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W. A. BEYER

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Inner product-magnitude-preserving transformations in complex Hilbert spaces [1]

by

W. A. Beyer

Abstract

Let B be a continuous transformation of a complex Hilbert space $\mathscr{H}$ onto a complex Hilbert space $\mathscr{R}$ such that

$$|(B\varphi, B\psi)| = |(\varphi, \psi)|$$

for all φ and ψ in $\mathscr{H}$. Let $\{\psi_i\}$ be a complete orthonormal set in $\mathscr{H}$. There exist complex constants $\{c_i\}$ of magnitude 1, two complex-valued continuous functions f_1, f_2 on $\mathscr{H}$, each of magnitude 1, and a subspace $\mathscr{H}_1$ of $\mathscr{H}$ such that for all $\varphi = \Sigma a_i \psi_i$ in $\mathscr{H}$:

$$B(\Sigma a_i \psi_i) = f_1(\Sigma a_i \psi_i) \sum_{\psi_i \in \mathscr{H}_1} a_i c_i B\psi_i + f_2(\Sigma a_i \psi_i) \sum_{\psi_i \in \mathscr{H}_1^{\perp}} \bar{a}_i c_i B\psi_i.$$

1. Introduction

The purpose of this note is to investigate continuous transformations which map a complex Hilbert space $\mathscr{H}$ onto a complex Hilbert space $\mathscr{R}$ and preserve the magnitude of the inner product of any two vectors $\varphi, \psi \in \mathscr{H}$. Such transformations are important in the study of time reversal symmetries in quantum mechanics. These transformations are discussed in Wigner's book (1959), page 233 and Wigner (1939), page 150.

2. Theorem and proof

$\bar{a}$ denotes the conjugate of the complex number a; $|a|$ denotes its absolute value. (φ, ψ) denotes the inner product of two vectors φ, ψ in a complex Hilbert space $\mathscr{H}$. If $\mathscr{H}_1 \subset \mathscr{H}$, then $\mathscr{H}_1^{\perp} = \{\varphi | \varphi \in \mathscr{H},\ (\varphi, \psi) = 0$ for all $\psi \in \mathscr{H}_1\}$. We assume the subscripts labelling the basis elements are well ordered.

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THEOREM. *Let B be a continuous transformation of a complex Hilbert space $\mathscr{H}$ onto a complex Hilbert space $\mathscr{R}$ such that $|(B\varphi, B\psi)| = |(\varphi, \psi)|$ for all φ and ψ in $\mathscr{H}$. Let $\{\psi_i\}$ be a complete orthonormal set in $\mathscr{H}$. There exist complex constants $\{c_i\}$ of magnitude 1, two complex-valued continuous functions f_1, f_2 on $\mathscr{H}$, each of magnitude 1, and a subspace $\mathscr{H}_1$ of $\mathscr{H}$ such that for all $\varphi = \Sigma a_i\psi_i$ in $\mathscr{H}$:*

$$B(\Sigma a_i\psi_i) = f_1(\Sigma a_i\psi_i)\sum_{\psi_i\in\mathscr{H}_1} a_i c_i B\psi_i + f_2(\Sigma a_i\psi_i)\sum_{\psi_i\in\mathscr{H}_1^{\perp}} \bar{a}_i c_i B\psi_i.$$

REMARK. The theorem says that B is "almost linear" on $\mathscr{H}_1$ and "almost antilinear" on $\mathscr{H}_1^{\perp}$.

PROOF. The set $\{B\psi_i\}$ is orthonormal since

$$|(B\psi_i, B\psi_j)| = |(\psi_i, \psi_j)| = \delta_{ij}$$

and thus $(B\psi_i, B\psi_j) = \delta_{ij}$. Suppose φ is in $\mathscr{R}$ and $(\varphi, B\psi_i) = 0$ for all i. Let ψ in $\mathscr{H}$ be such that $B\psi = \varphi$. Then

$$0 = |(\varphi, B\psi_i)| = |(\psi, \psi_i)|$$

for all i and therefore $\psi = \theta$ (the null vector). Hence $\varphi = 0$. Therefore $\{B\psi_i\}$ is complete in $\mathscr{R}$. Hence $\mathscr{H}$ and $\mathscr{R}$ are isomorphic.

Put

$$\psi^i = \psi_1 + \psi_i.$$

Since $(B\psi^i, B\psi_k) = (\psi^i, \psi^k) = 0$ for $k \neq 1$ or i, there exist constants a_1^i and a_2^i such that

$$B\psi^i = a_1^i B\psi_1 + a_2^i B\psi_i,$$

with

$$|a_1^i| = |(B\psi^i, B\psi_1)| = |(\psi^i, \psi_1)| = 1$$

and

$$|a_2^i| = 1.$$

Put

$$c_i = a_1^i / a_2^i$$

and

$$B\varphi = \Sigma b_i B\psi_i \qquad (\varphi = \Sigma a_i\psi_i \in \mathscr{H}).$$

Then

$$|b_j| = |(B\varphi, B\psi_j)| = |(\varphi, \psi_j)| = |a_j|.$$

Now suppose $a_1 \neq 0$. We have

$$\begin{aligned}|a_1+a_i| &= |(\varphi,\, \psi_1+\psi_i)| = |(B\varphi,\, B(\psi_1+\psi_i))| \\ &= |(B\varphi,\, a_1^i B\psi_1+a_2^i B\psi_i)| \\ &= \left|\left(\sum_{j=1}^{\infty} b_j B\psi_j,\, a_1^i B\psi_1+a_2^i B\psi_i\right)\right| \\ &= |b_1\overline{a_1^i}+b_i\overline{a_2^i}|.\end{aligned}$$

Hence

$$(a_1+a_i,\, a_1+a_i) = (b_1\overline{a_1^i}+b_i\overline{a_2^i},\, b_1\overline{a_1^i}+b_i\overline{a_2^i})$$

or

$$|a_1|^2+\bar{a}_1a_i+\bar{a}_ia_1+|a_i|^2 = |b_1a_1^i|^2+\bar{b}_1\bar{a}_1^ib_ia_2^i+\bar{b}_i\bar{a}_2^ib_1a_1^i+|b_ia_2^i|^2,$$

or

$$\bar{a}_1a_i+\bar{a}_ia_1 = \overline{b_1a_1^i}b_ia_2^i+\overline{b_ia_2^i}b_1a_1^i.$$

Observe that

$$a_2^ib_i\overline{a_2^ib_i} = a_i\bar{a}_i.$$

So

$$\bar{a}_1a_i+\bar{a}_ia_1 = \overline{b_1a_1^i}b_ia_2^i+a_i\bar{a}_i(a_2^ib_i)^{-1}b_1a_1^i$$

or

$$(\bar{a}_1a_i+\bar{a}_ia_1)a_2^ib_i = \overline{b_1a_1^i}(b_ia_2^i)^2+a_i\bar{a}_ib_1a_1^i$$

or

$$\overline{b_1a_1^i}(b_ia_2^i)^2-(\bar{a}_1a_i+\bar{a}_ia_1)a_2^ib_i+b_1a_1^i|a_i|^2 = 0. \tag{1}$$

Since

$$(\overline{b_1a_1^i})(b_1a_1^i) = |a_1|^2$$

or

$$(\overline{b_1a_1^i}) = |a_1|^2(b_1a_1^i)^{-1}$$

we obtain from (1) that

$$|a_1|^2(b_1a_1^i)^{-1}(b_ia_2^i)^2-(\bar{a}_1a_i+\bar{a}_ia_1)a_2^ib_i+b_1a_1^i|a_i|^2 = 0$$

or

$$|a_1|^2(b_ia_2^i)^2-[\bar{a}_1a_i+\bar{a}_ia_1](a_2^ib_i)(b_1a_1^i)+(b_1a_1^i)^2|a_i|^2 = 0$$

or

$$(a_1b_ia_2^i-b_1a_1^ia_i)(\bar{a}_1b_ia_2^i-b_1a_1^i\bar{a}_i) = 0.$$

Therefore either

$$b_i = a_i\frac{a_1^i}{a_2^i}\frac{b_1}{a_1} = a_ic_i\frac{b_1}{a_1}$$

or

$$b_i = \bar{a}_i\frac{a_1^i}{a_2^i}\frac{b_1}{\bar{a}_1} = \bar{a}_ic_i\frac{b_1}{\bar{a}_1}.$$

If $a_1 = 0$, replace in the above calculation the subscript 1 by the first subscript (in the well ordering) $l = i$ for which $a_i \neq 0$.

Thus we conclude that if $\varphi = \Sigma a_i \psi_i$, then (replacing b_i by $b_i c_i$):

$$B\varphi = \Sigma b_i c_i B\psi_i \tag{2}$$

where

(i) $|c_i| = 1$;

(ii) if l is the first index i for which $a_i \neq 0$ in a well-ordering of the subscripts i of the basis vectors $\{\psi_i\}$, then

$$b_l = g(\varphi)$$

with $g(\varphi)$ a continuous mapping of $\mathscr{H}$ to the complex plane and

$$|g(\varphi)| = |a_l|;$$

(iii) for $i \neq l$ either

$$\frac{b_i}{b_l} = \frac{a_i}{a_l} \tag{3}$$

or

$$\frac{b_i}{b_l} = \overline{\left(\frac{a_i}{a_l}\right)}. \tag{4}$$

For fixed i and l, a_i/a_l and b_i/b_l define a continuous and onto mapping of

$$\mathscr{H}_l = \{\varphi | \varphi \in \mathscr{H}, a_l(\varphi) \neq 0, a_1(\varphi) = a_2(\varphi) = \cdots = a_{l-1}(\varphi) = 0\}$$

onto the complex plane C. Let

$$\mathscr{H}_l^U = \{\varphi | \varphi \in \mathscr{H}_l, \operatorname{Im} a_i/a_l > 0\},$$
$$\mathscr{H}_l^L = \{\varphi | \varphi \in \mathscr{H}_l, \operatorname{Im} a_i/a_l < 0\},$$
$$C^U = \{z | \operatorname{Im} z > 0\},$$

and

$$C^L = \{z | \operatorname{Im} z < 0\}.$$

The function b_i/b_l, being continuous, must map $\mathscr{H}_l^U$ either onto all of C^U or all of C^L. Hence in all of $\mathscr{H}_l^U$, b_i/b_l can have only one of the two forms, (3) or (4). If in $\mathscr{H}_l^L$, b_i/b_l has the other form, then b_i/b_l is not an onto mapping. Let $\mathscr{H}_1$ be the subspace in which form (3) holds. Then from (2) we have

$$B(\Sigma a_i \varphi_i) = \frac{g(\varphi)}{a_l} \sum_{\psi_i \in \mathscr{H}_1} a_i c_i B\psi_i + \frac{g(\varphi)}{\bar{a}_l} \sum_{\psi_i \in \mathscr{H}_1^\perp} \bar{a}_i c_i B\psi_i.$$

Putting $f_1(\varphi) = g(\varphi)/a_l(\varphi)$ and $f_2(\varphi) = g(\varphi)/\bar{a}_l(\varphi)$ concludes the proof of the theorem.

3. Corollaries

A transformation $U : \mathscr{H} \xrightarrow{\text{onto}} \mathscr{H}$ is unitary if

$$(U\varphi_1, U\varphi_2) = (\varphi_1, \varphi_2)$$

for all φ_1, φ_2 in $\mathscr{H}$. A transformation $A : \mathscr{H} \xrightarrow{\text{onto}} \mathscr{H}$ is unitary conjugate if $(A\varphi_1, A\varphi_2) = (\overline{\varphi_1}, \overline{\varphi_2})$ for all φ_1, φ_2 in $\mathscr{H}$. A transformation $C : \mathscr{H} \xrightarrow{\text{onto}} \mathscr{H}$ is a multiplier transformation if for each φ in $\mathscr{H}$ there exists a complex constant $a_\varphi \neq 0$ such that $C\varphi = a_\varphi \varphi$.

COROLLARY 1. *For each inner product magnitude preserving continuous transformation* B: $\mathscr{H} \xrightarrow{\text{onto}} \mathscr{H}$ *such that* $\mathscr{H}_1^\perp = \theta$ *there exists a multiplier transformation* C^U *such that* $C^U B$ *is unitary.*

COROLLARY 2. *For each inner product magnitude preserving continuous transformation* B: $\mathscr{H} \xrightarrow{\text{onto}} \mathscr{H}$ *such that* $\mathscr{H}_1 = \theta$ *there exists a multiplier transformation* C^A *such that* $C^A B$ *is unitary conjugate.*

$C^U(C^A)$ is the operator which multiplies all vectors of the form $aB\psi_i$ by c_i^{-1} followed by a multiplication of all vectors $\varphi = \Sigma a_i \psi$ in $\mathscr{H}$ by $f_1^{-1}(\Sigma a_i \varphi_i)$ $\left(f_2^{-1}(\Sigma a_i \varphi_i)\right)$.

4. Remarks

1. It follows easily that a unitary transformation is linear: $U(a\varphi_1 + b\varphi_2) = aU(\varphi_1) + bU(\varphi_2)$. A unitary conjugate transformation A is semi-linear (or anti-linear):

$$A(a\varphi_1 + b\varphi_2) = \bar{a}A(\varphi_1) + \bar{b}A(\varphi_2).$$

2. For literature on semi-linear (or anti-linear) transformations see Stone (1932), page 357, Jacobson (1943), page 26, and Dunford and Schwartz (1963), page 1231.

3. A unitary conjugate transformation K is called a conjugation if $K^2 = I$. A transformation A: $\mathscr{H} \xrightarrow{\text{onto}} \mathscr{H}$ is unitary conjugate if and only if there is a conjugation K and a unitary transformation U such that

$$A = UK.$$

This is shown by observing that for any conjugation K, AK is unitary since

$$(AK\varphi, AK\psi) = (\overline{K\varphi}, \overline{K\psi}) = (\varphi, \psi).$$

Hence $AK = U$ and $A = UK^{-1} = UK$.

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University of California
Los Alamos Scientific Laboratory
Los Alamos, New Mexico