) = 2^{-p/2}(x-y, x_{k'});$$

therefore,

$$(x-y, x_{k'}) = 2^{p/2}(x-y, z_{n(m(k',p))}),$$

for each p. However, since there is an i for which both x and y are in X_i, and $m(k', p) \geqq 2^{k'+p}$ for each p, if $p \geqq i$, then

$$\begin{aligned} |(x-y, z_{n(m(k',p))})| &\leqq |(x, z_{n(m(k',p))})| + |(y, z_{n(m(k',p))})| \\ &\leqq 2a_{i,n(m(k',p))} \leqq 2a_{m(k',p),n(m(k',p))} \leqq 2^{1-k'-p}. \end{aligned}$$

Thus, for each $p \geqq i$,

$$|(x-y, x_{k'})| = 2^{p/2}|(x-y, z_{n(m(k',p))})| \leqq 2^{1-k'-p/2}.$$

This shows that for each k, $(x-y, x_k) = 0$ and, thus, that $x = y$ and P_1 is one-to-one on X, which proves the lemma.

Lemma 2. *If H_1 is a closed, linear subspace of H, P_1 is the projection of H onto H_1, X is a closed subset of H, and $P_1|X$ is a homeomorphism of X onto a closed subset of H_1, then for any sequence $\{f_i\}_{i=1}^{\infty}$ of C^{∞}-diffeomorphisms of H onto itself satisfying the four conditions below, the uniform limit f of $\{f_i \cdots f_1\}_{i=1}^{\infty}$ is a homeomorphism of H onto itself such that $f|P_1(X) = (P_1|X)^{-1}$ and $f|H \setminus P_1(X)$ is a C^{∞}-diffeomorphism.*

a) *$\{f_i \cdots f_1\}_{i=1}^{\infty}$ is uniformly Cauchy.*

b) *$P_1 f_i = P_1$, for all i.*

c) *f_{i+j} is the identity outside the open 2^{1-i}-neighborhood of X and the image under $f_i \cdots f_1$ of the open 2^{1-i}-neighborhood of $P_1(X)$, for each j.*

d) *For each x in X, $x = \text{limit}_{i \to \infty} f_i \cdots f_1(x)$.*

Proof: Condition (a) provides the existence of f as defined and its continuity; condition (c) ensures that $f|H \setminus P_1(X)$ is a C^{∞}-diffeomorphism onto its image, and condition (d) is the statement that $f|P_1(X) = (P_1|X)^{-1}$. Therefore, the only remaining things to establish are that f is one-to-one, that $f(H) = H$, and that f^{-1} is continuous on X. Conditions (c) and (d) immediately yield that $f(H) = H$. To see that f is one-to-one, observe that if there were two points, x and y, of H for which $f(x) = f(y)$, then one of them, say x, would have to be in $P_1(X)$, and the other would have to be in $H \setminus P_1(X)$. Condition (c) would then specify that there is a positive integer i and an open set containing $f_i \cdots f_1(y)$ on which f_{i+j} is the identity for each $j \geqq 1$, which, together with the fact that $f_i \cdots f_1$ is a homeomorphism of H, shows that $f(x) \neq f(y)$, after all. The continuity of f^{-1} at points of X is assured by (b), (c), and (d) together, for if x is in X and U is an open set containing $P_1(x)$, then for an i such that the open 2^{2-i}-neighborhood of $P_1(x)$ lies in U, (b), (c), and (d) give that the open set

$$2^{1-i}B_1^0 + 2^{1-i}B_2^0 + P_1(x)$$

is carried by $f_i \cdots f_1$ onto an open set V containing x (where B_1^0 is the open unit ball in H_1 about the origin and B_2^0 is the open unit ball in the orthogonal complement of H_1 about the origin). Now, conditions (b) and (c) guarantee that f_{i+j} carries V onto itself for each $j \geqq 1$. Therefore $f^{-1}(V)$ is contained in U, and f^{-1} is continuous.

Lemma 3. *If Y is a separable, metric space, G is an open cover of Y, and X is a closed, locally compact subset of Y, then there is a star-*

finite open cover $\{U_i\}_{i=1}^{\infty}$ *of* Y *and a cover* $\{X_i\}_{i=1}^{\infty}$ *of* X *by compacta of* X *such that for each* i, X_i *is contained in* U_i, *and* $\{\overline{U}_i\}_{i=1}^{\infty}$ *refines* G.

PROOF: This may be done easily by embedding Y in the Hilbert cube, taking the closure of the image of Y, and using the compactness of the Hilbert cube after the fashion of Theorem 1 of [2] and Theorem 1 of [4].

LEMMA 4. *If* X *is a closed, locally compact subset of* H *and* H_1 *is a closed, linear subspace of* H *for which* (a) $\overline{X \setminus H_1}$ *is compact and* (b) *if* P_1 *is the projection of* H *onto* H_1, *then* $P_1|X$ *is a homeomorphism, then for any positive real number* ε *and open set* W *of* H_1 *containing* $P_1(X)$ *there is a pair,* f *and* g, *of homeomorphisms of* H *onto itself satisfying the following*:

1) g *is the identity off the intersection of* $P_1^{-1}(W)$ *with the open* ε*-neighborhood of* X *and moves no point more than* ε;

2) $g|(H\setminus X)$ *is a* C^∞*-diffeomorphism onto* $g(H\setminus X)$;

3) f *is a* C^∞*-diffeomorphism and is the identity off* $P_1^{-1}(W)$;

4) $P_1 f = P_1 g = P_1$, *and* $f = fP_1 + I - P_1$, *where* I *is the identity, and*

5) $fg|X = P_1|X$.

PROOF: Let H_2 be the orthogonal complement of H_1, let P_2 be the projection of H onto H_2 ($P_2 = I - P_1$), and, for convenience, assume that $\varepsilon \leqq 1$. Let a be a non-increasing C^∞-function from the real numbers into $[0,1]$ for which $a^{-1}(0) \supset [1, \infty)$ and $a^{-1}(1) \supset (-\infty, 0]$. Let $b = \sup\{|a'(t)|\,|t \text{ real}\}$, and observe that from the Mean Value Theorem, $b \geqq 1$. For each positive number c, define g_c to be the function from H_1 to the real numbers such that $g_c(x) = a(||x||^2/c^2)$. Each g_c is a C^∞-function, and for each x in H, $||g_c'(x)|| \leqq 2b/c$. Let $c_1 \in (0, \varepsilon/4)$ be small enough that the closed $2c_1$-neighborhood in $P_1(X)$ of $P_1\overline{(X\setminus H_1)}$ is compact. Let G_1 be an open cover of H_1 refining $\{W, H_1\setminus P_1(X)\}$ for which (1) $U \in G_1$ implies that

$$\sup\{||P_2(P_1|X)^{-1}(x) - P_2(P_1|X)^{-1}(y)||\,|x, y \in P_1(X) \cap U\} \leqq c_1/4b,$$

and (2) each element of G_1 has diameter less than c_1. Let y_1 be a function from G_1 into H_2 such that $y_1(U)$ is in

$$P_2(P_1|X)^{-1}(U \cap P_1(X)) \text{ if } U \cap P_1(X) \neq \emptyset$$

and is the origin otherwise. Let S_1 be a C^∞-partition of unity on H_1 subordinate to G_1, and let u be a function from S_1 into G_1 such that

$u(s) \supset \overline{s^{-1}((0, \infty))}$ for each s in S_1. Set $\bar{f}_1$ to be the function from H_1 into H_2 defined by $\bar{f}_1(x) = \sum_{s \in S_1} s(x)y(u(s))$, and let $f_1 = I + \bar{f}_1 P_1$. It is immediate that f_1 is a C^∞-diffeomorphism of H onto itself and is the identity off $P_1^{-1}(W)$. Let $f = f_1^{-1} = I - \bar{f}_1 P_1$.

Let K_1 be the closure of $\{x \in P_1(X) | f_1(x) \neq x\}$, and note that by (2) and the choice of c_1, K_1 is compact. Let

$$p_1 : P_1(X) \to H_2 \text{ by } p_1(x) = (P_1|X)^{-1}(x) - f_1(x).$$

If B_i is the closed unit ball of H_i centered about the origin, then

$$p_1 P_1(X) \subset (c_1/4b)B_2.$$

The homeomorphism g will be constructed as a uniform limit of C$^\infty$-diffeomorphisms of H onto itself defined below.

Let $\{f_i\}_{i=1}^\infty$ be a sequence of C^∞-diffeomorphisms of H onto itself with f_1 as above satisfying the following five conditions:

1) there is a set $\{c_i\}_{i=1}^\infty$ of positive numbers, with c_1 as above, such that for each i, $2c_{i+1} < c_i < 2^{-2i}\varepsilon$;

2) if $i > 1$, then $f_i(x) = x + g_{c_{i-1}}(x - f_{i-1} \cdots f_1 P_1(x))\bar{f}_i P_1(x)$, where $\bar{f}_i$ is a C^∞-function from H_1 into $(c_{i-1}/4b)B_2$ for which the open c_i-neighborhood of $P_1(X)$ and W both contain $\bar{f}_i^{-1}(H_2 \setminus \{0\})$;

3) for each i, $p_i = (P_1|X)^{-1} - f_i \cdots f_1 | P_1(X)$ carries $P_1(X)$ into $(c_i/4b)B_2$;

4) $c_i B_2 + f_i \cdots f_1(H_1) \subset f_i \cdots f_1(2^{-i}B_2 + H_1)$, and

5) if y is in H_1 and x is in $P_1(X)$, then $||y - x|| < c_i$ implies that $||P_2 f_{i-1} \cdots f_1(y) - P_2 f_{i-1} \cdots f_1(x)|| < 2^{-2i}\varepsilon$.

Such a sequence $\{f_i\}_{i=1}^\infty$ exists, for the conditions are arranged to provide an easy inductive construction as follows:
If a collection $\{f_i\}_{i=1}^n$, $n \geqq 1$, of diffeomorphisms is given satisfying conditions (1)—(5), let c'_{n+1} be a positive number so small that for y in H_1 and x in $P_1(X)$ with $||y - x|| < c'_{n+1}$,

$$||P_2 f_n \cdots f_1(y) - P_2 f_n \cdots f_1(x)|| < 2^{-2n-2}\varepsilon.$$

Let $t_1, \cdots$, and t_n be in (0,1) such that if $0 < ||\, x \,|| < t_i$, then $|1 - g_{c_i}(x)| < 2(b/c_i)||x||$, for each $i \leqq n$. (This can be done because $g'_{c_i}(0)$ is the zero functional, and hence

$$\lim_{||x|| \to 0} (g_{c_i}(x) - g_{c_i}(0)) / ||x|| = 0.)$$

Let

$$c_{n+1} \in (0, \min \{c'_{n+1}, \tfrac{1}{2}c_n, 2^{-2n-2}t_1 \cdots t_n \varepsilon\}),$$

and let $d_{n+1} \in (0, c_{n+1})$ be so small that for x and y in $P_1(X)$ with

$$||x-y|| < d_{n+1},\ ||p_n(x)-p_n(y)|| < c_{n+1}/4b.$$

Now, let G_{n+1} be an open cover of H_1 which refines $\{W, H_1\setminus P_1(X)\}$ and is of mesh less than d_{n+1}; let $y_{n+1} : G_{n+1} \to H_2$ be a function such that $y_{n+1}(U)$ is in

$$p_n(P_1(X) \cap U) \text{ if } P_1(X) \cap U \neq \emptyset$$

and is the origin otherwise; let S_{n+1} be a C^∞-partition of unity on H_1 subordinate to G_{n+1}, and let $u_{n+1} : S_{n+1} \to G_{n+1}$ be a function such that for s in S_{n+1}, $u_{n+1}(s) \supset \overline{s^{-1}((0, \infty))}$. Let

$$\bar{f}_{n+1}(x) = \sum_{s\in S_{n+1}} s(x) y_{n+1}(u_{n+1}(s)),$$

for each x in H_1, and define

$$f_{n+1} : H \to H \text{ by } f_{n+1}(x) = x + g_{c_n}(x - f_n \cdots f_1 P_1(x))\bar{f}_{n+1} P_1(x).$$

At this point, it will be shown that f_{n+1} is a C^∞-diffeomorphism of H onto itself. The proof is a standard argument involving the Inverse Function Theorem, the Mean Value Theorem, and the Banach Contraction Principle (for explicit statements of these theorems and for proofs, see pages 11 and 12 of [6]). The Inverse Function Theorem implies that in order to show that f_{n+1} is a C^∞-diffeomorphism of H onto itself, it is sufficient to show that f_{n+1} is one-to-one, that $f_{n+1}(H) = H$, and that for each x in H, $f'_{n+1}(x)$ is a linear homeomorphism of H onto itself. In order to show that f_{n+1} is one-to-one and carries H onto itself, it suffices to show that for each y in H, the function o_y of H_2 into itself defined by the formula $o_y(x) = P_2(y) - g_{c_n}(x - P_2 f_n \cdots f_1 P_1(y))\bar{f}_{n+1} P_1(y)$ has a unique fixed-point, since if x_y is a fixed point of o_y, then

$$o_y(x_y) = x_y = P_2(y) - g_{c_n}(x_y - P_2 f_n \cdots f_1 P_1(y))\bar{f}_{n+1} P_1(y).$$

Thus,

$$P_2(y) = x_y + g_{c_n}(x_y - P_2 f_n \cdots f_1 P_1(y))\bar{f}_{n+1} P_1(y),$$

and

$$y = P_1(y) + x_y + g_{c_n}(P_1(y) + x_y - f_n \cdots f_1 P_1(P_1(y) + x_y)) \cdot \bar{f}_{n+1} P_1(P_1(y) + x_y) = f_{n+1}(P_1(y) + x_y).$$

In order to show that o_y has a unique fixed point, the Banach Contraction Principle asserts that it suffices to find a k in (0,1) such that for each x and x' in H_2, $||o_y(x) - o_y(x')|| \leqq k||x - x'||$. The Mean Value Theorem shows that if k is a uniform bound on the norm of the derivative of o_y, then this happens. For each y in H, the constant k may be taken to be $\frac{1}{2}$, for if y is in H and x is in H_2,

$$o'_y(x) = -g'_{c_n}(x-P_2f_n\cdots f_1P_1(y))\cdot \bar{f}_{n+1}P_1(y),$$

where "$\cdot$" denotes the scalar multiplication of the linear functional and the element of H. Since

$$\begin{aligned}||o'_y(x)|| &\leqq ||g'_{c_n}(x-P_2f_n\cdots f_1P_1(y))||\ ||\bar{f}_{n+1}P_1(y)||\\ &\leqq 2(b/c_n)(c_n/4b) = \tfrac{1}{2},\end{aligned}$$

f_{n+1} is a one-to-one map of H onto itself.

To complete the verification that f_{n+1} is a C^∞-diffeomorphism of H onto itself, there only remains to show that for each x in H, $f'_{n+1}(x)$ is a linear homeomorphism of H onto itself. By the Closed Graph Theorem, this is equivalent to showing that for each x, $f'_{n+1}(x)$ is one-to-one and carries H onto itself. For each x and y in H,

$$\begin{aligned}f'_{n+1}(x)(y) = y&+g'_{c_n}(x-f_n\cdots f_1P_1(x))\,(y-(f_n\cdots f_1)'(P_1(x))\,(P_1(y)))\\ &\cdot \bar{f}_{n+1}P_1(x)+g_{c_n}(x-f_n\cdots f_1P_1(x))\cdot \bar{f}'_{n+1}(P_1(x))\,(P_1(y)).\end{aligned}$$

Because both $\bar{f}_{n+1}P_1(x)$ and $\bar{f}'_{n+1}(P_1(x))\,(P_1(y))$ lie in H_2, the kernel of $f'_{n+1}(x)$ must also lie in H_2. However, H_2 is an invariant subspace of $f'_{n+1}(x)$, and $f'_{n+1}(x)|H_2$ is a linear homeomorphism of H_2 onto itself, since for y in H_2,

$$f'_{n+1}(x)(y) = y+g'_{c_n}(x-f_n\cdots f_1P_1(x))\,(y)\cdot \bar{f}_{n+1}P_1(x),$$

and

$$||g'_{c_n}(x-f_n\cdots f_1P_1(x))\,(y)\cdot \bar{f}_{n+1}P_1(x)\,|| \leqq 2(b/c_n)(c_n/4b)||y|| = \tfrac{1}{2}||y||.$$

Thus, $f'_{n+1}(x)$ is one-to-one, and since

$$(f'_{n+1}(x))^{-1} = P_1-(f'_{n+1}(x)\,|\,H_2)^{-1}(P_2f'_{n+1}(x)P_1-P_2),$$

f_{n+1} is a C^∞-diffeomorphism of H onto itself.

The collection $\{f_i\}_{i=1}^{n+1}$ satisfies conditions (1)—(5) rather easily. Conditions (1) and (5) are met explicitly by the choice of c_{n+1}, and condition (2) is satisfied by the construction of f_{n+1}, the fact that, by (3), $p_nP_1(X)$ lies in $(c_n/4b)B_2$, and the fact that no element of G_{n+1} meeting $P_1(X)$ contains points of $H_1\setminus W$ or farther than c_{n+1} from $P_1(X)$.

Condition (3) is met because if x is in $P_1(X)$, then

$$\begin{aligned}p_{n+1}(x) &= (P_1|X)^{-1}(x)-f_{n+1}\cdots f_1(x) = (P_1|X)^{-1}(x)-f_n\cdots f_1(x)\\ &\quad -g_{c_n}(f_n\cdots f_1(x)-f_n\cdots f_1P_1f_n\cdots f_1(x))\bar{f}_{n+1}P_1f_n\cdots f_1(x)\\ &= (P_1|X)^{-1}(x)-f_n\cdots f_1(x)-g_{c_n}(f_n\cdots f_1(x)-f_n\cdots f_1(x))\bar{f}_{n+1}(x)\\ &= p_n(x)-\sum_{s\in S_{n+1}} s(x)y_{n+1}(u_{n+1}(s))\end{aligned}$$

and, therefore,

$$\|p_{n+1}(x)\| \leqq \sup\{\|y_{n+1}(U)-p_n(x)\| \,|\, x \in U \in G_{n+1}\} \leqq c_{n+1}/4b,$$

by the choice of d_{n+1}.

In order to see that (4) is satisfied, observe that for each x in H_1, $f_{n+1}|(H_2+x)$ is a C^∞-diffeomorphism of H_2+x onto itself. Hence, $f_{n+1}\cdots f_1(2^{-n-1}t_1\cdots t_n B_2^0+x)$ is an open neighborhood in H_2+x of $f_{n+1}\cdots f_1(x)$. The argument below shows that it contains $c_{n+1}B_2^0+f_{n+1}\cdots f_1(x)$. If y is in H_2+x, $y \neq x$, and

$$\|f_i\cdots f_1(y)-f_1\cdots f_1(x)\| < t_i,$$

then

$$\begin{aligned}(\tfrac{3}{2})\|f_i\cdots f_1(y)-f_i\cdots f_1(x)\| &\geqq \|f_{i+1}\cdots f_1(y)-f_{i+1}\cdots f_1(x)\|\\ &\geqq \tfrac{1}{2}\|f_i\cdots f_1(y)-f_i\cdots f_1(x)\|.\end{aligned}$$

This is true because

$$\begin{aligned}
&\|f_{i+1}\cdots f_1(y)-f_{i+1}\cdots f_1(x)\| = \|f_i\cdots f_1(y)-f_i\cdots f_1(x)\\
&\quad -(g_{c_i}(f_i\cdots f_1(x)-f_i\cdots f_1P_1f_i\cdots f_1(x))\bar{f}_{i+1}P_1f_i\cdots f_1(x)\\
&\quad -g_{c_i}(f_i\cdots f_1(y)-f_i\cdots f_1P_1f_1\cdots f_1(y))\bar{f}_{i+1}P_1f_i\cdots f_1(y))\|\\
&= \|f_i\cdots f_1(y)-f_i\cdots f_1(x)\\
&\quad -(\bar{f}_{i+1}(x)-g_{c_i}(f_i\cdots f_1(y)-f_i\cdots f_1(x))\bar{f}_{i+1}(x))\|\\
&\geqq \|f_i\cdots f_1(y)-f_i\cdots f_1(x)\|\\
&\quad -(1-g_{c_i}(f_i\cdots f_1(y)-f_i\cdots f_1(x)))\|\bar{f}_{i+1}(x)\|\\
&\geqq \|f_i\cdots f_1(y)-f_i\cdots f_1(x)\|(1-(\|\bar{f}_{i+1}(x)\|/\|f_i\cdots f_1(y)-f_i\cdots f_1(x)\|)\\
&\quad (1-g_{c_i}(f_i\cdots f_i(y)-f_i\cdots f_1(x))))\\
&\geqq \|f_i\cdots f_1(y)-f_i\cdots f_1(x)\|(1-(c_i/4b)/(c_i/2b))\\
&= \tfrac{1}{2}\|f_i\cdots f_1(y)-f_i\cdots f_1(x)\|,
\end{aligned}$$

by the choice of t_i. A similar argument yields the other part of the inequality. Thus, if $\|y-x\| = 2^{-n-1}t_1\cdots t_n$, then since

$$\|y-x\| = \|f_1(y)-f_1(x)\|$$

(for y in H_2+x), an induction shows that

$$\|f_{n+1}\cdots f_1(y)-f_{n+1}\cdots f_1(x)\| \geqq c_{n+1}.$$

The set of all such y in H_2+x is the boundary in H_2+x of

$$2^{-n-1}t_1\cdots t_n B_0^2+x,$$

so its image under $f_{n+1}\cdots f_1$ must be the boundary of

$$f_{n+1}\cdots f_1(2^{-n-1}t_1\cdots t_n B_2^0+x)$$

in H_2+x, which must therefore contain

$$c_{n+1}B_2^0+f_{n+1}\cdots f_1(x).$$

Since conditions (1)—(5) are satisfied, an induction shows the existence of an infinite sequence $\{f_i\}_{i=1}^{\infty}$ of C^{∞}-diffeomorphisms of H onto itself which meets all five of the conditions. These five conditions imply the four conditions of Lemma 2. Conditions (1) and (2) imply (a); (2) and the definition of f_1 imply (b), and (1) and (3) imply (d). To show that (c) holds, let i and j be positive integers, and let y be in H. If $f_{i+j}(y)\neq y$, then, by (2), $||P_1(y)-P_1(x)|| < c_{i+j}$ for some x in X and, by (5),

$$||P_2f_{i+j-1}\cdots f_1P_1(y)-P_2f_{i+j-1}\cdots f_1P_1(x)|| < 2^{-2i-2j}\varepsilon.$$

Also, since

$$g_{c_{i+j-1}}(y-f_{i+j-1}\cdots f_1P_1(y))\neq 0,\ ||y-f_{i+j-1}\cdots f_1P_1(y)|| < c_{i+j-1};$$

furthermore, by (3),

$$||x-f_{i+j-1}\cdots f_1P_1(x)|| \leqq c_{i+j-1}/4b \leqq \tfrac{1}{4}c_{i+j-1}.$$

Combining these inequalities gives

$$\begin{aligned}
||x-y|| &= ||P_1(x)-P_1(y)+P_2(x)-P_2(y)||\\
&= ||P_1(x)-P_1(y)+P_2(x)-P_2f_{i+j-1}\cdots f_1P_i(x)+P_2f_{i+j-1}\cdots f_1P_1(x)\\
&\quad - P_2f_{i+j-1}\cdots f_1P_1(y)+P_2f_{i+j-1}\cdots f_1P_1(y)-P_2(y)||\\
&= ||P_1(x)-P_1(y)+x-f_{i+j-1}\cdots f_1P_1(x)+P_2f_{i+j-1}\cdots f_1P_1(x)\\
&\quad -P_2f_{i+j-1}\cdots f_1P_1(y)+f_{i+j-1}\cdots f_1P_1(y)-y||\\
&\leqq ||P_1(x)-P_1(y)||+||x-f_{i+j-1}\cdots f_1P_1(x)||\\
&\quad +||P_2f_{i+j-1}\cdots f_1P_1(x)-P_2f_{i+j-1}\cdots f_1P_1(y)||\\
&\quad +||y-f_{i+j-1}\cdots f_1P_1(y)|| < c_{i+j}+\tfrac{1}{4}c_{i+j-1}+2^{-2i-2j}\varepsilon\\
&\quad +c_{i+j-1} < 2^{3-2i-2j}\varepsilon \leqq 2^{-i}\varepsilon.
\end{aligned}$$

Therefore, f_{i+j} is the identity outside the open $2^{-i}\varepsilon$-neighborhood of X. Also, since if $f_{i+j}(y)\neq y$, then y is in $c_{i+j-1}B_2^0+f_{i+j-1}\cdots f_1(H_1)$, (4) yields that y is in $f_{i+j}\cdots f_1(2^{-i}B_2^0+H_1)$, so

$$||f_1^{-1}\cdots f_{i+j-1}^{-1}(y)-P_1(y)|| \leqq 2^{-i}.$$

Now, (2) shows that $||P_1(y)-P_1(x)|| < c_{i+j}$ for some x in X; thus,

$$||f_1^{-1}\cdots f_{i+j-1}^{-1}(y)-P_1(x)|| < 2^{-i}+c_{i+j} < 2^{-i}+2^{-2i-2j} < 2^{1-i}$$

and (c) holds.

By Lemma 2, the uniform limit h of $\{f_i\cdots f_1\}_{i=1}^{\infty}$ is a homeomorphism of H onto itself which is a C^∞-diffeomorphism off $P_1(X)$ and which on $P_1(X)$ agrees with $(P_1|X)^{-1}$. Let $g = f_1 h^{-1}$. Since each f_i for $i > 1$ is the identity off the open ε-neighborhood of X, g is the identity off the ε-neighborhood of X, and because

$$||g^{-1}(x)-x|| \leqq \sum_{i=2}^{\infty} ||\bar{f}_i P_1(x)|| \leqq \tfrac{1}{4}\varepsilon,$$

g moves no point as much as ε. It is easy to verify that f and g are the desired homeomorphisms of H.

Lemma 5. *If X is a compact subset of H lying in the open set U, then there is a real-valued function f of H into $[0,1]$ of class C^∞ such that $X = f^{-1}(0)$ and $H \setminus U \subset f^{-1}(1)$.*

Proof: This is an easy generalization from the well-known result in the case that H is finite-dimensional. (Or see [8], chapter V, for a discussion of carriers.)

Lemma 6 (Bessaga). *If H_1 is any closed, infinite-dimensional, linear subspace of the real Hilbert space E, then there is a C^∞-diffeomorphism h of $E\setminus\{0\}$ onto E which is the identity off the unit ball of E centered at the origin and has the property that $(I-h)(E\setminus\{0\})$ is contained in H_1.*

A proof of this lemma may be found in [3].

Lemma 7. *If X is a closed, locally compact subset of the closed, linear subspace H_1 of H and if H_2 is a closed, infinite-dimensional, linear subspace of the orthogonal complement $H_1^\perp$ of H_1, then for any open set U containing X, there is a C^∞-diffeomorphism h of $H\setminus X$ onto H which is the identity off U and has the property that $(I-h)$ $(H\setminus X)$ is contained in H_2.*

Proof: The internal direct sum of two closed, orthogonal, linear subspaces, H_i and H_j, of H will be denoted by H_i+H_j; the symbol "$C+D$" will continue to denote the set of all sums of pairs of elements, one from C and the other from D, for any subsets, C and D, of H. As before, P_i will denote the projection of H onto the closed, linear subspace H_i, and B_i will denote the closed unit ball of H_i centered at the origin. Let H_4 be a one-dimensional, linear subspace of H_2, and let H_3 be its orthogonal complement in H_2. Let e be an element of H_4 of norm one, and let T be a linear isomorphism of H onto H_1+H_3 for which $P_1 T = P_1$.

There is a C^∞-diffeomorphism f of H onto itself such that

1) $f(H_1+H_3) \cap H_1 = T(X) = X$,

2) $f^{-1}(H_1+H_3) \cap (H_1+H_3) = X+H_3 = T(X+H_1^{\perp})$,

3) $P_4 f|y+H_3$ is constant for each y in H_1, and

4) for each y in $(H_1+H_3)\setminus T(U)$, $||P_2 f(y)|| \geqq 1$. The folllowing three paragraphs provide a construction of such a function.

For each x in X, let V_x be a relatively open set in H_1 containing x for which there is a d_x in $(0,1)$ such that $V_x + d_x B_3$ is contained in $T(U)$. By Lemma 3, there exist open covers, $\{U_i\}_{i=1}^{\infty}$ and $\{W_i\}_{i=1}^{\infty}$, of H_1 and a cover $\{X_i\}_{i=1}^{\infty}$ of X by compact subsets of X such that, for each i, $X_i \subset U_i \subset \overline{U}_i \subset W_i$ and such that $\{W_i\}_{i=1}^{\infty}$ is a star-finite refinement of $\{H_1\setminus X\} \cup \{V_x\}_{x\in X}$.

Let, for each i, a_i be a C^∞-function from H_1 into $[0,1]$ for which $a_i^{-1}(0) \supset (H_1\setminus W_i)$ and $a_i^{-1}(1) \supset \overline{U}_i$, unless $U_i \cap X = \emptyset$, in which case $a_i(H_1) = \{0\}$. (If $U_i \cap X \neq \emptyset$, a_i may be obtained from a C^∞-partition of unity of H_1 subordinate to $\{W_i, H_1\setminus U_i)$ by summing all elements which vanish on a neighborhood of $H_1\setminus W_i$.) For each i such that $U_i \cap X \neq \emptyset$, let $x(i)$ be an element of X for which $V_{x(i)} \supset W_i$, and, for each i, let

$$d_i = \begin{Bmatrix} d_{x(i)}, & \text{if } U_i \cap X \neq \emptyset \\ 1, & \text{if } U_i \cap X = \emptyset \end{Bmatrix}.$$

Define a: $H_1 \to [0, \infty)$ by $a(x) = \prod_{i=1}^{\infty} (1+(1/d_i)a_i(x))$, where $\prod$ denotes real multiplication. Let $g : H \to H$ be defined by

$$g(x) = (I-P_3)(x) + a(P_1(x))P_3(x).$$

Now, if y is in $(H_1+H_3)\setminus T(U)$ and $P_1(y)$ is in

$$A = \cup \{U_i | U_i \cap X \neq \emptyset\},$$

then $||P_3 g(y)|| \geqq 1$ because there is an i for which $P_1(y)$ is in U_i and, so, $||P_3(y)|| \geqq d_i$. (Thus,

$$||a(P_1(y))P_3(y)|| \geqq (1+1/d_i)||P_3(y)|| > 1.)$$

By Lemma 5, for each i there is a C^∞-function b_i from H_1 into $[0,1]$ such that $b_i^{-1}(0) = X_i$ and $b_i^{-1}(1) \supset H_1\setminus U_i$, with the proviso that if $X_i = \emptyset$, then $b_i(H_1) = \{1\}$. Let $b : H_1 \to [0,1]$ be defined by $b(x) = \prod_{i=1}^{\infty} b_i(x)$, and note that $b^{-1}(1) \supset H_1\setminus A$; furthermore, $b^{-1}(0) = X$. The function f may now be defined by

$$f(x) = g(x) + b(P_1(x))e.$$

By Lemma 6, there exists a C^∞-diffeomorphism p of $H_1^{\perp}\setminus\{0\}$ onto $H_1^{\perp}$ which is the identity off the unit ball of $H_1^{\perp}$ centered at the origin and has the property that $(I-p)(H_1^{\perp}) \subset H_3$. Let

$h = T^{-1}f^{-1}(P_1+p(I-P_1))f(T|H\setminus X)$. This is the desired diffeomorphism of $H\setminus X$ onto H.

LEMMA 8. *If X is a closed, locally compact subset of H, U is an open subset of H containing X, and ε is a positive real number, then there is a C^∞-diffeomorphism of $H\setminus X$ onto H which is the identity off U and moves no point more than ε.*

PROOF: By Lemma 3, there is a star-finite open cover $\{V_i\}_{i=1}^\infty$ refining $\{H\setminus X, U\}$ and a cover $\{X_i\}_{i=1}^\infty$ of X by compact subsets of X which have the properties that (1) $\bar{V}_i \cap X$ is compact, for each i, and (2) for each i, $X_i \subset V_i$. By Lemma 1, there are three closed, linear subspaces, H_1, H_2, and H_3, of H such that each two are orthogonal, $H = H_1+H_2+H_3$, H_2 and H_3 are infinite-dimensional, and P_1 is a homeomorphism on each compact subset of X.

Let $A_1 = \{V_{i_1}\}$, where i_1 is the least integer i for which $V_i \cap X \neq \emptyset$, and, assuming $A_1, \cdots, A_{n-1}$ to be defined with A_j^* denoting the union of all elements of A_j, let $A_n = \{V_i | V_i \notin \bigcup_{k=1}^{n-1} A_k$, $V_i \cap X \neq \emptyset$, and either $V_i \cap A_{n-1}^* \neq \emptyset$ or i is the least integer for which V_i satisfies the first two conditions$\}$. Let $Y_n = \cup\{X_i | V_i \in A_n\}$. The collection $\{A_n^*\}_{n=1}^\infty$ has the property that $|n-m| > 1$ implies that $A_m^* \cap A_n^* = \emptyset$, and each Y_n is compact.

Let d be the function from the set of pairs of subsets of H to the real numbers defined by $d(A, B) = \inf\{\|a-b\| \,|\, a \in A, b \in B\}$, and for each n, let d_{2n-1} be a positive number less than

$$\min\{(\tfrac{1}{6})\varepsilon,\ d(Y_{2n-2}, H\setminus A_{2n-2}^*),\ d(Y_{2n-1}, H\setminus A_{2n-1}^*),\ d(Y_{2n}, H\setminus Y_{2n}^*)\}.$$

Now, for each n, set Z_{2n-1} to be the closed $\frac{1}{2}d_{2n-1}$-neighborhood of Y_{2n-1} in X, and note that each Z_{2n-1} is compact and its open $\frac{1}{2}d_{2n-1}$-neighborhood lies in A_{2n-1}^*. Consider $\{P_1(Z_{2n-1})\}_{n=1}^\infty$. Because P_1 is an open map and is a homeomorphism on $Z_{2n-3} \cup Z_{2n-1} \cup Z_{2n+1}$, for each n, there is a collection $\{W_{2n-1}^*\}_{n=1}^\infty$ of open sets in H_1 for which $P_1(Z_{2n-1}) \subset W_{2n-1} \subset P_1(A_{2n-1}^*)$ and $W_{2n-3} \cap W_{2n-1} = W_{2n-1} \cap W_{2n+1} = \emptyset$, for each n; furthermore, there is a collection $\{W_{2n}\}_{n=1}^\infty$ of open sets of H_1 for which $P_1(Y_{2n}) \subset W_{2n} \subset P_1(A_{2n}^*)$ and $W_{2n-2} \cap W_{2n} = W_{2n} \cap W_{2n+2} = \emptyset$, for each n. By Lemma 4, there is, for each n, a pair, f_{2n-1} and g_{2n-1}, of homeomorphisms of H onto itself such that f_{2n-1} is a C^∞-diffeomorphism of H which is the identity off $P_1^{-1}(W_{2n-1})$ and is a translation of each hyperplane parallel to H_2+H_3 into itself, g_{2n-1} is a C^∞-diffeomorphism on $H\setminus Z_{2n-1}$, is the identity on the complement of $P_1^{-1}(W_{2n-1})$ and on the complement of the open $\frac{1}{2}d_{2n-1}$-neighborhood of Z_{2n-1}, and moves no point more than $\frac{1}{2}d_{2n-1}$, $f_{2n-1}g_{2n-1}|Z_{2n-1} = P_1|Z_{2n-1}$, and $P_1 g_{2n-1} = P_1$.

Let $d'_{2n-1} \in (0, d_{2n-1})$ be small enough that (a) the open d'_{2n-1}-neighborhood of $f_{2n-1}g_{2n-1}(Y_{2n-1})$ in $f_{2n-1}g_{2n-1}(X)$ lies in $f_{2n-1}g_{2n-1}(Z_{2n-1}) = P_1(Z_{2n-1})$, (b) the open $4d'_{2n-1}$-neighborhood of $f_{2n-1}g_{2n-1}(Y_{2n-j})$ lies in $f_{2n-1}(A^*_{2n-j})$ and $f_{2n-1}g_{2n-1}(A^*_{2n-j})$, for $j = 0$, 1, and 2, and (c) the open $4d'_{2n-1}$-neighborhood of $f_{2n-1}g_{2n-1}(Z_{2n-1})$ lies $P_1^{-1}(W_{2n-1})$. By Lemma 7, there is, for each n, a C^∞-diffeomorphism h_{2n-1} of $H\setminus P_1(Y_{2n-1})$ onto H which is the identity off the open d'_{2n-1}-neighborhood of $P_1(Y_{2n-1})$ and has the property that $(P_1+P_3)h_{2n-1} = P_1+P_3$.

Each h_{2n-1} is the identity on $f_{2n-1}g_{2n-1}\overline{(Y_{2n-2}\setminus Z_{2n-1})}$ and $f_{2n-1}g_{2n-1}\overline{(Y_{2n}\setminus Z_{2n-1})}$ and carries the open d'_{2n-1}-neighborhoods of $f_{2n-1}g_{2n-1}(Y_{2n-2}\setminus Y_{2n-1})$ and $f_{2n-1}g_{2n-1}(Y_{2n}\setminus Y_{2n-1})$ in $H\setminus P_1(Y_{2n-1})$ into $f_{2n-1}(A^*_{2n-2})$ and $f_{2n-1}(A^*_{2n})$, respectively. This is because if z is in H and

$$\inf\{\|z-x\| \,|\, x \in f_{2n-1}g_{2n-1}(Y_{2n-2}\setminus Y_{2n-1})\} < d'_{2n-1},$$

then, by (b), the open $3d'_{2n-1}$-neighborhood of z lies in $f_{2n-1}(A^*_{2n-2})$. Since $z-h_{2n-1}(z)$ is in H_2, if $h_{2n-1}(z) \neq z$, then, as h_{2n-1} is the identity off the open d'_{2n-1}-neighborhood of $P_1(Y_{2n-1})$, $\|z-h_{2n-1}(z)\| < 2d'_{2n-1}$ and, hence, $h_{2n-1}(z)$ is in $f_{2n-1}(A^*_{2n-2})$. The same argument gives that h'_{2n-1} carries the open d'_{2n-1}-neighborhood of $f_{2n-1}g_{2n-1}(Y_{2n}\setminus Y_{2n-1})$ into $f_{2n-1}(A^*_{2n})$.

Let $F_{2n-1} = (f^{-1}_{2n-1}h_{2n-1}f_{2n-1}g_{2n-1})|H\setminus Y_{2n-1}$. By (b) and (c), each F_{2n-1} is the identity off the intersection of $P_1^{-1}(W_{2n-1})$ with A^*_{2n-1} and is a homeomorphism of $H\setminus Y_{2n-1}$ onto H which is a C^∞-diffeomorphism off Z_{2n-1}. Define $F : H\setminus\bigcup_{n=1}^\infty Y_{2n-1} \to H$ by $F(x) = \text{limit}_{n\to\infty} F_{2n-1}\cdots F_1(x)$, for each x in $H\setminus\bigcup_{n=1}^\infty Y_{2n-1}$. Since $\{\overline{A^*_{2n-1}}\}_{n=1}^\infty$ is a locally finite collection of sets (by virtue of of the fact that $\{V_i\}_{i=1}^\infty$ is star-finite) and the sets A^*_{2n-1} are pairwise disjoint, F is a homeomorphism which on $H\setminus\bigcup_{n=1}^\infty Z_{2n-1}$ is a C^∞-diffeomorphism. Because each A^*_n lies in U, F is the identity off U.

Consider, now, the collection of sets

$$\{Z_{2n} = f_{2n-1}f_{2n+1}F(Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1})\}_{n=1}^\infty.$$

Each of these sets lies, except for a subset with compact closure, in H_1+H_2, and $(P_1+P_2)|Z_{2n}$ is a homeomorphism of Z_{2n} into H_1+H_2. This statement may be verified as follows: Because

$Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1})$ is a closed, locally compact subset of $H\setminus\bigcup_{n=1}^{\infty} Y_{2n-1}$, Z_{2n} is a closed, locally compact subset of H. Because F_{2n-1} is the identity off A^*_{2n-1}, for each n, these functions commute, and

$$F|Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1}) = F_{2n+1}F_{2n-1}|Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1}).$$

By the condition on the sets $\{W_{2m-1}\}_{m=1}^{\infty}$, each of the functions h_{2n+1}, f_{2n+1}, and g_{2n+1} commutes with all of the functions h_{2n-1}, f_{2n-1}, and g_{2n-1}. Therefore

$$Z_{2n} = h_{2n+1}h_{2n-1}f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1})),$$

and since

$$f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}((Z_{2n-1}\cup Z_{2n+1})\setminus(Y_{2n-1}\cup Y_{2n+1}))$$

lies in H_1 and $h_{2n+1}h_{2n-1}(H_1)$ lies in H_1+H_2,

$$f_{2n+1}f_{2n-1}F((Z_{2n-1}\cup Z_{2n+1})\setminus(Y_{2n-1}\cup Y_{2n+1}))$$

lies in H_1+H_2. However, Z_{2n} is the union of this set with

$$f_{2n+1}f_{2n-1}F(\overline{Y_{2n}\setminus(Z_{2n-1}\cup Z_{2n+1})}),$$

which is compact; so, $\overline{Z_{2n}\setminus(H_1+H_2)}$ is compact. To see that $(P_1+P_2)|Z_{2n}$ is a homeomorphism of Z_{2n} into H_1+H_2, observe that, from the definitions of the functions involved,

$$P_1|Z_{2n} = (P_1g_{2n-1}^{-1}f_{2n-1}^{-1}g_{2n+1}^{-1}f_{2n+1}^{-1}h_{2n-1}^{-1}h_{2n+1}^{-1})|Z_{2n},$$

and

$$(g_{2n-1}^{-1}f_{2n-1}^{-1}g_{2n+1}^{-1}f_{2n+1}^{-1}h_{2n-1}^{-1}h_{2n+1}^{-1})|Z_{2n}$$

is a homeomorphism of Z_{2n} onto $Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1})$. Because P_1 is, by Lemma 1, a homeomorphism on $Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1})$, P_1 is a homeomorphism on Z_{2n}. Therefore, since

$$(P_1+P_2)|Z_{2n} = (P_1|(P_1+P_2)(Z_{2n}))^{-1}(P_1|Z_{2n}),$$

$(P_1+P_2)|Z_{2n}$ is also a homeomorphism.

For each n, let $d_{2n}\in(0, \min\{d'_{2n-1}, d'_{2n+1}\})$ be small enough that the open d_{2n}-neighborhood of

$$f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(Y_{2n}\setminus(Z_{2n-1}\cup Z_{2n+1}))$$

is contained in $f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(A^*_{2n})$. This requirement is sufficient to guarantee that the open d_{2n}-neighborhood of Z_{2n} also lies in $f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(A^*_{2n})$. This follows because if a point x of H is within d_{2n} of a point y of Z_{2n}, then in the case that y is in

$$f_{2n-1}f_{2n+1}F(\overline{Y_{2n}\setminus(Z_{2n-1}\cup Z_{2n+1})})$$
$$= f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(\overline{Y_{2n}\setminus(Z_{2n-1}\cup Z_{2n+1})}),$$

the specific choice of d_{2n} shows that x is in $f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(A^*_{2n})$, while in the case that y is in $f_{2n-1}f_{2n+1}F(Z_{2n\pm1}\setminus(Y_{2n\pm1}\cup Y_{2n\pm2}))$, then, by condition (c) on the choice of the set $\{d'_{2m-1}\}_{m=1}^{\infty}$, both x and y lie in $P_1^{-1}(W_{2n\pm1})$, so $h_{2n\mp1}(y) = y$, and from the fact that $\|h^{-1}_{2n\pm1}(y)-y\| < d'_{2n\pm1}$, it is true that $\|x-h^{-1}_{2n\pm1}(y)\| < 4d'_{2n\pm1}$, which, from conditions (b) and (c) on the choice of $d'_{2n\pm1}$ and the fact that $f_{2n\mp1}$ and $g_{2n\mp1}$ are the identity off $P_1^{-1}(W_{2n\mp1})$, gives that x is in $f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(A^*_{2n})$.

Lemma 4 gives a collection $\{f_{2n}, g_{2n}\}_{n=1}^{\infty}$ of pairs of homeomorphisms of H onto itself such that each f_{2n} is a C^∞-diffeomorphism of H, $f_{2n}(x+y) = f_{2n}(x)+y$ for each x in H and y in H_3, $(P_1+P_2)f_{2n} = P_1+P_2 = (P_1+P_2)g_{2n}$, f_{2n} and g_{2n} are the identity off $P_1^{-1}(W_{2n})$, g_{2n} is the identity off the open $\frac{1}{2}d_{2n}$-neighborhood of Z_{2n}, g_{2n} moves no point more than d_{2n}, and

$$f_{2n}g_{2n}|Z_{2n} = (P_1+P_2)|Z_{2n}.$$

Lemma 7 gives a collection $\{h_{2n}\}_{n=1}^{\infty}$ of functions such that each h_{2n} is a C^∞-diffeomorphism of $H\setminus f_{2n}g_{2n}(Z_{2n})$ onto H, $h_{2n}(x)-x$ is in H_3 for each x in $H\setminus f_{2n}g_{2n}(Z_{2n})$, for each n, and each h_{2n} is the identity off the intersection of $P_1^{-1}(W_{2n})$ with the image under f_{2n} of the open d_{2n}-neighborhood of Z_{2n} and with the open d_{2n}-neighborhood of H_1+H_2. Let

$$F_{2n} = (f^{-1}_{2n-1}f^{-1}_{2n+1}f^{-1}_{2n}h_{2n}f_{2n}g_{2n}f_{2n+1}f_{2n-1})$$
$$|H\setminus F(Y_{2n}\setminus(Y_{2n-1}\cup Y_{2n+1})),$$

and note that F_{2n} is the identity off

$$P_1^{-1}(W_{2n}) \cap (A^*_{2n-1}\cup A^*_{2n}\cup A^*_{2n+1}).$$

(This is true because f_{2n}, g_{2n}, and h_{2n} are the identity off $P_1^{-1}(W_{2n})$, g_{2n} is the identity off the open d_{2n}-neighborhood of Z_{2n}, which lies in $f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(A^*_{2n})$, and h_{2n} is the identity off the image under f_{2n} of the open d_{2n}-neighborhood of Z_{2n}, which together yield that F_{2n} is the identity off

$$f^{-1}_{2n-1}f^{-1}_{2n+1}(P_1^{-1}(W_{2n})\cap f_{2n+1}g_{2n+1}f_{2n-1}g_{2n-1}(A^*_{2n})),$$

which is $P_1^{-1}(W_{2n})\cap g_{2n+1}g_{2n-1}(A^*_{2n})$. Since $g_{2n\pm1}$ is the identity off $A^*_{2n\pm1}$, F_{2n} is the identity off $P_1^{-1}(W_{2n})\cap(A^*_{2n-1}\cup A^*_{2n}\cup A^*_{2n+1})$.) Thus, the F_{2n}'s are the identity off a collection of pairwise disjoint open sets, the closures of which form a locally finite collection of sets in H, so $G(x) = \text{limit}_{n\to\infty} F_{2n} \cdots F_2(x)$ is a C^∞-diffeomorphism of $H \setminus F(X \setminus \bigcup_{n=1}^\infty Y_{2n-1})$ onto H which is the identity off U. Let $h = G(F|H\setminus X)$. Now h is a C^∞-diffeomorphism of $H\setminus X$ onto H which is the identity off U. To verify that $||h(x)-x|| < \varepsilon$ for each x in $H\setminus X$, observe that if x is in $A^*_{2n-1} \cap P_1^{-1}(W_{2n-1})$, then

$$\begin{aligned}||F(x)-x|| = ||F_{2n-1}(x)-x|| &= ||f_{2n-1}^{-1}h_{2n-1}f_{2n-1}g_{2n-1}(x)-x|| \\ &< ||f_{2n-1}^{1-}h_{2n-1}f_{2n-1}(g_{2n-1}(x))-g_{2n-1}(x)||+d_{2n-1},\end{aligned}$$

and since $h_{2n-1}(y)-y$ is in H_2, for all y in the domain of h_{2n-1}, and $f_{2n-1}(x+P_2(y)) = f_{2n-1}(x)+P_2(y)$ for each x and y in H, $||F(x)-x|| < d_{2n-1}+2d'_{2n-1}$. If, on the other hand, x is not in any $A^*_{2n-1} \cap P_1^{-1}(W_{2n-1})$, then $F(x) = x$. Now, if y is in

$$P_1^{-1}(W_{2n}) \cap (A^*_{2n-1} \cap A^*_{2n} \cup A^*_{2n+1}),$$

then

$$||G(y)-y|| = ||f_{2n-1}^{-1}f_{2n+1}^{-1}f_{2n}^{-1}h_{2n}f_{2n}g_{2n}f_{2n+1}f_{2n-1}(y)-y||,$$

and since $f_{2n}(x)-x$, $g_{2n}(x)-x$, and $h_{2n}(x)-x$ all lie in H_3, for each x in the domains of these functions,

$$\begin{aligned}&||f_{2n-1}^{-1}f_{2n+1}^{-1}f_{2n}^{-1}h_{2n}f_{2n}g_{2n}f_{2n+1}f_{2n-1}(y)-y|| \\ &\quad = ||f_{2n}^{-1}h_{2n}f_{2n}g_{2n}(f_{2n+1}f_{2n-1}(y))-f_{2n+1}f_{2n-1}(y)|| \leqq 2d_{2n}+d_{2n};\end{aligned}$$

therefore, for each x in $H\setminus X$,

$$||h(x)-x|| < 3d_{2m}+2d'_{2n-1}+d_{2n-1},$$

for some m and n, which is less than ε.

THEOREM 1. *Eeach matrizable C^p-manifold modelled on separable, infinite-dimensional Hilbert spaces is C^p-diffeomorphic to the complement of each of its closed, locally compact subsets; moreover, the diffeomorphism may be required to be the identity off any open set containing the locally compact set in question and may be limited by any open cover of the manifold.*

PROOF: Let M, X, U, and G be the manifold, locally compact set, open set, and open cover in question. Because the diffeomorphism to be constructed may be defined on each component of M separately, it may be assumed that M is connected and, hence, separable and modelled on the separable, infinite-dimensional Hilbert space H.

It suffices to prove the following statement (Statement A):

If $V_0, \cdots, V_n$ are open subsets of H and $X_0, \cdots, X_n$ are locally compact subsets of $V_0, \cdots,$ and V_n, respectively, which are relatively closed in $\bigcup_{i=0}^n V_i$, then there is a C^∞-diffeomorphism of $(\bigcup_{i=0}^n V_i \setminus X_0$ onto $\bigcup_{i=0}^n V_i$ which is the identity off $V_0 \setminus X_0$ and carries $X_i \setminus X_0$ into V_i, for each $i = 1, \cdots, n$. This is true because then using the definition of M and Lemma 3, there are collections, $\{V_i\}_{i=1}^\infty$, $\{W_i\}_{i=1}^\infty$, and $\{X_i\}_{i=1}^\infty$, of subsets of M such that $\{V_i\}_{i=1}^\infty$ is an open cover of M which is a star-finite refinement of G and is a refinement of $\{U, M \setminus X\}$, each element of which is C^p-diffeomorphic to an open subset of H by a function f_i, each X_i is a compact subset of X, $X = \bigcup_{i=1}^\infty X_i$, $\{W_i\}_{i=1}$ is an open cover of M, and, for each i, $X_i \subset W_i \subset \overline{W}_i \subset V_i$. Now, Statement A gives a C^∞-diffeomorphism h_1 of $f_1(V_1 \setminus X_1)$ onto $f_1(V_1)$ which is the identity off $f_1(W_1)$ and carries $f_1(X_i \setminus X_1) \cap V_1)$ into $f_1(W_i \cap V_1)$, for each i. Let g_1 be the natural extension of $f_1^{-1}h_1(f_1|V_1 \setminus X_1)$ to $M \setminus X_1$. Inductively, for each $i > 1$, let h_i be a C^∞-diffeomorphism of

$$f_i(g_{i-1} \cdots g_1(V_i \setminus \bigcup_{j \leqq i} X_j) \cap V_i)$$

onto

$$f_i(g_{i-1} \cdots g_1(V_i \setminus \bigcup_{j < i} X_j) \cap V_i)$$

which is the identity off

$$f_i(g_{i-1} \cdots g_1(W_i \setminus \bigcup_{j \leqq i} X_j) \cap W_i)$$

and carries

$$f_i(g_{i-1} \cdots g_1(X_k \setminus \bigcup_{i \leqq j} X_j) \cap V_i)$$

into $f_i(W_k \cap V_i)$, for each $k > i$. Define g_i to be the natural extension of

$$f_i^{-1}h_i(f_i|V_i \cap g_{i-1} \cdots g_1(V_i \setminus \bigcup_{j \leqq i} X_j))$$

to $g_{i-1} \cdots g_1(M \setminus \bigcup_{j \leqq i} X_j)$. Require that if $X_i = \emptyset$, then h_i, hence g_i, be the identity. Since g_i is the identity except when $V_i \subset U$ and since $g_i \cdots g_1(x) \neq g_{i-1} \cdots g_1(x)$ implies x is in V_i, there is a well-defined C^p-diffeomorphism $g(x) = \text{limit}_{i \to \infty} g_i \cdots g_1(x)$ from $M \setminus X$ onto M. The function g^{-1} is the identity off U, and, because $\{V_i\}_{i=1}^\infty$ is a refinement of G, g^{-1} is limited by G.

In order to prove Statement A, first note that Lemma 8 easily implies that (Statement B) if U and V are two open subset of H, V is contained in U, and Y is a locally compact subset of V

which is relatively closed in U, then there is a C^∞-diffeomorphism of $U \setminus Y$ onto U which is the identity off V. To see this, let, by Lemma 3, $\{0_i\}_{i=1}^\infty$ and $\{Y_i\}_{i=1}^\infty$ be collections of subsets of U such that $\{0_i\}_{i=1}^\infty$ is a star-finite open cover of U refining $\{V, U \setminus Y\}$, $\{Y_i\}_{i=1}^\infty$ is a cover of Y by compact subsets of it, and each Y_i lies in 0_i. Let $\varepsilon_i = 1/2n_i \min \{d(Y_j, U \setminus 0_j) | 0_j \cap 0_i \neq \emptyset\}$, where n_i is the number of 0_j's which intersect 0_j. Now, by Lemma 8, there is a C^∞-diffeomorphism h_1 of $H \setminus Y_1$ onto H which is the identity off 0_1 and moves no point more than ε_1. Define h_i, for $i > 1$, inductively so that each h_i is a C^∞-diffeomorphism of

$$H \setminus h_{i-1} \cdots h_1(Y_i \setminus \bigcup_{j<i} Y_j)$$

onto H, is the identity off

$$0_i \cap h_{i-1} \cdots h_1(0_i \setminus \bigcup_{j \leq i} Y_j),$$

moves no point more than ε_i, and is the identity if $X_i = \emptyset$. Since $\{0_i\}_{i=1}^\infty$ is star-finite, no point has infinitely many distinct successive images and $h(x) = \text{limit}_{i \to \infty} h_i \cdots h_1(x)$ is a well-defined function which is the identity off $V \setminus Y$ and which is a C^∞-diffeomorphism on $U \setminus Y$.

Now proceeding with the proof of Statement A, let $V_0, \cdots, V_n$ be open subsets of H and $X_0, \cdots, X_n$ be locally compact subsets of $V_0, \cdots, V_n$, respectively, which are relatively closed in $\bigcup_{i=0}^n V_i$. For each $j = 0, \cdots, n$, let $Q_j = \{Z | Z = \bigcap_{i=0}^{n-j} X_{p(i)} \setminus \bigcup_{i=n-j+1}^n X_{p(i)}$, for some permutation p of $\{0, \cdots, n\}$ carrying 0 to $0\}$; let $Q = \bigcup_{j=0}^n Q_j$, and order $Q = \{Z_m\}_{m=1}^{2^n}$ in such a manner that if $j < k$, then all elements of Q_j precede those of Q_k. For each $m = 1, \cdots, 2^n$, let $N_m = \bigcap_{i=0}^{n-j} V_{p(i)}$, where Z_m is in Q_j and p is a permutation for which $Z_m = \bigcap_{i=0}^{n-j} X_{p(i)} \setminus \bigcup_{i=n-j+1}^n X_{p(i)}$. Let Q_k^* denote the union of the elements of Q_k. For each $j > 0$, the elements of Q_j form a set of pairwise disjoint, relatively closed, locally compact subsets of $\bigcup_{i=0}^n V_i \setminus \bigcup_{k=0}^{j-1} Q_k^*$, and each Z_m in Q_j lies in N_m. Therefore, for each $j > 0$, there exists a collection of pairwise disjoint open sets M_m in $\bigcup_{i=0}^n V_i \setminus \bigcup_{k=0}^{j-1} Q_k^*$, one for each Z_m in Q_j, such that for each m, $Z_m \subset M_m \subset N_m \setminus \bigcup_{i=n-j+1}^n X_{p(i)}$, where j is such that Z_m is in Q_j and p is a permutation defining Z_m as above. By Statement B, there is a C^∞-diffeomorphism h_1 of $\bigcup_{i=0}^n V_i \setminus Z_1$ onto $\bigcup_{i=0}^n V_i$ which is the identity off N_1. Inductively, for $1 < m \leqq 2^n$, let h_m be a C^∞-diffeomorphism of

$$\bigcup_{i=0}^n V_i \setminus h_{m-1} \cdots h_1(Z_m)$$

onto $\bigcup_{i=0}^{n} V_i$ which is the identity off $h_{m-1} \cdots h_1(M_m) \cap N_m$. Let $h = h_{2^n} \cdots h_1$. This is a C^∞-diffeomorphism of $\bigcup_{i=0}^{n} V_i \setminus X_0$ onto $\bigcup_{i=0}^{n} V_i$ which is the identity off V_0; furthermore, if x is in $X_i \setminus X_0$ for some $i = 1, \cdots$, or n, then $h_m \cdots h_1(x) \neq h_{m-1} \cdots h_1(x)$ implies that x is in M_m and $h_m \cdots h_1(x)$ is in N_m, so because M_m must lie in V_i, $h_m \cdots h_1(x)$ must also lie in V_i, and by induction, so must $h(x)$. Therefore, Statement A, and hence Theorem 1, is proved.

REMARK. Since all of the functions f constructed in the Lemmas may easily be required to have the property that for a given one-dimensional linear subspace H_0 of H, $P_0 f = P_0$, where P_0 is the projection of H onto H_0, the proof of Theorem 1 easily generalizes to manifolds with boundary and the following corollary is true, since each paracompact Hilbert manifold is metrizable [7].

COROLLARY 1. *Each paracompact C^p-manifold with boundary modelled on separable, infinite-dimensional Hilbert spaces is C^p-diffeomorphic to the complement of each of its closed, locally compact subsets; moreover, the diffeomorphism may be required to be the identity off any open set containing the locally compact set in question and may be limited by any open cover of the manifold.*

In fact, since the above remark applies to the orthogonal complement of any infinite-dimensional linear subspace, one may require the diffeomorphism of Theorem 1 and Corollary 1 to carry a given closed submanifold into itself provided that each of its components which intersects the locally compact set in question is infinite-dimensional.

REFERENCES

R. D. ANDERSON

[1] On a theorem of Klee, Proc. AMS 17 (1966), 1401—4.

R. D. ANDERSON, D. W. HENDERSON and J. E. WEST

[2] Negiligible subsets of infinite-dimensional manifolds, Compositio Math. (to appear).

Cz. BESSAGA

[3] Any Hilbert space of infinite dimension is diffeomorphic with its unit sphere, Bull. Acad. Polon. Sci. Ser., Sci. Math. Astron. Phys. 14 (1966), 27—30.

S. KAPLAN

[4] Homology properties of arbitrary subsets of Euclidean spaces, Trans. AMS 62 (1947), 248—271.

V. L. KLEE, JR.

[5] Convex bodies and periodic homeomorphisms in Hilbert space, Trans. AMS 74 (1953), 10—43.

S. LANG

[6] Introduction to Differentiable Manifolds, Interscience, New York, 1962.

R. S. PALAIS

[7] Homotopy theory of infinite dimensional manifolds, Topology 5 (1966), 1—16.

R. S. PALAIS

[8] Lectures on the Differential Topology of Infinite Dimensional Manifolds, Mimeographed Notes by S. Greenfield, Brandeis University, 1964—1965.

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