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Analytic sheaf cohomology with compact supports

by

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Among many other results Andreotti and Grauert proved in [2] the following:

(1) Suppose n is a non-negative integer and $\mathscr{F}$ is a coherent analytic sheaf on a Stein space X such that codh $\mathscr{F} \geqq n$ (where codh $\mathscr{F}$ = homological codimension of $\mathscr{F}$). Then $H^p_*(X, \mathscr{F}) = 0$ for $p < n$. (Cf. Prop. 25, [2]).

Reiffen proved in [6] the following:

(2) Suppose n is a non-negative integer and $\mathscr{F}$ is a coherent analytic sheaf on a complex space X such that dim Supp $\mathscr{F} \leqq n$ (where Supp $\mathscr{F}$ = support of $\mathscr{F}$). Then $H^p_*(X, \mathscr{F}) = 0$ for $p > n$. (Cf. Satz 3, [6]).

In this note we prove converses of these statements:

Theorem 1. *Suppose n is a non-negative integer. If $\mathscr{F}$ is a coherent analytic sheaf on an open subset G of a Stein space X and $H^p_*(G, \mathscr{F}) = 0$ for $p < n$, then* codh $\mathscr{F}_x \geqq n$ *for $x \in G$.*

Theorem 2. *Suppose n is non-negative integer, $\mathscr{F}$ is a coherent analytic sheaf on a Stein space X, and G is an open subset of X. If $H^p_*(G, \mathscr{F}) = 0$ for $p > n$, then* dim $(G \cap \text{Supp}\, \mathscr{F}) \leqq n$.

For the proofs of Theorems 1 and 2 we need the following Lemmata:

Lemma 1. *Suppose G is an open subset of $\mathbb{C}^N$, $x \in G$, and A is an at most countable subset of $G - \{x\}$. Then there exists a holomorphic function f on $\mathbb{C}^N$ such that $f(x) = 0$ and $f(y) \neq 0$ for $y \in A$.*

Proof. Let F be the vector space of all holomorphic functions on $\mathbb{C}^N$ vanishing at x. F is a Fréchet space with the topology of uniform convergence on compact subsets of $\mathbb{C}^N$. For $y \in A$ let $\varphi_y : F \to \mathbb{C}$ be defined by $\varphi_y(f) = f(y)$ for $f \in F$. Let $K_y = \text{Ker}\, \varphi_y$. K_y is a nowhere dense closed subspace of F. For, if we take $g \in F$ such that $g(y) \neq 0$, then for any open neighborhood U in F of

$h \in K_y$ we have $\lambda g + h \in U - K_y$ for $\lambda \in \mathbf{C} - \{0\}$ with $|\lambda|$ sufficiently small. By Baire category theorem $\bigcup_{y \in A} K_y \neq F$. $f \in F - \bigcup_{y \in A} K_y$ satisfies the requirement. q.e.d.

LEMMA 2. *Suppose $\mathscr{G}$ is a coherent analytic sheaf on an open subset G of $\mathbf{C}^N$. There exist subvarieties X_p in G, either empty or of pure* $\dim p$, $0 \leqq p \leqq N-1$, *such that, for every $x \in G$, if a non-identically-zero holomorphic function-germ f at x does not vanish identically on any non-empty branch-germ of X_p at x for any p, then f is not a zero-divisor for the stalk $\mathscr{G}_x$ of $\mathscr{G}$ at x.*

PROOF. For $0 \leqq p \leqq N-1$, define a subsheaf $\mathscr{G}_p$ of $\mathscr{G}$ on G as follows: for $x \in G$, $(\mathscr{G}_p)_x = \{s \in \mathscr{G}_x |$ for some subvariety A_s of dimension $\leqq p$ in some open neighborhood U_s of x in G there exists $t \in \Gamma(U_s, \mathscr{G})$ such that $t_x = s$ and $t_y = 0$ for $y \notin A_s\}$. $\mathscr{G}_p$ is a coherent analytic subsheaf of $\mathscr{G}$ and $\dim \operatorname{Supp} \mathscr{G}_p \leqq p$. For, if $\varphi : {}_N\mathscr{O}^q \to \mathscr{G}$ is a sheaf-epimorphism on an open subset D of G (where ${}_N\mathscr{O}$ = structure-sheaf of $\mathbf{C}^N$) and $(\operatorname{Ker} \varphi)_p$ is the p^{th} step gap-sheaf of $\operatorname{Ker} \varphi$ in the sense of Thimm (Def. 9, [9]), then $\mathscr{G}_p = \varphi((\operatorname{Ker} \varphi)_p)$ on D and by Satz 3, [9] $(\operatorname{Ker} \varphi)_p$ is coherent and $\dim \{x \in D | ((\operatorname{Ker} \varphi)_p)_x \neq (\operatorname{Ker} \varphi)_x\} \leqq p$. Let X_p be the union of p-dimensional branches of $\operatorname{Supp} \mathscr{G}_p$. We claim that these satisfy the requirement.

Suppose f is a non-identically-zero holomorphic function-germ at a point x of G not vanishing identically on any non-empty branch-germ of X_p at x for any p. We have to prove that f is not a zero-divisor for $\mathscr{G}_x$. Suppose the contrary. Then there exist $g \in \Gamma(U, {}_N\mathscr{O})$ and $h \in \Gamma(U, \mathscr{G})$ for some connected open neighborhood U of x in G such that $g_x = f$, $h_x \neq 0$, and $gh = 0$. Let $Z = \operatorname{Supp} h$ and let p be the dimension of the germ of Z at x. $0 \leqq p \leqq N-1$. By shrinking U we can assume that $\dim Z = p$. $h \in \Gamma(U, \mathscr{G}_p)$ and $Z \subset \operatorname{Supp} \mathscr{G}_p$. Since $\dim \operatorname{Supp} \mathscr{G}_p \leqq p$ and at x Z has dimension p, Z and X_p have a branch-germ A in common at x. $gh = 0$ implies that f vanishes identically on A. Contradiction. q.e.d.

LEMMA 3. *Suppose $\mathscr{S}$ is a torsion-free coherent analytic sheaf on a normal reduced irreducible complex space Z_0. Then the set E of points in Z_0 where $\mathscr{S}$ is not locally free is a subvariety of codimension $\geqq 2$.*

PROOF. Let $m = \dim Z_0$. D is a subvariety in Z_0 (Prop. 8, [1]). Suppose the Lemma is false. Then D contains an $(m-1)$-dimensional branch A. Let M be the set of all regular points of Z_0.

Since dim $(Z_0-M) \leqq m-2$, there exists $x \in M \cap A$. There is a non-identically-zero holomorphic function f on some connected open neighborhood U of x in M such that f vanishes identically on $A \cap U$. Since $\mathscr{S}$ is torsion-free, for $y \in U$ f_y is not a zero-divisor for $\mathscr{S}_y$. Let $\mathscr{I} = \mathscr{S}/f\mathscr{S}$ on U. $F = \{y \in U |\ \text{codh}\ \mathscr{I}_y \leqq m-2\}$ is of dimension $\leqq m-2$ (Satz 5, [7]). There exists $z \in U \cap A - F$. codh $\mathscr{S}_z = m$. $\mathscr{S}$ is locally free at z, contradicting that $z \in D$. q.e.d.

Lemma 4. *Suppose P is an m-dimensional complex manifold. Suppose $\mathcal{O}$ is the structure-sheaf of P, $\mathscr{S}$ is a locally free sheaf on P, and $\mathscr{L}$ is the sheaf of germs of holomorphic $(m, 0)$-forms on P. If $H^m_*(P, \mathscr{S}) = 0$, then $\Gamma(P, \text{Hom}_{\mathcal{O}}(\mathscr{S}, \mathscr{L})) = 0$.*

Proof. Let B and B^* be respectively the holomorphic vector-bundles canonically associated with the locally free sheaves $\mathscr{S}$ and $\text{Hom}_{\mathcal{O}}(\mathscr{S}, \mathscr{L})$. For $0 \leqq p \leqq m$ let $\lambda(0, p)$ denote the vector-bundle of $(0, p)$-forms on P. Let $\mathscr{A}^{(0,p)}(B)$ denote the sheaf of germs of infinitely differentiable sections in $B \otimes \lambda(0, p)$ and let $\mathscr{D}^{(0,p)}(B^*)$ denote the sheaf of germs of distribution-sections in $B^* \otimes \lambda(0, p)$. Let $\Gamma_*(P, \mathscr{A}^{(0,p)}(B))$ denote the set of all global sections in $\mathscr{A}^{(0,p)}(B)$ with compact supports.

$$0 \to \mathscr{S} \to \mathscr{A}^{(0,0)}(B) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}^{(0,m-1)}(B) \xrightarrow{\bar\partial} \mathscr{A}^{(0,m)}(B) \to 0$$

and

$$0 \to \text{Hom}_{\mathcal{O}}(\mathscr{S}, \mathscr{L}) \to \mathscr{D}^{(0,0)}(B^*) \xrightarrow{\bar\partial} \mathscr{D}^{(0,1)}(B^*) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{D}^{(0,m)}(B^*) \to 0$$

are fine-sheaf-resolutions for $\mathscr{S}$ and $\text{Hom}_{\mathcal{O}}(\mathscr{S}, \mathscr{L})$ respectively. $H^m_*(P, \mathscr{S}) = 0$ means that

$$\alpha : \Gamma_*(P, \mathscr{A}^{(0,m-1)}(B)) \to \Gamma_*(P, \mathscr{A}^{(0,m)}(B))$$

induced by

$$\bar\partial : \mathscr{A}^{(0,m-1)}(B) \to \mathscr{A}^{(0,m)}(B)$$

is surjective. $\Gamma(P, \mathscr{D}^{(0,0)}(B^*))$ and $\Gamma(P, \mathscr{D}^{(0,1)}(B^*))$ are respectively the duals of $\Gamma_*(P, \mathscr{A}^{(0,m)}(B))$ and $\Gamma_*(P, \mathscr{A}^{(0,m-1)}(B))$.

$$\beta : \Gamma(P, \mathscr{D}^{(0,0)}(B^*)) \to \Gamma(P, \mathscr{D}^{(0,1)}(B^*))$$

induced by $\bar\partial : \mathscr{D}^{(0,0)}(B^*) \to \mathscr{D}^{(0,1)}(B^*)$ is the transpose of α (Cf. [8]). β is therefore injective. $\Gamma(P, \text{Hom}_{\mathcal{O}}(\mathscr{S}, \mathscr{L})) = 0$. q.e.d.

Proof of Theorem 1: Since X is Stein, by imbedding X and extending $\mathscr{F}$ trivially we can assume w.l.o.g. that $X = \mathbf{C}^N$ and

$n > 0$. Fix $x \in G$. For $0 \leqq m \leqq n$ we are going to construct by induction on m holomorphic functions $f_0 \equiv 0, f_1, \cdots, f_m$ on G such that $f_1(x) = \cdots = f_m(x) = 0$, $(f_1)_x \neq 0, \cdots, (f_m)_x \neq 0$, and for $1 \leqq k \leqq m$

$$(3) \qquad 0 \to \mathscr{F}/\textstyle\sum_{i=0}^{k-1} f_i \mathscr{F} \xrightarrow{\varphi_k} \mathscr{F}/\sum_{i=0}^{k-1} f_i \mathscr{F} \to \mathscr{F}/\sum_{i=0}^{k} f_i \mathscr{F} \to 0$$

is an exact sequence on G, where φ_k is defined by multiplication by f_k.

The case $m = 0$ is trivial. Suppose we have constructed $f_0 \equiv 0$, $f_1, \cdots, f_m$ for some $0 \leqq m < n$. (3) implies that

$$(4) \quad H_*^p(G, \mathscr{F}/\textstyle\sum_{i=0}^{k-1} f_i \mathscr{F}) \to H_*^p(G, \mathscr{F}/\sum_{i=0}^{k} f_i \mathscr{F}) \to H_*^{p+1}(G, \mathscr{F}/\sum_{i=0}^{k-1} f_i \mathscr{F}) \text{ is exact for } p \geqq 0.$$

Since $H_*^p(G, \mathscr{F}) = 0$ for $p < n$, by induction on k we obtain from (4) that, for $0 \leqq k \leqq m$

$$(5)_k \qquad H_*^p(G, \mathscr{F}/\textstyle\sum_{i=0}^{k} f_i \mathscr{F}) = 0 \quad \text{for} \quad p < n-k.$$

Let $\mathscr{G} = \mathscr{F}/\sum_{i=0}^{m} f_i \mathscr{F}$. For the coherent analytic sheaf $\mathscr{G}$ on G we have in G subvarieties X_p, of pure $\dim p$ or empty, $0 \leqq p \leqq N-1$, satisfying the requirement of Lemma 2. Since $H_*^0(G, \mathscr{G}) = 0$ by $(5)_m$, from the construction in the proof of Lemma 2 we can choose $X_0 = \emptyset$. Let $X_p = \bigcup_{i \in I_p} X_p^i$ be the decomposition into irreducible branches, $1 \leqq p \leqq N-1$. For $X_p \neq \emptyset$ take $x_p^i \in X_p^i - \{x\}$. Let $G - \{x\} = \bigcup_{j \in J} G_j$ be the decomposition into topological components. Take $x_j \in G_j$. Let

$$A = \{x_p^i | i \in I_p, 1 \leqq p \leqq N-1, X_p \neq \emptyset\} \cup \{x_j | j \in J\}.$$

A is at most countable. There exists by Lemma 1 a holomorphic function f on G such that $f(x) = 0$ and $f(y) \neq 0$ for $y \in A$. For $z \in G$ f_z cannot vanish identically in any non-empty branch-germ of X_p at z for any p. Therefore for $z \in G$ f_z is not a zero-divisor for $\mathscr{G}_z$. Set $f_{m+1} = f$. The sequence $f_0 \equiv 0, f_1, \cdots, f_m, f_{m+1}$ satisfies the construction requirement. The construction is complete. $(f_1)_x, \cdots, (f_n)_x$ is an $\mathscr{F}_x$-sequence in the sense of (27.1), [5]. $\operatorname{codh} \mathscr{F}_x \geqq n$. q.e.d.

Proof of Theorem 2. Again w.l.o.g. we can assume that $X = \mathbf{C}^N$. Let $Y = \operatorname{Supp} \mathscr{F}$, $D = G \cap Y$, and $\dim D = m$. We have to prove that $m \leqq n$. Suppose the contrary. Then $n < m$ and $H_*^p(G, \mathscr{F}) = 0$ for $p \geqq m$.

Let $\mathscr{I}$ be the annihilating ideal-sheaf for $\mathscr{F}$, i.e. for $x \in \mathbf{C}^N$, $\mathscr{I}_x = \{s \in {}_N\mathscr{O}_x | s\mathscr{F}_x = 0\}$. Let $\mathscr{H} = {}_N\mathscr{O}/\mathscr{I}$. The sheaf of modules

$\mathscr{F}$ can be regarded as over the sheaf of rings $\mathscr{H}$. Let $\mathscr{K}$ be the subsheaf of all nilpotent elements of $\mathscr{H}$. The exactness of

$$0 \to \mathscr{K}\mathscr{F} \to \mathscr{F} \to \mathscr{F}/\mathscr{K}\mathscr{F} \to 0$$

implies the exactness of

$$H^p_*(G, \mathscr{F}) \to H^p_*(G, \mathscr{F}/\mathscr{K}\mathscr{F}) \to H^{p+1}_*(G, \mathscr{K}\mathscr{F}) \quad \text{for} \quad p \geqq 0.$$

Since

$$\dim G \cap (\operatorname{Supp} \mathscr{K}\mathscr{F}) \leqq m, H^{p+1}_*(G, \mathscr{K}\mathscr{F}) = 0 \quad \text{for} \quad p \geqq m$$

by Satz 3, [6]. Hence

$$H^p_*(G, \mathscr{F}/\mathscr{K}\mathscr{F}) = 0 \quad \text{for} \quad p \geqq m.$$

$\operatorname{Supp}(\mathscr{F}/\mathscr{K}\mathscr{F}) = \operatorname{Supp} \mathscr{F}$. For, if for some $x \in \mathbf{C}^N$ $\mathscr{F}_x = \mathscr{K}_x\mathscr{F}_x$, then, since $\mathscr{K}_x$ is contained in the maximal-ideal of the local ring $\mathscr{H}_x$, we have $\mathscr{F}_x = 0$ by Krull-Azumaya Lemma ((4.1), [5]).

Let $\mathscr{G} = (\mathscr{F}/\mathscr{K}\mathscr{F})|Y$ and $\tilde{\mathscr{O}} = (\mathscr{H}/\mathscr{K})|Y$. $\mathscr{G}$ is a coherent analytic sheaf on the *reduced* Stein space $(Y, \tilde{\mathscr{O}})$. $\operatorname{Supp} \mathscr{G} = Y$ and $H^p_*(D, \mathscr{G}) = 0$ for $p \geqq m$.

Let $\pi : Z \to Y$ be the normalization of $(Y, \tilde{\mathscr{O}})$. Let $\mathscr{G}'$ be the inverse image of $\mathscr{G}$ under π (Def. 8, [3]) and let $\mathscr{G}''$ be the zero$^{\text{th}}$ direct image of $\mathscr{G}'$ under π. There exists a natural sheaf-homomorphism $\lambda : \mathscr{G} \to \mathscr{G}''$ (Satz 7 (b), [3]). λ is bijective at regular points of Y. Let $\mathscr{R} = \operatorname{Ker} \lambda$ and $\mathscr{L} = \lambda(\mathscr{G})$. The exactness of $0 \to \mathscr{R} \to \mathscr{G} \to \mathscr{L} \to 0$ implies the exactness of

$$H^p_*(D, \mathscr{G}) \to H^p_*(D, \mathscr{L}) \to H^{p+1}_*(D, \mathscr{R}) \quad \text{for} \quad p \geqq 0.$$

Since $\dim D \cap \operatorname{Supp} \mathscr{R} < m$, $H^{p+1}_*(D, \mathscr{R}) = 0$ for $p \geqq m-1$. $H^p_*(D, \mathscr{L}) = 0$ for $p \geqq m$. The exactness of

$$0 \to \mathscr{L} \to \mathscr{G}'' \to \mathscr{G}''/\mathscr{L} \to 0$$

implies the exactness of

$$H^p_*(D, \mathscr{L}) \to H^p_*(D, \mathscr{G}'') \to H^p_*(D, \mathscr{G}''/\mathscr{L}) \quad \text{for} \quad p \geqq 0.$$

Since $\dim D \cap \operatorname{Supp} \mathscr{G}''/\mathscr{L} < m$, $H^p_*(D, \mathscr{G}''/\mathscr{L}) = 0$ for $p \geqq m$. $H^p_*(D, \mathscr{G}'') = 0$ for $p \geqq m$. Let $L = \pi^{-1}(D)$. Since

$$H^p_*(L, \mathscr{G}') \approx H^p_*(D, \mathscr{G}'') \quad \text{for} \quad p \geqq 0,$$

$H^p_*(L, \mathscr{G}') = 0$ for $p \geqq m$.

Let $\mathscr{I}$ be the torsion subsheaf of $\mathscr{G}'$ and let $\mathscr{S} = \mathscr{G}'/\mathscr{I}$. On Z $\mathscr{S}$ is coherent and torsion-free (Prop. 6, [1]). Since $\operatorname{Supp} \mathscr{G} = Y$, Supp $\mathscr{S} = Z$. The exact sequence $0 \to \mathscr{I} \to \mathscr{G}' \to \mathscr{S} \to 0$ gives rise to the exact sequence

$$H^p_*(L, \mathscr{G}') \to H^p_*(L, \mathscr{S}) \to H^{p+1}_*(L, \mathscr{I}) \quad \text{for} \quad p \geqq 0.$$

Since $\dim L \cap \operatorname{Supp} \mathscr{I} < m$, $H^{p+1}_*(L, \mathscr{I}) = 0$ for $p \geqq m-1$. $H^p_*(L, \mathscr{S}) = 0$ for $p \geqq m$. Let Z_0 be an m-dimensional branch of Z intersecting L. $H^p_*(L \cap Z_0, \mathscr{S}) = 0$ for $p \geqq m$. Let M be the set of all regular points of Z_0 and let E be the set of points in Z_0 where $\mathscr{S}$ is not locally free. By Lemma 3 $\dim E \leqq m-2$. Since Z_0 is normal, $\dim (Z_0 - M) \leqq m-2$. By Satz 3, [6],

$$H^p_*(L \cap (M-E), \mathscr{S}) = 0 \quad \text{for} \quad p \geqq m.$$

Let $\mathscr{O}$ be the structure-sheaf of Z_0 and let $\mathscr{L}$ be the sheaf of germs of holomorphic $(m, 0)$-forms on M. By Lemma 4 $\Gamma(L \cap (M-E), \operatorname{Hom}_{\mathscr{O}} (\mathscr{S}, \mathscr{L})) = 0$. Take $x \in L \cap (M-E)$. Since $\mathscr{S}_x \neq 0$ and Z_0 is Stein, there exists $s \in \Gamma(Z_0, \operatorname{Hom}_{\mathscr{O}} (\mathscr{S}, \mathscr{O}))$ such that $s_x \neq 0$. Since Z_0 is Stein, there exist holomorphic functions $g_1, \cdots, g_m$ on Z_0 such that the map $(g_1, \cdots, g_m) : Z_0 \to \mathbf{C}^m$ has rank m at x. $dg_1 \wedge \cdots \wedge dg_m$ defines an element f of $\Gamma(M, \mathscr{L})$. $f_x \neq 0$. Since $\operatorname{Hom}_{\mathscr{O}} (\mathscr{S}, \mathscr{L}) \approx \operatorname{Hom}_{\mathscr{O}} (\mathscr{S}, \mathscr{O}) \otimes_{\mathscr{O}} \mathscr{L}$ on M, $s \otimes f | L \cap (M-E)$ is a nonzero element of $\Gamma(L \cap (M-E), \operatorname{Hom}_{\mathscr{O}} (\mathscr{S}, \mathscr{L}))$. Contradiction. q.e.d.

Remark. In Theorems 1 and 2 the assumption that X is Stein cannot be dropped altogether. Counter-examples can easily be constructed by letting X be a complex projective space and by using Theorem von Serre in [3]. However, easy modifications in the proof can show that Theorem 1 holds under the weaker assumption that holomorphic functions on X separate points.

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