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Relativization with respect to formulas and its use in proofs of independence

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Leon Henkin [1]

In predicate logic the process of relativizing all the quantifiers of a formula φ is a familiar one.[2] The basic logical principle involving this process is as follows: If φ is logically valid and contains free variables $v_0, \cdots, v_{n-1}$, if μ is any formula containing exactly one free variable, and if $\varphi^{(\mu)}$ is the result of relativizing the quantifiers of φ to μ, then the formula

$$[\mu(v_0) \wedge \cdots \wedge \mu(v_{n-1})] \rightarrow \varphi^{(\mu)}$$

is also logically valid.[3]

This elementary principle, which can be demonstrated in a completely constructive manner, plays an important role in many independence proofs. For example, suppose σ is the conjunction of some finite system of axioms (without free variables) for set theory, and τ is a sentence whose independence from σ we are trying to demonstrate. Assume we have a formula $\mu(v_0)$ (e.g., μ might express the proposition that v_0 is a constructible set in the sense of Gödel [4]) for which we can prove the formulas

$$\exists v_0 \mu(v_0),\ \sigma^{(\mu)}, \quad \text{and} \quad \neg\, \tau^{(\mu)}$$

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[2] Cf. item [A] (page 24) of the bibliography, for example.

[3] In case φ is a sentence, i.e., has no free variables, the antecedent is replaced by $\exists v_0 \mu(v_0)$.

[4] See [B].

in some deductive system $\sum$. Thus any deduction of

$$\exists v_0 \mu(v_0) \rightarrow (\sigma^{(\mu)} \rightarrow \tau^{(\mu)})$$

within $\sum$ would give us a proof of contradiction within $\sum$. By the principle of relativization of quantifiers we thus show constructively how a proof of $\sigma \rightarrow \tau$ would give a contradiction within $\sum$. In this way we obtain a proof of the independence of τ from σ, relative to the consistency of $\sum$.

In the present paper we wish to describe a process of relativizing an arbitrary formula φ with respect to a formula π which may have more than one free variable, of which the ordinary process of relativizing quantifiers is a special case.[5] And we shall demonstrate the utility of the more general process by giving an independence proof of an unusual kind — in a system of logic containing only finitely many individual variables.

Let α be any of the ordinal numbers $1, 2, \cdots, \omega$. We shall consider an arbitrary grammar G_α having *individual variables* $v_0, v_1, \cdots, v_n, \cdots$ for each $n < \alpha$. Atomic formulas are constructed by using a k-placed sequence of these variables with any *k-place predicate symbol* of $G_\alpha (k < \alpha)$. We assume G_α contains a 2-place predicate symbol, $\approx$, the equality symbol. Further formulas are built up from the atomic ones in a familiar way, using the primitive sentential connectives $\wedge$ (*conjunction sign*) and $\neg$ (*negation sign*), and the *existential quantifier symbol* $\exists$. Other connectives such as $\vee$ (disjunction), $\rightarrow$ (conditional), and $\leftrightarrow$ (bi-conditional), and the universal quantifier symbol $\forall$, may be introduced into the system by definitions in terms of the primitive symbols. We assume known the distinction between free and bound occurrences of variables in a given formula.

Definition. Let π be any formula of G_α. To each formula φ of G_α we associate a formula φ_π, by the following recursive scheme: If φ is atomic, $\varphi_\pi = \varphi$; for all φ, $(\neg \varphi)_\pi = \neg (\varphi_\pi)$; for all φ and ψ, $(\varphi \wedge \psi)_\pi = (\varphi_\pi \wedge \psi_\pi)$; for all φ, and for every $n < \alpha$,

$$(\exists v_n \varphi)_\pi = \exists v_n (\pi \wedge \varphi_\pi).$$

We say that φ_π *results from φ by relativizing with respect to π.*

[5] The more general process is perhaps not unknown, although I have not found it set down in any standard work on logic. It is an application of the familiar notion of relativizing a boolean algebra to one of its elements (cf. [C]) applied to the boolean algebra of classes of equivalent formulas of a deductive system.

In the usual process of relativizing quantifiers we use, in place of π, a formula μ with just one free variable, obtaining a formula $\varphi^{(\mu)}$ which is defined in a similar manner except that we specify $(\exists v_n \varphi)^{(\mu)} = \exists v_n(\mu(v_n) \wedge \varphi^{(\mu)})$, where $\mu(v_n)$ results by substituting free occurrences of v_n for the single free variable of μ. The two notions of relativization are related as follows.

Suppose μ has just one free variable, $n < \alpha$, and φ is any formula of G_α all of whose variables (free and bound) are among $v_0, \cdots, v_{n-1}$. Let $\pi = \mu(v_0) \wedge \cdots \wedge \mu(v_{n-1})$. Then the formula $(\pi \wedge \varphi^{(\mu)}) \leftrightarrow (\pi \wedge \varphi_\pi)$ is logically valid, and indeed provable in the deductive system D_α to be described below. Thus the formula $\pi \rightarrow \varphi^{(\mu)}$ will be provable in D_α if, and only if, $\pi \rightarrow \varphi_\pi$ is provable. In this sense the process of relativizing φ with respect to an arbitrary formula π may be considered a generalization of the process of relativizing the quantifiers of φ to a formula μ with one free variable.

Based upon the grammar G_α we now define a deductive system, D_α, by providing a familiar set of axioms and rules of inference. We first describe the axioms in several groups. In A_2 (and elsewhere in the paper) we use φ^i_j to indicate the result of substituting free occurrences of v_j for all free occurrences of v_i in φ, simultaneously changing bound variables (in a manner we do not stop to describe in detail) so as to avoid collisions.

A_1. Any formula of G_α obtained by substitution in a sentential tautology, is an axiom of D_α.

A_2. If φ is a formula and $i, j < \alpha$, let $H(\varphi, i, j)$ hold if, and only if, no free occurrence of v_i in φ is within the scope of a quantifier involving v_j. Then the formula $\varphi^i_j \rightarrow \exists v_i \varphi$ is an axiom of D_α whenever $H(\varphi, i, j)$.

A_3. $\exists v_i(\varphi \wedge \psi) \rightarrow (\varphi \wedge \exists v_i \psi)$ is an axiom of D_α whenever v_i has no free occurrence in φ.

A_4. $v_i \approx v_i$ is an axiom of D_α for each $i < \alpha$.

A_5. $(v_i \approx v_j \wedge \varphi) \rightarrow \psi$ is an axiom of D_α whenever φ is an atomic formula of G_α and ψ is obtained from φ by replacing one or more occurrences of v_i by occurrences of v_j.

As rules of inference we take *detachment*, to infer a formula ψ from given formulas φ and $\varphi \rightarrow \psi$, and *generalization*, to infer $\forall v_i \varphi$ from a given formula φ where v_i is any variable, $i < \alpha$. By means of these axioms and rules we define the notions of a *formal proof* and a *formal theorem* for the system D_α, in the usual way; we write $\vdash_\alpha \varphi$, or sometimes simply $\vdash \varphi$, to indicate that a formula φ is such a formal theorem. Finally, if Γ is any set of formulas we write $\Gamma \vdash \varphi$ to indicate that

$$\vdash (\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow \varphi$$

for some $\psi_1, \cdots, \psi_n \in \Gamma$.

THEOREM 1. (*Principle of relativization.*) *Let* α *be any of the ordinal numbers* $1, 2, \cdots, \omega$, *let* Γ *be a set of formulas and* φ *a formula of* G_α. *Choose* $n < \alpha$, *and any formula* π *of* G_α *containing no free variables other than* $v_0, \cdots, v_{n-1}$, *and let* Γ_π *be the set of all relativized formulas* ψ_π *for all* $\psi \in \Gamma$. *Let* Δ *consist of*

(i) all formulas

$$\forall v_0 \cdots \forall v_m[(\exists v_i \exists v_j \varphi)_\pi \rightarrow (\exists v_j \exists v_i \varphi)_\pi]$$

where φ is any formula of D_α and $v_0, \cdots, v_m$ include all of its free variables, and

(ii) all formulas

$$\forall v_0 \cdots \forall v_m(\pi \rightarrow \pi^i_j), \quad \text{for any} \quad i, j < \alpha,$$

where $v_0, \cdots v_m$ include all variables occurring free in $\pi \rightarrow \pi^i_j$. *If* $\Gamma \vdash_\alpha \varphi$ *then* $\Delta \cup \Gamma \vdash_\alpha \pi \rightarrow \varphi_\pi$.

REMARK. In case μ is a formula with single free variable, v_0, and

$$\pi = \mu(v_0) \wedge \cdots \wedge \mu(v_{n-1}),$$

all of the formulas of Δ of type (i) or (ii) are provable in D_α; hence the above theorem generalizes the principle of relativization of quantifiers mentioned in the opening paragraph of this paper.

PROOF of theorem. We proceed by a series of lemmas.

LEMMA 1.1. *If* ψ *is any of the axioms* A_1, *i.e., if* φ *results by substitution in a sentential tautology, then* $\vdash \pi \rightarrow \psi_\pi$.

Since for all formulas φ, θ we have $(\neg \varphi_\pi) = \neg (\varphi_\pi)$ and $(\varphi \wedge \theta)_\pi = (\varphi_\pi \wedge \theta_\pi)$, we see that if ψ is built up from formulas $\nu^1, \cdots, \nu^k$ by negation and conjunction signs, then ψ_π can be obtained from ψ by substituting ν^i_π for ν^i, $i = 1, \cdots, k$. In particular, if ψ is obtained by substituting $\nu^1, \cdots, \nu^k$ for the variables of a sentential tautology, then ψ_π will be obtained from the same tautology by substituting $\nu^1_\pi, \cdots, \nu^k_\pi$ for its variables. Hence ψ_π is also one of the axioms A_1, and so $\pi \rightarrow \psi_\pi$ is also. This proves $\vdash \pi \rightarrow \psi_\pi$.

LEMMA 1.2. $\vdash \varphi^i_j \leftrightarrow \exists v_i(v_i \approx v_j \wedge \varphi)$, *for any formula* φ *and any* $i, j < \alpha$. For it is well known from predicate logic that axioms A_5 may be employed in a recursive fashion to give

(A) $$\vdash (v_i \approx v_j \wedge \varphi) \to \varphi^i_j,$$

and

(B) $$\vdash (v_i \approx v_j \wedge \varphi^i_j) \to \varphi.$$

From (A) we obtain

(A′) $$\vdash \exists v_i(v_i \approx v_j \wedge \varphi) \to \varphi^i_j,$$

since v_i is not free in φ^i_j. And from (B) we obtain first

$$\vdash \exists v_i(v_i \approx v_j \wedge \varphi^i_j) \to \exists v_i(v_i \approx v_j \wedge \varphi),$$

and then, since v_i is not free in

$$\varphi^i_j, \vdash (\exists v_i(v_i \approx v_j) \wedge \varphi^i_j) \to \exists v_i(v_i \approx v_j \wedge \varphi).$$

Using axioms A_4 we then get

(B′) $$\vdash \varphi^i_j \to \exists v_i(v_i \approx v_j \wedge \varphi)$$

which, together with A′, gives the desired result.

LEMMA 1.3. *For any formula* φ *of* G_α, *and any* $i, j < \alpha$, *either* $H(\varphi, i, j)$ *does not hold or*

$$\Delta \vdash \pi \to [(\varphi^i_j)_\pi \leftrightarrow (\varphi_\pi)^i_j].$$

We show this by induction on the length of φ.

Of course if φ is atomic then so is φ^i_j, so that $(\varphi^i_j)_\pi = (\varphi_\pi)^i_j = \varphi^i_j$, and hence $\pi \to [(\varphi^i_j)_\pi \leftrightarrow (\varphi_\pi)^i_j]$ is one of the axioms A_1.

Since $(\neg\, \varphi)^i_j = \neg\, (\varphi^i_j)$ and $(\varphi \wedge \theta)^i_j = (\varphi^i_j) \wedge (\theta^i_j)$ for all φ and θ, the induction steps in the case of negations or conjunctions are trivial. We proceed, therefore, to consider formulas of the form $\exists v_k \varphi$.

Assume that $H(\exists v_k \varphi, i, j)$ holds. Then either $k = i$ and $(\exists v_k \varphi)^i_j = \exists v_i \varphi$, or else $k \neq i, j$ and $H(\varphi, i, j)$ holds.

In case $k = i$ and $(\exists v_k \varphi)^i_j = \exists v_i \varphi$ we obtain

(1) $$((\exists v_k \varphi)^i_j)_\pi = \exists v_i(\pi \wedge \varphi_\pi),$$

and

(2) $$((\exists v_k \varphi)_\pi)^i_j = (\exists v_i(\pi \wedge \varphi_\pi))^i_j = \exists v_i(\pi \wedge \varphi_\pi).$$

Thus $\pi \to [((\exists v_k \varphi)^i_j)_\pi \leftrightarrow ((\exists v_k \varphi)_\pi)^i_j]$ is a tautology and so of course

(3) $$\Delta \vdash \pi \to [((\exists v_k \varphi)^i_j)_\pi \leftrightarrow ((\exists v_k \varphi)_\pi)^i_j]$$

as desired.

In the other case, where $k \neq i, j$ and $H(\varphi, i, j)$ holds, we have the induction hypothesis

(4) $$\Delta \vdash \pi \to [(\varphi^i_j)_\pi \leftrightarrow (\varphi_\pi)^i_j].$$

Since no variable is free in any formula of Δ, we obtain from (4):

$$\Delta \vdash \forall v_k(\pi \to [(\varphi^i_j)_\pi \leftrightarrow (\varphi_\pi)^i_j]),$$

and hence

$$\Delta \vdash \exists v_k(\pi \wedge (\varphi^i_j)_\pi) \leftrightarrow \exists v_k(\pi \wedge (\varphi_\pi)^i_j).$$

By definition of relativization, this gives

(5) $$\Delta \vdash (\exists v_k(\varphi^i_j))_\pi \leftrightarrow \exists v_k(\pi \wedge (\varphi_\pi)^i_j).$$

Now using Lemma 1.2, we see that

(6) $$\vdash \exists v_i(v_i \approx v_j \wedge \pi \wedge \varphi_\pi) \leftrightarrow (\pi \wedge \varphi_\pi)^i_j = \pi^i_j \wedge (\varphi_\pi)^i_j.$$

Since $\forall v_0 \cdots \forall v_m(\pi \to \pi^i_j)$ is one of the formulas (type (ii)) of Δ, where $v_0, \cdots, v_m$ include all variables free in $\pi \to (\pi^i_j)$, we infer from (6) and the definition of relativization that

$$\Delta \vdash [\pi \wedge (\exists v_i(v_i \approx v_j \wedge \varphi))_\pi] \leftrightarrow [\pi \wedge (\varphi_\pi)^i_j].$$

Since v_k is not free in any formula of Δ, this may be combined with (5) to yield

(7) $$\Delta \vdash (\exists v_k(\varphi^i_j))_\pi \leftrightarrow \exists v_k[\pi \wedge (\exists v_i(v_i \approx v_j \wedge \varphi))_\pi].$$

Now the right member of the equivalence in (7) is simply $(\exists v_k \exists v_i(v_i \approx v_j \wedge \varphi))_\pi$. Hence we may use the fact that

$$\Delta \vdash (\exists v_k \exists v_i(v_i \approx v_j \wedge \varphi))_\pi \leftrightarrow (\exists v_i \exists v_k(v_i \approx v_j \wedge \varphi))_\pi,$$

as we see from the form of the type (i) formulas of Δ, to obtain from (7):

(8) $$\Delta \vdash (\exists v_k(\varphi^i_j))_\pi \leftrightarrow (\exists v_i \exists v_k(v_i \approx v_j \wedge \varphi))_\pi.$$

Since $k \neq i, j$, we obtain (with the aid of the definition of relativization)

$$\vdash (\exists v_i \exists v_k(v_i \approx v_j \wedge \varphi))_\pi \leftrightarrow \exists v_i(\pi \wedge v_i \approx v_j \wedge \exists v_k(\pi \wedge \varphi_\pi))$$

which, with another application of Lemma 1.2, gives

(9) $$\vdash (\exists v_i \exists v_k(v_i \approx v_j \wedge \varphi))_\pi \leftrightarrow (\pi^i_j \wedge ((\exists v_k \varphi_\pi))^i_j).$$

Again using the fact that $\Delta \vdash \pi \rightarrow \pi^i_j$, from the form of type (ii) formulas of Δ, we can combine (8) and (9) to obtain

$$\Delta \vdash \pi \rightarrow [(\exists v_k(\varphi^i_j))_\pi \leftrightarrow ((\exists v_k \varphi)_\pi)^i_j],$$

which completes the inductive proof of Lemma 1.3.

LEMMA 1.4. *If ψ is any of the axioms A_2 of D_α, then $\Delta \vdash \pi \rightarrow \psi_\pi$.*

For suppose $\psi = (\varphi^i_j \rightarrow \exists v_i \varphi)$ for some φ, i, and j satisfying $H(\varphi, i, j)$. We have

$$\psi_\pi = (\varphi^i_j)_\pi \rightarrow \exists v_i(\pi \wedge \varphi_\pi)$$

by definition of relativization, and hence

$$(10) \qquad \Delta \vdash \pi \rightarrow [\psi_\pi \leftrightarrow [(\varphi_\pi)^i_j \rightarrow \exists v_i(\pi \wedge \varphi_\pi)]]$$

by Lemma 1.3.

But $[\pi^i_j \wedge (\varphi_\pi)^i_j] \rightarrow \exists v_i(\pi \wedge \varphi_\pi)$ is one of the axioms A_2, and $\Delta \vdash \pi \rightarrow \pi^i_j$ because of the type (ii) formulas of Δ, hence (10) implies $\Delta \vdash \pi \rightarrow \psi_\pi$, as claimed.

LEMMA 1.5. *If φ is any formula of G_α containing no free occurrence of the individual variable v_i, then $\Delta \vdash \pi \rightarrow (\varphi_\pi \leftrightarrow (\exists v_i \varphi)_\pi)$.*

We prove this by induction on the length of φ. In particular, if φ is atomic and does not contain v_i, then $\varphi_\pi = \varphi$ and $\vdash (\exists v_i \varphi)_\pi \leftrightarrow ((\exists v_i \pi) \wedge \varphi)$. Hence $\vdash \pi \rightarrow (\varphi_\pi \leftrightarrow (\exists v_i \varphi)_\pi)$.

Next suppose that v_i is not free in $\neg\varphi$, and make the induction hypothesis that $\Delta \vdash \pi \rightarrow (\varphi_\pi \leftrightarrow (\exists v_i \varphi)_\pi)$. Since $\neg(\varphi_\pi) = (\neg\varphi)_\pi$ we get

$$(11) \qquad \Delta \vdash \pi \rightarrow [(\neg\varphi)_\pi \leftrightarrow \neg(\exists v_i \varphi)_\pi].$$

As v_i is not free in any formula of Δ, this gives

$$\Delta \vdash \exists v_i(\pi \wedge (\neg\varphi)_\pi) \leftrightarrow \exists v_i(\pi \wedge \neg(\exists v_i \varphi)_\pi).$$

However, v_i is not free in $\neg(\exists v_i \varphi)_\pi$, since $(\exists v_i \varphi)_\pi = \exists v_i(\pi \wedge \varphi_\pi)$. Hence we get

$$\Delta \vdash \exists v_i(\pi \wedge (\neg\varphi)_\pi) \leftrightarrow [(\exists v_i \pi) \wedge \neg(\exists v_i \varphi)_\pi].$$

Combining this with (11) and the definition of relativization, we get

$$\Delta \vdash \pi \rightarrow [(\neg\varphi)_\pi \leftrightarrow (\exists v_i(\neg\varphi))_\pi]$$

as desired.

The induction step for conjunctions can be handled in a similar manner to the case of negations, and so we proceed to consider formulas of the form $\exists v_k \varphi$ which contain no free occurrence of v_i. Of course if $k = i$ then $(\exists v_k \varphi)_\pi = \exists v_i(\pi \wedge \varphi_\pi)$, so that

$$\vdash \pi \to [(\exists v_k \varphi)_\pi \leftrightarrow \exists v_i(\pi \wedge (\exists v_k \varphi)_\pi)]$$

in this case, in accordance with the lemma.

On the other hand if $k \neq i$ then v_i has no free occurrence in φ, and so our induction hypothesis yields

$$\Delta \vdash \pi \to (\varphi_\pi \leftrightarrow (\exists v_i \varphi)_\pi).$$

Since v_k is not free in any formula of Δ, this gives

$$\Delta \vdash \exists v_k(\pi \wedge \varphi_\pi) \leftrightarrow \exists v_k(\pi \wedge (\exists v_i \varphi)_\pi),$$

i.e.,

$$\Delta \vdash (\exists v_k \varphi)_\pi \leftrightarrow (\exists v_k \exists v_i \varphi)_\pi .$$

Since $\Delta \vdash (\exists v_k v_i \varphi)_\pi \leftrightarrow (\exists v_i \exists v_k \varphi)_\pi$, by the type (i) formulas of Δ, we obtain

$$\Delta \vdash (\exists v_k \varphi)_\pi \leftrightarrow (\exists v_i \exists v_k \varphi)_\pi ,$$

completing the inductive proof of the lemma.

LEMMA 1.6. *If ψ is any of the axioms A_3 of D_α, then $\Delta \vdash \pi \to \psi_\pi$.*

For suppose $\psi = \exists v_i(\varphi \wedge \theta) \to (\varphi \wedge \exists v_i \theta)$, where φ and θ are formulas of G_α and v_i is not free in φ. We have

$$\psi_\pi = \exists v_i(\pi \wedge \varphi_\pi \wedge \theta_\pi) \to (\varphi_\pi \wedge \exists v_i(\pi \wedge \theta_\pi)).$$

From Lemma 1.5 we get

$$\Delta \vdash \pi \to [(\varphi_\pi \wedge \theta_\pi) \leftrightarrow ((\exists v_i \varphi)_\pi \wedge \theta_\pi)],$$

and hence (since v_i is not free in any formula of Δ),

$$\Delta \vdash \exists v_i(\pi \wedge \varphi_\pi \wedge \theta_\pi) \leftrightarrow \exists v_i(\pi \wedge (\exists v_i \varphi)_\pi \wedge \theta_\pi).$$

Thus

$$\Delta \vdash \psi_\pi \leftrightarrow [\exists v_i(\pi \wedge (\exists v_i \varphi)_\pi \wedge \theta_\pi) \to (\varphi_\pi \wedge \exists v_i(\pi \wedge \theta_\pi)).$$

Applying Lemma 1.5 again, this becomes

$$\Delta \vdash \pi \to [\psi_\pi \leftrightarrow [\exists v_i(\pi \wedge (\exists v_i \varphi)_\pi \wedge \theta_\pi) \to ((\exists v_i \varphi)_\pi \wedge \exists v_i(\pi \wedge \theta_\pi))]].$$

But v_i is not free in $(\exists v_i \varphi)_\pi = \exists v_i(\pi \wedge \varphi_\pi)$, so that

$$\exists v_i(\pi \wedge (\exists v_i \varphi)_\pi \wedge \theta_\pi) \to ((\exists v_i \varphi)_\pi \wedge \exists v_i(\pi \wedge \theta_\pi))$$

is one of the axioms A_3.

Hence $\Delta \vdash \pi \to \psi_\pi$, as claimed

LEMMA 1.7. *If ψ is any of the axioms A_4 or A_5 of D_α, then* $\Delta \vdash \pi \rightarrow \psi_\pi$. The proof is trivial.

LEMMA 1.8. *If* $\Delta \vdash \pi \rightarrow \varphi_\pi$ *and* $\Delta \vdash \pi \rightarrow (\varphi \rightarrow \theta)_\pi$, *then* $\Delta \vdash \pi \rightarrow \theta_\pi$.

This is obvious, since $(\varphi \rightarrow \theta)_\pi = (\varphi_\pi \rightarrow \theta_\pi)$.

LEMMA 1.9. *If* $\Delta \vdash \pi \rightarrow \varphi_\pi$ *then also* $\Delta \vdash \pi \rightarrow (\forall v_i \varphi)_\pi$ *for any variable* v_i.

We have

$$(\forall v_i \varphi)_\pi = \neg\,(\exists v_i(\pi \wedge \neg\,(\varphi_\pi)),$$

so that

$$\vdash (\forall v_i \varphi)_\pi \leftrightarrow \forall v_i(\pi \rightarrow \varphi_\pi).$$

But from the hypothesis $\Delta \vdash \pi \rightarrow \varphi_\pi$ and the fact that v_i is not free in any formula of Δ, we get $\Delta \vdash \forall v_i(\pi \rightarrow \varphi_\pi)$. Thus $\Delta \vdash (\forall v_i \varphi)_\pi$, and the lemma follows.

With the lemmas at hand we can now quickly prove the theorem. By hypothesis, there are formulas $\gamma_1, \cdots, \gamma_n \in \Gamma$ such that $\vdash (\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \varphi$. Let $\theta_1, \cdots, \theta_k$ be the formulas, in order, making up a formal proof of $(\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \varphi$. For each $j = 1, \cdots k$ we can show, by induction, that $\Delta \vdash \pi \rightarrow (\theta_j)_\pi$. In case θ_j is an axiom, we obtain this by using one of the Lemmas 1.1, 1.4, 1.6, or 1.7; if θ_j is obtained from earlier θ's by one of the rules of inference, we use Lemma 1.8 or 1.9. Finally we see that $\Delta \vdash \pi \rightarrow (\theta_k)_\pi$, i.e., $\Delta \vdash \pi \rightarrow [(\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow \varphi]_\pi$, from which we readily obtain

$$\Delta \vdash \pi \rightarrow [((\gamma_1)_\pi \wedge \cdots \wedge (\gamma_n)_\pi) \rightarrow \varphi_\pi].$$

This shows that $\Delta \cup \Gamma_\pi \vdash \pi \rightarrow \varphi_\pi$, as asserted in the theorem.

REMARK. Neither the definition of relativized formulas φ_π, nor the formulation of the principle of relativization (Theorem 1) require the presence of the equality sign, $\approx$, in the deductive system. We call attention, however, to the essential use of Lemma 1.2 in proving Lemma 1.3. Although the latter deals only with the relativization of an axiom relating quantifiers with substitution, we brought in the equality sign in giving our proof. We do not know how to eliminate this incursion, and so we do not know whether Theorem 1 holds for systems of predicate logic without identity.

It is well known that the deductive system D_ω is complete: Any formula of G_ω which is logically valid (i.e., holds for all models) is provable in D_ω. D_1 is also complete. But if

$1 < \alpha < \omega$, there are provable formulas of G_α which cannot be proved in D_α. In this section we shall give a simple example for the case $\alpha = 3$.

Consider the completely obvious proposition that a function with at most two distinct elements in its domain can have at most two elements in its range. Note that this can be expressed by a formula of G_3, as follows.

We take a system G_3 having a single 2-place predicate symbol, F, and two 1-place predicate symbols, D and R. Only the individual variables v_0, v_1, v_2 are in the grammar, of course. Now we consider the following formulas of G_3:

$$\begin{aligned}
\text{Fun}\ (F) &= \forall v_0 \forall v_1 \forall v_2 [(Fv_0v_1 \wedge Fv_0v_2) \rightarrow v_1 \approx v_2] \\
\text{Dom}\ (F, D) &= \forall v_0 [Dv_0 \leftrightarrow \exists v_1 Fv_0v_1] \\
\text{Ran}\ (F, R) &= \forall v_1 [Rv_1 \leftrightarrow \exists v_0 Fv_0v_1] \\
D_{\leqq 2} &= \forall v_0 \forall v_1 \forall v_2 [(Dv_0 \wedge Dv_1 \wedge Dv_2) \\
&\quad \rightarrow (v_0 \approx v_1 \vee v_0 \approx v_2 \vee v_1 \approx v_2)] \\
R_{\leqq 2} &= \forall v_0 \forall v_1 \forall v_2 [(Rv_0 \wedge Rv_1 \wedge Rv_2) \\
&\quad \rightarrow (v_0 \approx v_1 \vee v_0 \approx v_2 \vee v_1 \approx v_2)].
\end{aligned}$$

Then the proposition that every function whose domain has at most 2 elements can have at most 2 elements in its range, is expressed by the following formula φ^*:

$$(\varphi^*) \quad [\text{Fun}\ (F) \wedge \text{Dom}\ (F, D) \wedge \text{Ran}\ (F, R) \wedge D_{\leqq 2}] \rightarrow R_{\leqq 2}.$$

Since φ^* is logically valid it can, of course, be proved in D_ω. Note, however, that the natural proof involves 6 variables, and so would have to be carried out formally within D_6.

Speaking very informally, to prove the proposition expressed by φ^* we would proceed by considering any elements u, v, w in the range R of F, and taking x, y, z in the domain D of F so that Fxu, Fyv, and Fzw. Using the hypothesis $D_{\leqq 2}$ we would conclude that $x = y$, $x = z$, or $y = z$. From this, and the hypothesis Fun (F), we would infer that $u = v$, $u = w$, or $v = w$. Since u, v, w were *arbitrary* elements of R, we would come to the conclusion $R_{\leqq 2}$, as desired.

Although this argument requires 6 individual variables, with a little care we can convert it to one in which only 4 variables are employed.

Theorem 2. $\vdash_4 \varphi^*$.

We shall outline a proof by means of a series of lemmas. To facilitate reading formulas we shall use the letters x, y, z, u in place of v_0, v_1, v_2, v_3. Proofs of lemmas will be omitted where they are very simple.

LEMMA 2.1.

$$D_{\leqq 2} \vdash_4 (Du \wedge Dz \wedge \neg z \approx u) \rightarrow \forall x(Dx \rightarrow (x \approx z \vee x \approx u)).$$

Proof omitted.

LEMMA 2.2.

$$\text{Dom}\,(F, D) \vdash_4 [\exists y(Fyx \wedge \neg y \approx u) \wedge \forall y(Dy \rightarrow (y \approx z \vee y \approx u)] \rightarrow Fzx$$

PROOF. Dom $(F, D) \vdash_4 Fyx \rightarrow Dy$. Hence

$$\text{Dom}\,(F, D) \vdash_4 (Fyx \wedge \neg\,(y \approx u)) \rightarrow (Dy \wedge \neg\, y \approx u)$$

so that

$$\text{Dom}\,(F, D) \vdash_4 [(Fyx \wedge \neg\,(y \approx u)) \wedge \forall y(Dy \rightarrow (y \approx z \vee y \approx u))] \rightarrow y \approx z$$

$$\text{Dom}\,(F, D) \vdash_4 [(Fyx \wedge \neg\,(y \approx u)) \wedge \forall y(Dy \rightarrow (y \approx z \vee y \approx u))] \rightarrow Fzx.$$

Since y is not free in Fzx we obtain

$$\text{Dom}\,(F, D) \vdash_4 [\exists y(Fyx \wedge \neg\,(y \approx u)) \wedge \forall y(Dy \rightarrow (y \approx z \vee y \approx u))] \rightarrow Fzx$$

as claimed.

LEMMA 2.3.

$$D_{\leqq 2} \wedge \text{Dom}\,(F, D) \vdash_4 [(Du \wedge Dz \wedge \neg z \approx u) \wedge \exists z(Fzx \wedge \neg(z \approx u)] \rightarrow Fzx$$

PROOF. Combine Lemmas 2.1 and 2.2 with change of bound variables.

LEMMA 2.4

$$\text{Fun}\,(F) \wedge D_{\leqq 2} \wedge \text{Dom}\,(D, F) \vdash_4 [(Du \wedge \exists z(Fzy \wedge \neg z \approx u)) \wedge \exists z(Fzx \wedge \neg z \approx u)] \rightarrow y \approx x.$$

PROOF.

$$\text{Dom}\,(F, D) \vdash_4 Fzy \rightarrow Dz.$$

Hence by Lemma 2.3,

$$D_{\leqq 2} \wedge \mathrm{Dom}\,(F, D)$$
$$\vdash_4 [(Du \wedge Fzy \wedge \neg\, z \approx u) \wedge \exists z(Fzx \wedge \neg\, z \approx u)] \to Fzx.$$

But Fun $(F) \vdash_4 (Fzy \wedge Fzx) \to y \approx x$.
Thus

$$\mathrm{Fun}\,(F) \wedge D_{\leqq 2} \wedge \mathrm{Dom}\,(F, D)$$
$$\vdash_4 [(Du \wedge Fzy \wedge \neg\, z \approx u) \wedge \exists z(Fzx \wedge \neg\, z \approx u)] \to y \approx x.$$

Since z is not free in $y \approx x$ we obtain

$$\mathrm{Fun}\,(F) \wedge D_{\leqq 2} \wedge \mathrm{Dom}\,(F, D)$$
$$\vdash_4 [Du \wedge \exists z(Fzy \wedge \neg\, z \approx u) \wedge \exists z(Fzx \wedge \neg\, z \approx u)] \to y \approx x$$

as claimed.

Lemma 2.5.

$$\mathrm{Fun}\,(F) \vdash_4 (Fuz \wedge Fxy \wedge \neg\, y \approx z) \to \neg\,(u \approx x).$$

Proof omitted.

Lemma 2.6.

$$\mathrm{Fun}\,(F) \wedge \mathrm{Ran}\,(F, R) \vdash_4 (Fuz \wedge Ry \wedge \neg\, y \approx z) \to \exists z(Fzy \wedge \neg\, z \approx u).$$

Proof. By Lemma 2.5,

$$\mathrm{Fun}\,(F) \vdash_4 (Fuz \wedge Fxy \wedge \neg\, y \approx z) \to \exists x(Fxy \wedge \neg\, x \approx u).$$

Changing bound variables and noting that x is not free in

$$\exists z(Fzy \wedge \neg\, z \approx u),$$

we obtain

$$\mathrm{Fun}\,(F) \vdash_4 (Fuz \wedge \exists x Fxy \wedge \neg\, y \approx z) \to \exists z(Fzy \wedge \neg\, z \approx u).$$

Since

$$\mathrm{Ran}\,(F, R) \vdash_4 Ry \to \exists x Fxy$$

we obtain

$$\mathrm{Fun}\,(F) \wedge \mathrm{Ran}\,(F, R) \vdash_4 (Fuz \wedge Ry \wedge \neg\, y \approx z) \to \exists z(Fzy \wedge \neg\, z \approx u)$$

as desired.

Lemma 2.7.

$$\mathrm{Fun}\,(F) \wedge \mathrm{Ran}\,(F, R) \vdash_4 [Fuz \wedge Ry \wedge Rx \wedge \neg\, y \approx z \wedge \neg\, x \approx z]$$
$$\to [\exists z(Fzy \wedge \neg\, z \approx u) \wedge \exists z(Fzx \wedge \neg\, z \approx u)].$$

Proof. This is immediate from Lemma 2.6.

LEMMA 2.8.

$$\text{Fun}\,(F) \wedge \text{Dom}\,(F, D) \wedge \text{Ran}\,(F, R) \wedge D_{\leqq 2} \vdash_4 [Fuz \wedge Ry \wedge Rx \wedge \neg\, y \approx z \wedge \neg\, x \approx z] \rightarrow y \approx x.$$

PROOF. We have

$$\text{Dom}\,(D, F) \vdash_4 Fuz \rightarrow Du.$$

Hence by Lemmas 2.4 and 2.7 we obtain the desired result.

Theorem 2 follows readily from Lemma 2.8. In fact, since u occurs free only once in the formulation of this lemma, we may replace the part Fuz by $\exists uFuz$ and then, since Dom (D, F) is a hypothesis, $\exists u\, Fuz$ may in turn be replaced by Rz. Thus we get

$$\text{Fun}\,(F) \wedge \text{Dom}\,(F, D) \wedge \text{Ran}\,(F, R) \wedge D_{\leqq 2} \vdash_4 (Rz \wedge Ry \wedge Rx) \rightarrow (y \approx z \vee x \approx z \vee y \approx x),$$

which leads directly to $\vdash_4 \varphi^*$.

Having reduced the "natural" 6-variable proof of φ^* to a 4-variable proof, one is of course led to wonder whether a 3-variable proof can be constructed. We shall show that it cannot.

THEOREM 3. *Not* $\vdash_3 \varphi^*$.

PROOF. The normal, direct way to show the unprovability of a logical formula is to provide a model in which it is false. But since φ^* is logically valid, we cannot hope to do this in the present case. However, the principle of relativization will serve us.

Let us consider the grammar G_3 with the predicate symbols F, D, and R which enter into the formula φ^*. In this grammar let us select a formula π, as follows:

$$(\pi)\quad \neg\, [Dv_0 \wedge Dv_1 \wedge Dv_2 \wedge \neg\, v_0 \approx v_1 \wedge \neg\, v_0 \approx v_2 \wedge \neg\, v_1 \approx v_2].$$

According to Theorem 1, *if* $\vdash_3 \varphi^*$ *then* $\Delta \vdash_3 \pi \rightarrow (\varphi^*)_\pi$, where Δ is a certain set of sentences of G_3 described in that theorem. We shall show, however, that in fact we do *not* have $\Delta \vdash_3 \pi \rightarrow (\varphi^*)_\pi$, by constructing a model M in which all sentences of Δ are true, and a 3-place sequence of elements of M which satisfies π but not $(\varphi^*)_\pi$. This will prove that we do *not* have $\vdash_3 \varphi^*$.

Let $U = \{0, 1, 2, 3, 4, 5\}$ be the universe of our model M. We shall choose subsets $\bar{D}$ and $\bar{R}$ of U, and a binary relation $\bar{F}$ over U, to serve as the denotations of the predicate symbols D, R, and F respectively.

$$\bar{D} = \{0, 1, 2\}$$
$$\bar{R} = \{3, 4, 5\}$$
$$\bar{F} = \{\langle 03\rangle, \langle 14\rangle, \langle 25\rangle\}.$$

With the model $M = \langle U, \bar{D}, \bar{R}, \bar{F}\rangle$ thus determined, we may interpret the grammar G_3 and obtain a definite truth value for each sentence φ. More generally, for any elements $x, y, z \in U$ and any formula φ of G_3, we may determine whether the sequence $\langle x, y, z\rangle$ *satisfies* φ in the model M, by assigning x, y, z as the values respectively of any free occurrences of v_0, v_1, v_2 in φ.

LEMMA 3.1. *For any* $x, y, z \in U$, $\langle x, y, z\rangle$ *satisfies* π *if, and only if,* $\{x, y, z\} \neq \{0, 1, 2\}$.

By the definition of satisfaction we know that $\langle x, y, z\rangle$ satisfies the formula

(12) $[Dv_0 \wedge Dv_1 \wedge Dv_2 \wedge \neg v_0 \approx v_1 \wedge \neg v_0 \approx v_2 \wedge \neg v_1 \approx v_2]$

if, and only if, $[x, y, z \in \bar{D}$ and x, y, z are distinct$]$. Since $\bar{D} = \{0, 1, 2\}$ and since π is the negation of the formula (12), we obtain the lemma.

LEMMA 3.2. *If* ψ *is any of the sentences of* Δ *of type* (*i*), *then* ψ *is true in the model* M.

By part (i) of the definition of Δ (given in Theorem 1),

$$\psi = \forall v_0 \forall v_1 \forall v_2 [(\exists v_i \exists v_j \varphi)_\pi \leftrightarrow (\exists v_j \exists v_i \varphi)_\pi]$$

for some formula φ of G_3 and some $i, j < 3$. Because of the symmetric way in which the variables v_0, v_1, v_2 enter into our formula π, it suffices to consider the case where $i = 0$ and $j = 1$. Then, because of the symmetric nature of ψ itself, we see that *to prove the lemma it suffices to show*:

(13) For any $\langle x, y, z\rangle$ which satisfies $(\exists v_0 \exists v_1 \varphi)_\pi$ in M we also have $\langle x, y, z\rangle$ satisfies $(\exists v_1 \exists v_0 \varphi)_\pi$ in M.

Consider, then any formula φ of G_3 and

(14) let x, y, z be any elements of U such that $\langle x, y, z\rangle$ satisfies $(\exists v_0 \exists v_1 \varphi)_\pi$ in M.

By definition of relativization,

$$(\exists v_0 \exists v_1 \varphi)_\pi = \exists v_0 (\pi \wedge \exists v_1 (\pi \wedge \varphi_\pi)).$$

Hence we infer from (14) and Lemma 3.1 that:

(15) For some $x' \in U$ we have $\{x', y, z\} \neq \{0, 1, 2\}$, and for some $y' \in U$ we have $\{x', y', z\} \neq \{0, 1, 2\}$ and $\{x', y', z\}$ satisfies φ_π in M.

Case 1. Suppose $\{x, y', z\} \neq \{0, 1, 2\}$. In this case $\langle x, y', z\rangle$ satisfies π by Lemma 3.1, and it satisfies $\exists v_0(\pi \wedge \varphi_\pi)$ since, by (15), $\langle x', y', z\rangle$ satisfies $\pi \wedge \varphi_\pi$. Hence $\langle x, y', z\rangle$ satisfies $\pi \wedge \exists v_0(\pi \wedge \varphi_\pi)$ in M. But then it follows that $\langle x, y, z\rangle$ satisfies $\exists v_1(\pi \wedge \exists v_0(\pi \wedge \varphi_\pi))$. By definition of relativization we thus obtain:

(16) In Case 1, $\langle x, y, z\rangle$ satisfies $(\exists v_1 \exists v_0 \varphi)_\pi$ in M.

Case 2. Suppose $\{x, y', z\} = \{0, 1, 2\}$. By symmetry it suffices to consider the case where:

(17) $$x = 0, y' = 1, z = 2.$$

In this case we consider the permutation p of U which interchanges 0 with 1, 3 with 4, and leaves 2 and 5 unaltered:

$$p(0) = 1, p(1) = 0, p(2) = 2, p(3) = 4, p(4) = 3, p(5) = 5.$$

Obviously this permutation leaves the sets $\bar{D}$ and $\bar{R}$, and the relation $\bar{F}$, invariant. (In other words, for any $s, t \in U$ we have $s \in \bar{D}$ if and only if $p(s) \in \bar{D}$, $t \in \bar{R}$ if and only if $p(t) \in \bar{R}$, and $\langle s, t\rangle \in \bar{F}$ if and only if $\langle p(s), p(t)\rangle \in \bar{F}$.)

As is well known (and easily established by induction), we can infer from the invariance of $\bar{D}$, $\bar{R}$, and $\bar{F}$ under p that for any elements $s, t, u \in U$, and any formula θ of G_3,

$\langle s, t, u\rangle$ satisfies θ in M if and only if

$\langle p(s), p(t), p(u)\rangle$ satisfies θ.

In particular we obtain from (15) that $\langle p(x'), p(y'), p(z)\rangle$ satisfies $\pi \wedge \varphi_\pi$ in M, which gives us, by (17) and the definition of p:

(18) $$\langle p(x'), 0, 2\rangle \text{ satisfies } \pi \wedge \varphi_\pi \text{ in } M.$$

From this it follows that $\langle x, 0, 2\rangle$ satisfies $\exists v_0(\pi \wedge \varphi_\pi)$. But since $x = 0$, by (17), we certainly have $\{x, 0, 2\} \neq \{0, 1, 2\}$ which means (by Lemma 3.1) that

(19) $$\langle x, 0, 2\rangle \text{ satisfies } (\pi \wedge \exists v_0(\pi \wedge \varphi_\pi)) \text{ in } M.$$

But then clearly $\langle x, y, 2\rangle$ satisfies $\exists v_1(\pi \wedge \exists v_0(\pi \wedge \varphi_\pi))$, and since $z = 2$, by (17), we have $\langle x, y, z\rangle$ satisfies $\exists v_1(\pi \wedge \exists v_0(\pi \wedge \varphi_\pi))$.

This implies, by definition of relativization:

(20) In Case 2, $\langle x, y, z\rangle$ satisfies $(\exists v_1 \exists v_0 \varphi)_\pi$ in M.

Since Cases 1 and 2 are exhaustive, (16) and (20) together show that in every case $\langle x, y, z\rangle$ satisfies $(\exists v_1 \exists v_0 \varphi)_\pi$. But by (14) we chose $\langle x, y, z\rangle$ to be an arbitrary sequence satisfying $(\exists v_0 \exists v_1 \varphi)_\pi$. Thus we have demonstrated the proposition (13) which we know is sufficient to establish our lemma.

LEMMA 3.3. *If ψ is any of the sentences of Δ of type (ii), then ψ is true in the model M.*

Actually, we shall show that ψ is true in *every* model, i.e., that it is logically valid. Indeed, by part (ii) of the definition of Δ (given in Theorem 1),

$$\psi = \forall v_0 \forall v_1 \forall v_2 (\pi \to \pi^i_j)$$

for some $i, j < 3$. If $i = j$, then $\pi^i_j = \pi$ and certainly ψ is logically valid in this case. On the other hand, if $i \neq j$ then the formula π^i_j is itself logically valid (which obviously implies the validity of ψ). By the symmetry of π, it suffices to consider the case where $i = 0$, $j = 1$. In this case:

$$\pi^0_1 = \neg\, [Dv_1 \wedge Dv_1 \wedge Dv_2 \wedge \neg\, v_1 \approx v_1 \wedge \neg\, v_1 \approx v_2 \wedge \neg\, v_1 \approx v_2],$$

and the logical validity of π^0_1 is evident.

LEMMA 3.4. *The formula $\pi \to (Fun\,(F))_\pi$ is valid in the model M.*

Referring to the definitions of $Fun\,(F)$ and of relativization, we see that:

$(Fun\,(F))_\pi$ is logically equivalent to
$\neg\, [\exists v_0[\pi \wedge \exists v_1[\pi \wedge \exists v_2(\pi \wedge Fv_0v_1 \wedge Fv_0v_2 \wedge \neg\, v_1 \approx v_2)]]]$.

Thus, using Lemma 3.1 we see that in order to show that $\pi \to (Fun\,(F))_\pi$ is valid in M *it suffices to show that*:

Given any $x, y, z \in U$ such that $\{x, y, z\} \neq \{0, 1, 2\}$ there do *not* exist elements $x', y', z' \in U$ such that $\{x', y, z\} \neq \{0, 1, 2\}$, $\{x', y', z\} \neq \{0, 1, 2\}$, $\{x', y', z'\} \neq \{0, 1, 2\}$, $\langle x', y'\rangle \in \bar{F}$, $\langle x', z'\rangle \in \bar{F}$, and $y' \neq z'$.

But this is obviously true because, by definition of $\bar{F}$, there do *not* exist elements $x', y', z' \in U$ such that $\langle x', y'\rangle \in \bar{F}$, $\langle x', z'\rangle \in \bar{F}$, and $y' \neq z'$.

LEMMA 3.5.
The formulas $\pi \to (\text{Dom}\,(F, D))_\pi$ *and* $\pi \to (\text{Ran}\,(F, R))_\pi$ *are valid in the model* M.

The two formulas are similar, and we shall only examine the first. Referring to the definitions of Dom (D, F) and relativization we see that:

$$(\text{Dom}\,(D, F))_\pi \text{ is logically equivalent to}$$
$$\neg\, \exists v_0[\pi \wedge \neg\, [Dv_0 \leftrightarrow \exists v_1(\pi \wedge Fv_0v_1)]].$$

Thus, using Lemma 3.1 we see that in order to show that $\pi \to (\text{Dom}\,(F, D))_\pi$ is valid in M *it suffices to show that*:

Given any $x, y, z \in U$ such that $\{x, y, z\} \neq \{0, 1, 2\}$, there is *no* $x' \in U$ such that $\{x', y, z\} \neq \{0, 1, 2\}$ and for which: *either* $x' \notin \bar{D}$ and for some $y' \in U$ we have $\{x', y', z\} \neq \{0, 1, 2\}$ and $\langle x', y'\rangle \in \bar{F}$, *or* $x' \in \bar{D}$ and there is no y' such that $\{x', y', z\} \neq \{0, 1, 2\}$ and $\langle x', y'\rangle \in \bar{F}$.

But this is certainly true. Referring to the definitions of $\bar{F}$ and $\bar{D}$, we see on the one hand that there is *no* $x' \in U$ for which $x' \notin \bar{D}$ and for some $y' \in U$ we have $\langle x', y'\rangle \in \bar{F}$. On the other hand, if $x' \in D$ then there must be some y such that $\langle x', y'\rangle \in \bar{F}$ — and furthermore $\{x', y', z\} \neq \{0, 1, 2\}$ since $y' \in \{3, 4, 5\}$. This proves the lemma.

LEMMA 3.6. *The formula* $\pi \to (D_{\leqq 2})_\pi$ *is valid in the model* M.

We first remark that this lemma differs from the preceding two in a very important respect. In the case of Lemma 3.4, for example, not only is the formula $\pi \to (\text{Fun}\,(F))_\pi$ valid in M, as stated, but so also is the formula Fun (F) itself. Similarly in the case of Lemma 3.5. On the other hand, the sentence $D_{\leqq 2}$ is certainly false in M. Nevertheless, we shall show that $\pi \to (D_{\leqq 2})_\pi$ is valid in M — and in fact, it is valid in *every* model (i.e., logically valid).

Referring to the definitions of π, of $D_{\leqq 2}$, and of relativization, we see that:

$$(D_{\leqq 2})_\pi \text{ is logically equivalent to}$$
$$\neg\, [\exists v_0(\pi \wedge \exists v_1(\pi \wedge \exists v_2(\pi \wedge \neg\, \pi)))],$$
and hence is logically valid.

Hence $\pi \to (D_{\leqq 2})_\pi$ is logically valid, and the lemma holds.

LEMMA 3.7. *The formula* $\pi \to (\neg\, R_{\leqq 2})_\pi$ *is valid in the model* M.

The proof is similar to that of Lemmas 3.4 and 3.5, and will be omitted. (Note that $\neg\, R_{\leqq 2}$ is itself valid in M.)

LEMMA 3.8. *The formula $\pi \to (\neg \varphi^*)_\pi$ is valid in the model M.*

This follows at once from Lemmas 3.4, 3.5, 3.6, and 3.7, since from the definitions of φ^* and of relativization we see that $(\neg \varphi^*)_\pi$ is logically equivalent to

$$(\mathrm{Fun}(F))_\pi \wedge (\mathrm{Dom}(F, D))_\pi \wedge (\mathrm{Ran}(F, D))_\pi \wedge (D_{\leq 2})_\pi \wedge (\neg R_{\leq 2})_\pi .$$

LEMMA 3.9. *It is not the case that $\Delta \vdash_3 \pi \to \varphi^*$.*

This follows from Lemmas 3.2, 3.3, and 3.8 since the logical axioms of the deductive system D_3 are valid in M (as in every model), and the rules of inference of D_3 preserve validity in M (as in every model).

Combining Lemma 3.9 with the principle of relativization (Theorem 1), we see at once that we have *not* $\vdash_3 \varphi^*$. This proves Theorem 3.

Theorem 3 is an independence proof, and it is customary for logicians to inquire about the nature of the deductive system in which an independence proof is carried out. Once Theorem 1 is granted — and we believe our proof of it is intuitionistically impeccable — the demonstration of the non-provability of φ^* in the system D_3 reduces to the choice of the formula π, the construction of the model M, and the demonstration that all formulas of Δ, as well as the formula $\pi \to (\neg \varphi^*)_\pi$, are valid in M. Since M has 6 elements in its domain, it is quite evident that we can describe M by means of the grammar G_6 in which F, D, and R are the predicate symbols. On the basis of this grammar we can construct a deductive system D_6' by adding to the logical axioms and rules of D_6 a single non-logical axiom which, in terms of the predicate symbols F, D, and R, completely describes our model M.

Let us write $\vdash_6' \varphi$ to indicate that a formula φ of G_6 is provable in this system D_6'. Although we do not pretend to have checked the matter in any detail, it seems plausible that our non-formalized proofs of Lemmas 3.2, 3.3, and 3.8 could be formalized within D_6', leading to the results:

$$\vdash_6' \psi \text{ for every } \psi \in \Delta,$$

and

$$\vdash_6' \pi \to (\neg \varphi^*)_\pi .$$

In this sense we may be said to have established the independence of φ^* in the system D_3, relative to the consistency of the system D_6'.

Despite the fact that I lived for a year in Amsterdam — and a very pleasant year it was, thanks in large part to Professor dr. Heyting — I do not think I ever met a mathematician who would have doubts about the consistency of the system D'_6.

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