A. S. TROELSTRA

The scientific work of A. Heyting

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Dedicated to A. Heyting on the occasion of his 70th birthday

by

A. S. Troelstra [1]

The bibliography of an author might be described as a linearly ordered set of titles of books and papers, with an order relation introduced by the time of appearance. As such, it is a rather colorless object, the chronological order being the only aspect linking it with the activity of a human being. Perhaps A. Heyting himself would be quite happy with a bibliography of his work, and nothing more, on such an occasion as a "Festschrift".

However, it seems to me that a "Festschrift" ought to recognize, in a dignified and therefore modest way, that there is something more to the scientific activity of a human being, than is apparent from such a very rough condensation as is a bibliography.

The following lines, therefore, are intended to accompany Heytings bibliography, not as a critical discussion, nor as an eulogy, but in order to add some touches of color and to provide some background.

Or, returning to a more mathematical mode of expression, next to the static information given by the ordered set, we want to show the existence of other relationships between the elements of the set, namely the various chains of thought linking a lifetime of scientific contemplation.

Heyting's work is almost exclusively concerned with intuitionism and philosophy of mathematics from an intuitionistic point of view. The contents of the bibliography might be classified under the headings: contributions to the technical development of intuitionistic mathematics; discussion of the philosophical position and the fundamental notions of intuitionism; expositions of the basic ideas and surveys of results in intuitionism; books and papers on other subjects.

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We shall adopt this crude classification in our survey [2]. The work concerned with the technical development of intuitionistic mathematics can be considered as a contribution to the execution of Brouwer's program of rebuilding mathematics in accordance with intuitionistic requirements.

Reconstruction of mathematics along intuitionistic lines is very often a matter of subtle distinctions and methods, which can be satisfactorily presented only in a detailed account of the subject. Therefore we have to limit ourselves to a short description of Heyting's contributions to various fields of intuitionistic mathematics.

Heyting's thesis [1] deals with the axiomatics of intuitionistic projective geometry. The purpose here is to present a complete system for projective geometry in two and three dimensions; therefore the system is made categorical by means of postulates on order and continuity. The account in [4] however, considers incidence postulates only. Heyting shows that the analytic projective geometries with coordinates taken from a skew field are models for his axioms. (The algebraic preliminaries needed are treated in [3].) Conversely, he shows that coordinates can be introduced in two and three-dimensional projective spaces. These coordinates constitute a skew field under suitable definitions of the operations; and Heyting proves intuitionistically that the validity of Pascal's theorem in the projective space is equivalent to the commutativity of the multiplication in the field of coordinates.

Later, in [51], Heyting returns to the subject of geometry; in this paper the connection between the intuitionistic affine and projective planes is investigated.[3]

Heyting's researches on intuitionistic algebra are contained in [3], [26], [54]. [3] is a self contained treatment of the theory of determinants with coefficients in a skew field; in [26], parts of the theory of fields are developed intuitionistically, among others divisibility of polynomials, transcendental extensions, elimination from binary equations, factorization of polynomials.

A survey of intuitionistic axiomatizations of various algebraic systems is given in [54].

[2] References to the bibliography of A. Heyting are indicated by a numeral between square brackets (e.g. [3]); other literature references are indicated by numerals between round brackets (e.g. (4)) and refer to the list at the end of this paper.

[3] Van Dalen's work (2) contains a further development of Heyting's researches on this subject.

We mention briefly three contributions of a technical nature: quadratic forms in separable Hilbert space ([35]), an intuitionistic version of von Neumann's proof of the Riesz-Fischer theorem ([33]), and a discussion of Brouwer's "predicates of countability" ([5])[4].

The next subject gradually leads us to discussions of a more fundamental nature, since Heyting's work on the formalization of intuitionistic logic and mathematics is important from a technical as well as from a philosophical point of view. The fundamental papers [6], [7] and [8] contain a formalization of intuitionistic predicate logic and arithmetic, and a partial formalization of intuitionistic analysis. They received much attention and instigated many investigations on formal systems representing intuitionistic logic and parts of intuitionistic mathematics. Initially, these researches dealt almost exclusively with logic. This gave rise to the false idea that intuitionistic mathematics consisted of a certain part of classical mathematics notwithstanding the fact that Brouwer had stressed time and again the essential divergence between the classical and the intuitionistic view point.

Lately, in consequence of the efforts of, among others, Kleene and Kreisel, the divergence between intuitionistic and classical mathematics has received more general attention. (See e.g. (4)).

Heyting's paper [27] deals with relationships between various "translations" of the quantifiers (x), (Ex), which are classically equivalent, but intuitionistically not so.

Among Heyting's most important contributions to the discussion of basic notions is his clarification of the interpretation of the logical constants and theorems of logic. ([9], [10], [12—15], [17], [36], [40], [42], [50], [53]; compare his own survey [48], pp. 106—110.)

Originally he proposed the interpretation of logical formulas as denoting intentions of constructions. This interpretation and the corresponding explanation of the logical constants are illustrated by the following quotations ([10], p. 113)

"A logical function is a method which transforms any given assertion into another assertion. Negation is such a function; its meaning has been described quite accurately by Becker, in agree-

[4] Intuitionistic Hilbert space is treated in the recent work of Ashvinikumar (1). A unified treatment of Brouwer's researches in (2) and Heyting's contributions in [5] can be found in (5).

ment with Husserl. According to him, negation is something throughout positive in character, namely the intention of a contradiction connected with the original intention. Here the assertion "C is not rational" expresses the expectation that one could derive a contradiction from the assumption "C is rational"."

and from [10], p. 114

"$p \vee q$ indicates the intention which is satisfied if and only if at least one of the intentions p and q is satisfied. The formula expressing the law of the excluded middle ought to be $p \vee \neg p$. For a given assertion p, this law can be asserted if and only if either p has been proved or p has been reduced to a contradiction.

Therefore a proof for the general law of the excluded third has to consist of a method which permits one given any assertion either to prove this assertion or to prove its negation".

The interpretation of the implication is illustrated by ([15], p. 14)

"$a \supset b$ therefore represents the intention of a construction which transforms every proof of a into a proof of b."

At that time, Heyting interpreted a logical formula as an intention of a construction, not as a construction. On this view, there is a difference between the assertion of "p" and the assertion "p has been proved". Heyting devotes some considerations to this distinction in [9] and [10], and he makes the remark that if every assertion is to express a construction, then the distinction vanishes.

Later, he interpretes logical formulas as assertions about constructions; accordingly, logical laws are viewed as mathematical laws of a very general nature, namely laws about certain simple operations on constructions ([36]). He characterizes intuitionistic logic in a general way as the logic of knowledge ([42]).

Heyting has commented repeatedly on Griss' objections to the use of negation in intuitionistic mathematics. ([32], [46], [48] pp. 110—112). Although he acknowledges in some respects the validity of Griss' criticism, he remarks ([32], p. 95)

"It is certainly too early to judge of the value of Griss' conception of mathematics. I venture one critical remark. It seems to me that it is impossible to banish all unrealized suppositions, for such suppositions are implicit in every general proposition. If we say "for all real numbers a and b, $a+b = b+a$" this means the same as "if we construct two real numbers a and b, then $a+b = b+a$", but we have not actually constructed them."

Griss' criticism induced Heyting to distinguish between various levels of constructive evidence. In [56] he enumerates as illustration the following notions and constructions in ascending order of complexity (or descending degree of evidence): assertions about small numbers, like $2+4 = 4+2$; assertions about large numbers, like $1002+4 = 1004+2$; the notion of order type ω, as it occurs in the definition of constructible ordinals; the use of quantified expressions in logical formulas; the introduction of indefinitely proceeding sequences (ips, choice sequences); notion of a species.

This demonstrates a certain shift from absolutism to relativism with respect to the conception of intuitionism.

Concerning the relation between language and mathematics, Heyting generally holds the same view as Brouwer did, namely that language is not essential to intuitionistic mathematics. This does not mean that the separation between language and mathematics is absolute. Heyting expresses this as follows: ([29], pp. 128—129)

"The intuitionist therefore looks for rigor not in language, but in mathematical thought itself. At the same time it seems to me to contradict reality, to suppose that intuitionistic mathematics in its pure form consists only of constructions in the thoughts of individual mathematicians, constructions which exist independently of each other and between which language provides no more than a very loose connection.

For that purpose, different mathematicians influence each other too much and understand each other too well."

"The intuitionist also uses ordinary language as well as the language of signs as an aid to the memory. We must beware of the fiction of the mathematician with the perfect memory, who could do without the support of language. In actual mathematical research language is essentially involved from the beginning; mathematics as it presents itself to us, converted into linguistic expressions is not preceded by a phase completely devoid of language, but it is preceded by a phase in which the role of language is much less important than in the communication."

With respect to the axiomatic method ([51], [54], [55]) Heyting distinguishes between a creative function and a descriptive function of axiomatization. According to his view the axiomatic method is used in its creative function in classical set theory, by requiring the existence of objects, which existence is not ensured by a construction. Only the descriptive use of the axiomatic method (i.e. the use for systematizing and abbreviating) is legitimate in intuitionistic mathematics. For this reason, Heyting is not opposed to formalization and axiomatization of parts of intuitionistic mathematics. (Brouwer on the other hand avoided formalization and axiomatization, in order to stress the independence of intuitionistic mathematics from logic and formal language.)

Heyting's use of formal language seems to have created a misunderstanding in some cases, namely that "intuitionism" was fully described by the formal systems given for parts of intuitionistic mathematics, notwithstanding the fact that Heyting himself warned his readers in [6], page 42, where he states

"Intuitionistic mathematics is a mental process, and every language, the formalistic one included, is an aid to communication only. It is in principle impossible to construct a system of formulas equivalent to intuitionistic mathematics, since the possibilities of thinking cannot be reduced to a finite number of rules constructed in advance."

Another subject considered repeatedly by Heyting in his papers is Church's thesis and the relation between the theory of recursive functions and intuitionism. ([46], p. 340, [52], [56].) Here Church's thesis is understood as the assertion: Every constructive (effectively calculable) number theoretic function is recursive.

Heyting remarks that (1) there are two possible interpretations of recursion theory, an intuitionistic and a classical one, (2) classical interpretation of recursion theory is a falsification of constructivist intentions, since assertions about the existence of algorithms might have non-constructive proofs on this interpretation. (3) if recursion theory is interpreted intuitionistically, the definition of effectively calculable function by means of Church's thesis is circular, and (4) if recursion theory is interpreted classically, certain intuitionistic distinctions, but not all, have their counterpart in recursion theory, if we translate effective by recursive. A simple illustration of the fourth remark: intuitionistically, not every subset of a finite set can be proved to be decidable; classically, every subset of a finite set is recursive.

The bibliography contains a number of papers and books of an expository character.

From these, especially [40], [41] and [48] deserve mentioning. [41] is intended as an introduction for mathematicians not acquainted with foundations in general and intuitionistic mathematics in particular, to enable them to see which results in their respective fields might be susceptible to intuitionistic treatment.

Among Heyting's writings on subjects outside the domain of intuitionism, there are three textbooks, to wit [28], [57], [58], which are fine examples of Heyting's didactical qualities.

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