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Assertions depending on time and corresponding logical calculi

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Dedicated to A. Heyting on the occasion of his 70th birthday

by

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S. A. Kripke on the Oxford 1963 Colloquium spoke about interpretations of intuitionistic logic in [3]. I did not attend this meeting; but I also published similar ideas in 1964 in [2] developing a bit more the philosophical interpretation. Now (using the classical logic in my reasoning) I shall display a similar philosophico-methodological analysis as leading to interpretations not only for intuitionistic logic but also for some other logical calculi.

Let the fixed structure $\langle T, \prec \rangle$ be called time. Let $\prec$ be the transitive and antisymmetric relation of time-succession between moments (i.e. elements of the set T). Every empirical inquiry E may be identified with the couple $\langle A_E, B_E \rangle$ where $B_E(t)$ is the set of objects observed (or investigated) till the moment t. The set $B_E(t)$ increases in time, so that we have

$$(1) \qquad t \prec s \rightarrow B_E(t) \subset B_E(s)$$

and $A_E(t)$ is the set of atomic empirical sentences we are forced to assert in performing experiments prescribed to the moment t by the programme of our inquiry E. The atomic sentences (which are empirical) if once asserted, cannot be rejected later, i.e.:

$$(2) \qquad t \prec s \rightarrow A_E(t) \subset A_E(s).$$

The basis for our analysis is the definition of strong assertion, i.e. of the expression: "in my inquiry E I must assert the sentence $\tilde{\Phi}$ at the moment t", in symbols $As_E(\tilde{\Phi}, t)$. The meaning of this formula may be considered as given by intuition only with respect to the atomic empirical statements: $P(a), R(a, b), \cdots$ where $P, R, \cdots$ are observational predicates and $a, b, c \cdots$ are names of observable objects. For a given inquiry E the range of As_E for atomic sentences is defined by the function A_E:

$$(3) \qquad \tilde{\Phi} \in \text{atomic} \rightarrow \left(As_E(\tilde{\Phi}, t) \equiv \tilde{\Phi} \in A_E(t) \right).$$

For compound sentences strong assertion is defined by induction on the number of operations; this is true even for negation. Several items of this definition seem to fit with some considerations of Carnap and Mehlberg [4] concerning e.g. empirical meaning of negation. I must assert $\neg \tilde{\Phi}$ in the moment t iff I am sure that my inquiry E will never force me to assert the sentence $\tilde{\Phi}$ in later moments:

$$(4) \qquad As_E(\neg \tilde{\Phi}, t) \equiv \bigwedge s \in T\left(t \prec s \to \sim \left(As_E(\Phi, s)\right)\right).$$

The assertion of implication and quantification has a similar a priori character:

$$(5) \quad As_E(\tilde{\Phi} \to \Psi, t) \equiv \bigwedge s \in T\left(t \prec s \to \left(\sim As_E(\tilde{\Phi}, s) \vee As_E(\Psi, s)\right)\right)$$

$$(6) \quad As_E(\bigwedge x\tilde{\Phi}(x), t) \equiv \bigwedge s \in T\left(t \prec s \to \bigwedge a \in B_E(s) As_E(\tilde{\Phi}(\lfloor a \rfloor), s)\right)$$

where $\lfloor a \rfloor$ is the name of the object a. For other connectives the strong assertion seems to fulfil the normal conditions of Tarski's notion [6] of classical satisfaction:

$$(7) \qquad As_E(\Phi \vee \Psi, t) \equiv As_E(\Phi, t) \vee As_E(\Psi, t),$$

$$(8) \qquad As_E(\tilde{\Phi} \wedge \Psi, t) \equiv As_E(\tilde{\Phi}, t) \wedge As_E(\Psi, t),$$

$$(9) \qquad As_E(\bigvee x\Phi(x), t) \equiv \bigvee a \in B_E(t) As_E(\Phi(\lfloor a \rfloor), t).$$

Supposing (in conformity with the physical relativistic theory of time) that every finitary tree ordering may be embedded in $\langle T, \prec \rangle$, we can reread Kripke's result as saying that:

(10) The valid formulas of the intuitionistic formal logical calculus of A. Heyting (with quantifiers) is identical with the set of formulas which must be asserted in every moment t of every inquiry E.

According to philosophical tradition, logic is a set of formulas which are assertible in every situation. Hence intuitionistic formal calculus constitutes the logic of strong assertion in the proper sense of the word.

This characterization of intuitionistic calculus suggests an inquiry concerning some other kinds of assertion and their logics. I shall present one example of the study going in this direction.

Strong assertion seems to occur especially in sciences operating with descriptive predicates. For investigations using more theoretical notions a kind of weak assertion was discovered by Popper [5]. Following Popper's analysis one can say that assertion in theoretical research may be expressed rather by using the utter-

ance: "In my investigation E I cannot refute the sentence $\tilde{\Phi}$ in the moment t", or in other words "I can admit Φ as supposition in situation t of my investigation E", in symbols: $Ad_E(\Phi, t)$. Hence, if we confine ourselves to the sentences without quantifiers, a theoretical investigation E may be identified with the triple $\langle A_E, R_E, L_E \rangle$ where $A_E(\Phi)$, for Φ atomic, is the set of moments in which I can admit $\tilde{\Phi}$, $R_E(\tilde{\Phi})$ for $\tilde{\Phi}$ atomic is the set of moments in which I can admit $\neg\tilde{\Phi}$, and L_E is the conjunction of all theories previously accepted as indubitable theoretical background for the investigation E. Then

(11) $$(t \in T \wedge \tilde{\Phi} \in L_E) \to Ad_E(\tilde{\Phi}, t).$$

Atomic theoretical sentences usually concern equalities or order relations between theoretical quantities (electric field, temperature, length). The following postulates seem to be relevant:

An atomic sentence if once refuted can not be admitted later:

(12) $$\left(\tilde{\Phi} \in \text{atomic} \wedge s \prec t \wedge Ad_E(\tilde{\Phi}, t)\right) \to Ad_E(\Phi, s).$$

The same holds for negations of atomic sentences:

(13) $$\left(\tilde{\Phi} \in \text{atomic} \wedge s \prec t \wedge Ad_E(\neg \Phi, t)\right) \to Ad_E(\neg \tilde{\Phi}, s).$$

We are never obliged to refute both $\tilde{\Phi}$ and $\neg\tilde{\Phi}$:

(14) $$T' = A_E(\tilde{\Phi}) \cup R_E(\tilde{\Phi})$$

But there are many moments in which we can admit $\tilde{\Phi}$ as well as $\neg\tilde{\Phi}$, especially at the beginning of the research when we have no information. Hence we can assume that:

(15) $\tilde{\Phi} \in \text{atomic} \to \bigwedge t \in T \bigvee s \in T$

$$s \prec t \wedge \bigwedge s' \in T\left(s' \prec s \to (Ad_E(\tilde{\Phi}, s') \wedge Ad_E(\neg \tilde{\Phi}, s'))\right)$$

The definition of weak assertion begins of course by the conditions for atomic sentences:

(16) $$(\tilde{\Phi} \in \text{atomic} \wedge t \in T) \to \begin{cases} Ad_E(\tilde{\Phi}, t) \equiv t \in A_E(\tilde{\Phi}), \\ Ad_E(\neg\tilde{\Phi}, t) \equiv t \in R_E(\tilde{\Phi}). \end{cases}$$

Hencefor every atomic $\tilde{\Phi}$ the functions A_E (and R_E) determine at which moment of time we must refute $\tilde{\Phi}$ (respectively $\neg\tilde{\Phi}$).

Then for compound sentences $\alpha(\tilde{\Phi}, \Psi)$ we must define inductively admissibility of $\alpha(\tilde{\Phi}, \Psi)$ together with the admissibility of its negation $\neg\alpha(\tilde{\Phi}, \Psi)$ for $\alpha = \vee, \wedge, \to$. The simplest case is that of alternative:

(17) $Ad_E(\widetilde{\varPhi} \vee \varPsi, t) \equiv Ad_E(\widetilde{\varPhi}, t) \vee Ad_E(\varPsi, t)$

(18) $Ad_E(\neg(\widetilde{\varPhi} \vee \varPsi), t) \equiv Ad_E(\neg\, \widetilde{\varPhi}, t) \wedge Ad_E(\neg\, \varPsi, t)$

The conjunction of two admissible sentences is admissible if they are consistent with the theory L_E assumed as the background for the investigation E:

(19) $Ad_E(\widetilde{\varPhi} \wedge \varPsi, t) \equiv Ad_E(\varPhi, t) \wedge Ad_E(\varPsi, t) \wedge \neg(\widetilde{\varPhi} \wedge \varPsi) \notin L_E$

(20) $Ad_E(\neg(\varPhi \wedge \varPsi), t) \equiv Ad_E(\neg\varPhi, t) \vee Ad_E(\neg\varPsi, t) \vee \neg(\widetilde{\varPhi} \wedge \varPsi) \in L_E$

The implication $\widetilde{\varPhi} \to \varPsi$ is admissible in the situation t iff for every former situation s, if s allows us to not refute $\widetilde{\varPhi}$, then s allows us to not refute $\varPsi$. Thus implication does not decide anything about the future, but only generalizes former situations:

(21) $Ad_E(\widetilde{\varPhi} \to \varPsi, t) \equiv \bigwedge s \in T\big((s \prec t \wedge Ad_E(\widetilde{\varPhi}, s)) \to Ad_E(\varPsi, s)\big).$

The negation of an implication is admissible if we have had an admissible counterexample:

(22) $Ad_E(\neg(\widetilde{\varPhi} \to \varPsi), t) \equiv \bigvee s \in T(s \prec t \wedge Ad_E(\widetilde{\varPhi}, s) \wedge Ad(\neg\varPsi, s))$

To complete the definition we must define the admissibility of the negation of a negation:

(23) $Ad_E(\neg\,\neg\, \widetilde{\varPhi}, t) \equiv Ad_E(\widetilde{\varPhi}, t)$

For the above notion of admissibility one can prove that:

(24) If we consider the investigations E such that L_E contains the intuitionistic calculus, then the set of formulas which may be admitted in every moment of every investigation is identical with classical calculus.

The proof consists in the verification (by induction on the length of the formula) of the rule of excluded middle.

If we take no assumption about the theory L_E then the set of formulas which may be admitted in every moment of every investigation is much poorer (but it contains e.g. the system of strict implication of Anderson and Belnap [1]).

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