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Dedicated to A. Heyting on the occasion of his 70th birthday

by

G. Kreisel

Introduction

The two main subjects of a theory of *choice sequences* are, of course, the notion (or, perhaps, class of notions) of choice sequence, and the *operations* applied to such sequences. Both points are well illustrated by the theory of convergent sequences of rational intervals $[r_n, r'_n]$, $\boldsymbol{r}$ for short, and, say, the (usual) *sum* operation. These $\boldsymbol{r}$ obey the condition, for $n = 0, 1, 2, \cdots$,

$$r_n \leqq r_{n+1} < r'_{n+1} \leqq r'_n < r_n + 2^{-n}.$$

Thus, r_0 is chosen arbitrarily, r'_0 must lie in the interval (r_0, r_0+1), r_1 in $[r_0, r'_0)$, r'_1 in $(r_0, r'_0] \cap (r_1, r_1+2^{-1})$ and so on. The *sum*, say $\boldsymbol{t}$, of $\boldsymbol{r}$ and $\boldsymbol{s}$ is given by $t_n = r_{n+1}+s_{n+1}$, $t'_n = r'_{n+1}+s'_{n+1}$. To verify that $\boldsymbol{t}$ satisfies the basic condition

$$(t_n \leqq t_{n+1} < t'_{n+1} \leqq t'_n < t_n + 2^{-n})$$

we use

$$r_{n+1} \leqq r_{n+2} < r'_{n+2} \leqq r'_{n+1} < r_{n+1} + 2^{-(n+1)},$$
$$s_{n+1} \leqq s_{n+2} < s'_{n+2} \leqq s'_{n+1} < s_{n+1} + 2^{-(n+1)}$$

and

$$2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.$$

For any n, the values of t_n and t'_n are given by r_{n+1} and s_{n+1}, respectively r'_{n+1} and s'_{n+1}. No use is made of any *rule* or *law* that may (or may not) be involved in the calculation of $\boldsymbol{r}$ and $\boldsymbol{s}$. Specifically, the values of t_n and t'_n depend only on (a finite number of) values of $\boldsymbol{r}$ and $\boldsymbol{s}$ and not, for instance in the case of recursive rules for $\boldsymbol{r}$ and $\boldsymbol{s}$, on the defining equations.

The example shows that even if we *started* with a specific class of rules for constructing sequences of rational intervals, we should be led to consider operations, such as the *sum* above, which are quite meaningful without the restriction to this class, or, in fact, without any restriction on the rules. In short, such operations apply to *choice sequences.* The reader may compare this situation with the case of algebraic operations on the real numbers. Even when we start with the specific structure of the reals, we see that the algebraic operations use little of this structure; as we say nowadays, they apply in any real closed field. In contrast to the notion of real closed fields, the notion of choice sequence has a rather clear informal meaning, and it is therefore useful to describe more fully in informal terms the objects we are going to study. (The description will be made more precise by the axiomatic analysis in the body of the text.)

Informal destinctions. Let us look more closely at the *restrictions* on the sequences $\boldsymbol{r}$ in the example above. What is typical here, at least of most examples in analysis, is that we have a rule, given in advance, telling us for each $\boldsymbol{n}$ what (finite) sequences $[r_0, r'_0], \cdots, [r_n, r'_n]$ are "admitted"; and that any such sequence can be continued, i.e., there is also a pair $[r_{n+1}, r'_{n+1}]$ such that the rule admits

$$[r_0, r'_0], \cdots, [r_n, r'_n], [r_{n+1}, r'_{n+1}];$$

e.g.,

$$r_{n+1} = (2r_n+r'_n)/3,\ r'_{n+1} = \min\,[r_{n+1}+2^{-(n+1)}, (r_n+2r'_n)/3].$$

This type of restriction is called a *spread.*[1]

We may reformulate the restriction used in the example, and thus obtain a useful generalization. Instead of imposing a condition on a sequence $\boldsymbol{r}$ directly, we start with an *unrestricted sequence of pairs of rationals,* say (a_n, b_n) and associate with it, in a canonical manner, a sequence $\bar{a}_n, \bar{a}'_n$ such that

$[\bar{a}_n, \bar{a}'_n]$ always satisfies our condition,
if $[a_0, b_0], \cdots, [a_n, b_n], \cdots, [a_m, b_m]$ satisfies our condition for $n \leqq m$ then $\bar{a}_n = a_n$, $\bar{a}'_n = b_n$ for $n \leqq m$.

Definition. Put $\bar{a}_m = a_m$, $\bar{a}'_m = b_m$ if $[a_0, b_0], \cdots, [a_m, b_m]$ satisfies our condition. If not, let n_0, necessarily $\leqq m$, be the first n such that $[a_0, b_0], \cdots, [a_{n_0}, b_{n_0}]$ does not satisfy the condition. ($n_0 = 0$ means that $b_0 \leqq a_0$ or $b_0 \geqq a_0+1$.) Put $\bar{a}_m = a_{n_0-1}$ if $n_0 \neq 0$, and $\bar{a}_m = a_0$ if $n_0 = 0$, for all $m \geqq n_0$. Put $\bar{a}'_m = a_0+2^{-(m+1)}$

[1] The particular spread here considered is *stochastic,* i.e., the restriction on $[r_{n+1}, r'_{n+1}]$ depends only on $[r_n, r'_n]$ and not on any $[r_m, r'_m]$ for $m < n$. Though stochastic spreads are not of special foundational importance, they are technically interesting. (I owe this information to Dr. Troelstra.)

if $n_0 = 0$, and otherwise

$$\bar{a}'_m = \min(a_{n_0-1}+2^{-(m+1)}, b_{n_0-1}).$$

Note that the values of each $\bar{a}_m$, $\bar{a}'_m$ still only depend on a *finite* number of values of (a_i, b_i) $(i \leqq m)$, but not merely on, say, a_m, b_m.

The general principle involved is that we start with an unrestricted sequence, and apply an operation to it to form a new sequence. For any given spread, as shown in the example, we have an operation which maps an unrestricted sequence into the spread and leaves invariant any sequence that is already in the spread. These operations are of the same kind as the *sum* operation above. We shall return in Section 4 to the *technical* question what operations other than mappings into spreads are useful.

Once we think of restrictions we are led, as in Brouwer's [2], p. 323 (2), to consider more sophisticated versions: restrictions *on* restrictions, so-called second order restrictions; third order restrictions on second order restrictions and so forth. Brouwer himself did not pursue his ideas, perhaps because he realized too quickly that a theory of *general* higher order restrictions might be hopelessly complicated (and inelegant even when compared to the horrors of the worst kind of intuitionistic mathematics); cf. footnote, p. 142, in [3].[2]

The sequences to be considered in detail in the present paper are those where the *simplest kind of restriction on restrictions is made*, namely some finite initial segment of values is prescribed, and, beyond this, no restriction is to be made.[3] I expressed this idea by *absolutely free* in [6], but shall call these sequences *lawless*

[2] I am indebted to A. S. Troelstra for the reference to [2] and to S. A. Kripke for the reference to [3]. *Correction.* On p. 180 of [8] I misinterpreted Brouwer's footnote on p. 142 of [3] to refer to lawless sequences. Dr. Troelstra pointed out to me that *higher order* restrictions are meant.

[3] More precisely one first makes a *general* restriction on the species from which the elements of the sequence are chosen; here natural numbers. The work extends directly to any countable decidable species. To avoid a possible misunderstanding it is as well to note the following *distinction.* The restriction involved in the notion of lawless sequence is intended to mean that in any *particular* context only a finite initial segment is used, not that the sequence is *given* by an initial segment once and for all. (Formally, such independence of context would be expressed by modifying axiom 2.3 on p. 15. putting "$\exists n$" between "$\forall\alpha$" and "$\forall\alpha_1$"; evidently several of the axioms of Section 2, e.g. 2.4, are false for this second kind of notion.) In brief, the theory of lawless sequences cannot be said to be about operations on a "finite amount of information".

here.[4] (We consider sequences of natural numbers; so in the examples above, pairs of rationals would be coded by a natural number.)

Principal result (Section 2). We consider the context in which the subject of sequences first presents itself, namely we have variables for natural numbers, constructive number theoretic functions (i.e., rules), lawless sequences, and *species* of such objects, and consider compound expressions built up logically. We find a *complete analysis of lawless sequences in this context.* Precisely, we give enough basic properties of the objects discussed, to construct, for any assertion A in the present context, a A' not containing symbols for lawless sequences at all, which is equivalent to A. More formally, $A \leftrightarrow A'$ follows from the particular properties (axioms) of our basic notions.[5] The theory of the notions other than lawless sequences, i.e., of the notions involved in A', is given in Section 1. The theory of Section 2 is specialized to *binary* sequences in Section 3, superseding results on a fragment given in [6].

General discussion. The following comments on the principal results, may, I believe, be useful before reading the technical sections; of course the latter can be read independently.

Elsewhere ([8], 2.523 on p. 136 and 2.622 on p. 140). I have described theorems similar to the principal result above as *elimination results.* There I thought of the results as means of "getting rid" or "analyzing away" certain notions of choice sequence (for reasons given in footnote 12 below). But the principal result is *not* to be interpreted in this way. We have simply discovered

[4] The term *lawless* was proposed by Gödel (in conversation) after he had seen the properties of absolutely free sequences given in my paper [8]. (Neither of us knew at the time Brouwer's earlier analysis.) In choosing between the two terms one faces the familiar issue between freedom and licence (lawlessness). Ten years ago I certainly felt that even the thought of restreint, e.g., the possibility of a diet, was a restriction of freedom, and so I naturally used "absolutely free". With the years the lure of licence has diminished: hence the present title. May healthier and livelier (nowadays called "hippier") readers not be misled by it!

[5] Two "refinements" are worth noting. First (for reasons set out in 1.1 below) our formal language contains not only specific species, explicitly defined from constants, but species *variables*. Second, there is also a proof theoretic result (but not established in the present paper): if A can be formally derived from our axioms for lawless sequences and constructive objects (numbers and constructive functions) then A' can be derived from our axioms for constructive objects. This syntactic result is quite elementary, for instance derivable on primitive recursive arithmetic. This provides the classical consistency proof left open in [6], p. 386.

enough about lawless sequences to be able to assert $A \leftrightarrow A'$: this does not "get rid" of lawless sequences since they are involved in A itself! In other words, the situation is parallel to the elimination of quantifiers in certain axiomatic theorems. More specifically, recall that our formal systems are not complete. So not all statements A' are decided. It may well turn out that some A is **evident** for our interpretation, while A' is not: in this case we should convince ourselves of A' by *using* the result $A \leftrightarrow A'$. (Note however, by the second result in footnote 5, that, at the *present* time, any such use of lawless sequences seems to be eliminable since the axioms of Section 2 seem to codify all known informal principles valid for lawless sequences of natural numbers.)

An "unusual" feature of our exposition of the subject is the *explicit* use of "constructive function" as a *primitive* concept. Of course, it is used implicitly when the notion of constructive function is "defined" as in recursion theory, since the notion is involved in the quantifier combination $\forall x \exists y$ in $\forall x \exists y T(e, x, y)$. Besides certain technical advantages, this explicit use is relevant to the informal discussions of the notion of choice sequence particularly by Myhill [12] and Troelstra [13]. They have considered certain *problematic* forms of the axiom of choice (problematic, because one considers not *arbitrary* selection operators, but extensional or even continuous ones). To disentangle the roles of constructive functions and of choice sequences in these discussions, one must avoid a premature identification between constructive functions and some defined concept. (The literature is discussed briefly at the end of Sections 1 and 4.) [6]

A principal tool in our analysis is the notion of *Brouwer operation* (Section 1) which is intended to formulate generally the *continuity*

[6] *Correction.* Another context where questions about constructive functions were misstated as being about choice sequences, is in connection with Gödel's result on Heyting's predicate calculus (see e.g. [8], p. 146, 2.741). If the latter is complete then, for each primitive recursive property $A(n)$ we have

$$(*) \qquad (\forall a_B) \neg\neg \exists x A(\bar{a}_B x) \rightarrow \exists y \forall a_B (\exists x \leqq y) A(\bar{a}_B x),$$

where a_B ranges over all constructive functions taking the values 0 or 1, and $\bar{a}_B x$ denotes the sequence $\langle a_B 0, \cdots, a_B(x-1)\rangle$. Note that $(\forall a_B)(\exists x \leqq y)A(\bar{a}_B x)$ is a decidable property of y, and equivalent to $(\forall \alpha_B^*)(\exists x \leqq y)A(\bar{\alpha}_B^* x)$, where α_B^* ranges over choice sequences taking values 0 or 1 only. Gödel's *argument* establishes (*), but only the weaker result

$$(\forall \alpha_B^*) \neg\neg \exists x A(\bar{\alpha}_B^* x) \rightarrow \exists y (\forall \alpha_B^*)(\exists x \leqq y) A(\bar{\alpha}_B^* x)$$

was stated. NB. The α_B^* are *not* lawless sequences, but choice sequences of the kind used in analysis and illustrated in the introduction.

property, i.e., dependence on an initial segment illustrated by the *sum* operation at the beginning of the introduction. The basic question, implicit in [1], is whether *all* completely defined operations on choice sequences are Brouwer operations. The present paper does not attempt to answer this question; but it verifies that Brouwer operations are continuous, and that the familiar, independently defined operations on choice sequences are Brouwer operations. More generally, we give *closure properties* of this class (species) of operations. Perhaps most important, we reduce the problem, in the case of lawless sequences, to the question, whether all *continuous* operations are Brouwer operations (by virtue of the evident axioms of Section 2). Though the problem of non-extensional operations discussed in [12] does not arise for lawless sequences, it appears in Section 4 in connection with certain *derived* notions of choice sequence involving both constructive functions and lawless sequences.

Throughout this paper Heyting's formal rules of intuitionistic logic are used. These rules are valid not only for the Brouwer-Heyting interpretation of the logical particles, but also for Gödel's interpretation in [4]. The axioms of the theory of (completed) constructive objects in Section 1 hold for both interpretations (when Gödel's is extended to a formalism with inductively defirmed *species*). But Gödel's interpretation is not valid for the theory of lawless sequences in Section 2 since, on his interpretation, we have

$$[\forall x(Ax \vee \neg Ax) \wedge \neg\neg \exists x Ax] \rightarrow \exists x Ax$$

by footnote p. 113 of [7], which does not apply when we take $\alpha x = 0$ for Ax, where α is a variable for lawless sequences of natural numbers.

1. Constructive number theoretic functions and Brouwer operations

We confine the description to essentials.

Variables: $x, y, z, \cdots$ for natural numbers; $a, b, c, \cdots$ for monadic number theoretic functions; X, Y, Z for species, sometimes written $X_{n,m}$ to indicate that X has n number and m function arguments (this will be $X_{n,m,0}$ in the notation of the next section to indicate that there are no arguments of lawless sequences).

Constants: 0 and $^+$ for the successor (we write t^+ for ^+t); j_1 and j_2 for inverse pairing functions and $K_{0,1}$ for the species of Brouwer operations.

Relations: $=$ (at least between numerical terms), $V(a, x, y)$ for function evaluation, also written $a(x) = y$ or $ax = y$.

The usual formation rules for *terms* and *formulae* are used where terms are understood to be numerical valued. The only function "terms" are the function variables and function constants.

The *axioms* are divided into groups.

1.1. *Closure conditions on the notion of species* expressing that the basic relations are species, that logical operations can be used to form species and that specialization of some arguments of a species yields a species. (Definitions of species by quantification of species variables are not used in the *basic* theory; see below.)

These axioms are exactly parallel to the *class* formation rules in the theory of classes (see e.g. App. A of [9]). Proof theoretically the theory below is equivalent to a system obtained by replacing axioms involving universal species quantification by axiom schemata applied to *definable* relations. But such a schema leaves open whether the corresponding axiom is valid for arbitrary species or whether it depends on some special property of the definable relations (such as extensionality of V and $K_{0,1}$ in our case).

1.2. *Successor axioms.* $\forall x \neg x^+ = 0$, $\forall x \forall y (x^+ = y^+ \to x = y)$, and

$$\forall X_{1,0}([X0 \wedge \forall x(Xx \to Xx^+)] \to \forall x Xx).$$

One then derives in the usual way induction for $X_{n+1,m}$, and the theorems

$$\forall x \forall y\ (x = y \vee \neg x = y),\ \ \forall x \neg (x^+ = x),\ \ \forall x \exists y\ (\neg x = 0 \to x = y^+),$$

etc.

1.3. *Pairing axioms*: $\forall x \forall y \exists ! z\ (j_1 z = x \wedge j_2 z = y)$.

For any function term t, we can "regard" t as defining a function of two variables by using the *convention* (eliminable abbreviation)

$$t(x, y) = z \text{ for } \exists u\ (j_1 u = x \wedge j_2 u = y \wedge tu = z),$$

whence by use of the pairing axioms we also have

$$t(x, y) = z \leftrightarrow \forall u\ [(j_1 u = x \wedge j_2 u = y) \to tu = z].$$

The fact that $t(x, y) = z$ is in Δ_1^0-form will be of use in Section 2.

1.4. *Closure conditions on the species of constructive functions*, expressing the existence of a function, say $\dot{0}$, such that $\forall x\,(\dot{0}x = 0)$, closure under composition, substitution, and permutation of variables (when functions are regarded as binary); alternatively one could use λ-terms.

To state the two elementary axioms of choice (the countable axiom of choice $AC-NF$, from numbers "N" to functions "F"; and the axiom of dependent choices $DC-F$) we need an *abbreviation* for X containing at least one function argument b:

$$X(a_x) \text{ stands for: } \exists b \forall y\,[(j_1 y = x \to b j_2 y = ay) \wedge Xb].$$

Note that our closure conditions imply

$$\forall x \forall a \exists b \forall y\,(j_1 y = x \to b j_2 y = ay).$$

$$\begin{aligned} AC-NF &: \forall X_{0,1}[\forall x \exists a X(x,a) \to \exists b \forall x X(x, b_x)] \\ DC-F &: \forall X_{1,1}[\forall a \exists b X(a,b) \to \forall c \exists d \forall x (d_0 x = cx \wedge X(d_x, d_{x+1}))]. \end{aligned}$$

By use of $AC-NF$ and induction, one derives closure under *primitive recursion*.

1.4.1. Note that this derivation is much less elementary than the usual proof that the recursion equations together with the usual computation procedure provide a mechanical rule for computing primitive recursive functions. For the function b in $AC-NF$ is thought of as defined from a *proof* of the premise which definition is not *prima facie* mechanical or recursive ([8], p. 131, 2.35). One of the interesting consequences of certain recursive realizability interpretations is that (within a limited context), $AC-NF$ or $DC-FF$ are satisfied, in the sense of these interpretations, even if one restricts oneself to mechanical rules. But, while $AC-NF$ is evident for the general notion of constructive function, the verification of a realizability interpretation is never immediate.

1.4.2. We shall need *concatenation theory* to state the axioms for Brouwer operations. We use the coding of [10], which is slightly different from Kleene's, but use Kleene's $\bar{a}n$ for the (canonical) representation of the sequence $a0, \cdots, ai, \cdots$ for $i < n$.

We use variables $m, n, \cdots$ for natural numbers when we think of them as codes for finite sequences, say $\xi_1, \cdots, \xi_k$. The empty sequence has number 0.

$j_1 m = k$, the length of the sequence,

$j_2 m = \nu(\xi_1, \cdots, \xi_k)$ where $j_1 \nu(\xi_1, \cdots, \xi_k) = \xi_1$, $j_2 \nu(\xi_1, \cdots, \xi_k) = \nu(\xi_2, \cdots, \xi_k)$ if $k > 1$, $= 0$ if $k = 1$. (Thus ν is a many-one numbering of finite sequences which is made $1-1$ by giving the length of the sequence.)

As usual we shall use $*$ for concatenation, but write $n * m$ for $*(n, m)$ (which in turn is to be interpreted according to convention 1.3).

Note that if $\forall x (j_2 x \leqq x)$ we can decide primitive recursively whether any given n is in the range of ν.

1.5. *Brouwer operations*, denoted by $e, f, \cdots$. These are intended to be *neighborhood functions* defined on sequence numbers m so as to induce an assignment of numbers to functions. Specifically, $em = x^+$ is to indicate that for all functions a "belonging" to the neighborhood $(\xi_1, \cdots, \xi_k)$ with number m, i.e., for a such that $ay = \xi_{y+1}$ for $y < k$ or, again for $\bar{a} j_1 m = m$ ($j_1 m$ being the length of the sequence m!), we assign the value x. If $em = 0$ we have left open whether all functions in $(\xi_1, \cdots, \xi_k)$ get the same value. An elementary *consistency* condition is therefore

$$\forall m \forall n \forall x [em = x^+ \rightarrow e(m * n) = x^+].$$

The crucial *continuity* condition which generalizes the essential properties of the sum operation in the introduction is

1.51 $$\forall a \exists m \exists x (\bar{a} j_1 m = m \wedge em = x^+).$$

The critical point, stressed in the introduction, is the following requirement on the quantifier combination $\forall a \exists m$. The existence of m should not be derived by use of delicate properties or assumptions on possible rules for a, but, as in the case of the sum operation, should be insensitive to the class of rules considered. In the present section, after having defined the species of Brouwer operations, we shall do justice to this requirement by proving 1.51 for such operations e, *using nothing about a except that its values are determined!* We do not even use such elementary closure conditions as $\forall a \exists b \forall x (bx = ax^+)$. In the next section we shall reformulate the requirement; instead of making restrictions on the kind of proof of 1.51 to be used, we shall ask for a proof of a different theorem, namely

$$\forall \alpha \exists m \exists x (\bar{\alpha} j_1 m = m \wedge em = x^+)$$

where α ranges over choice sequences.[7]

Axioms for the species $K_{0,1}$ of Brouwer operations. Let $A[Y_{0,1}]$ denote the conjunction of

$$\forall y \forall e [\forall n(en = y^+) \rightarrow Ye]$$
$$\forall e([e0 = 0 \wedge \forall y \exists f(Yf \wedge \forall m[fm = e(\hat{y} * m)])] \rightarrow Ye)$$

where $\hat{y}$ is the number of the sequence consisting of the single element y.

1.52 The axioms for $K_{0,1}$ are: $A[K]$ and

$$\forall X_{0,1}(A[X] \rightarrow \forall e(Ke \rightarrow Xe)).$$

Note that this axiom expresses an *induction principle* which is seen to hold if we think of K as *generated* according to the two clauses $A[K]$. First we start with the neighborhood functions $\dot{1}$, $\dot{2}$, $\dot{3}$, which assign outright the value 0, 1, 2, $\cdots$ respectively (to any a). Second, if e enumerates the sequence $e^0, e^1, e^2, \cdots$ in the sense

$$\forall y \forall m [e(\hat{y} * m) = e^y(m)]$$

then e assigns to the sequence $(a0, a1, \cdots)$ the value which e^{a0} assigns to $(a1, a2, \cdots)$. If K is generated in this way, and X is closed under these closure conditions, then X must contain K.

Note also that the induction principle, together with the other axioms for constructive functions, implies a further strong closure property of K, namely the axiom of choice[8]

$$\forall X_{1,1}(\forall x \exists e [Ke \wedge X_{1,1}(x, e)] \rightarrow \exists e \forall x [Ke \wedge X_{1,1}(x, e^x)]).$$

This closure condition makes the general theory of Brouwer operations more elegant than giving an explicit list of Brouwer operations.

[7] We have here an instance of a very general principle, often applied without analysis. We begin with an idea of a particular kind of argument or *restriction* to a particular kind of evidence or proof. We then discover that this idea leads to the same results, i.e., the same set of theorems (in a given language), as considering a wider class of *objects* and allowing arbitrary proofs. A well known example of this situation (already mentioned in the introduction) concerns the old idea of *algebraic proofs* about real numbers stressed, for instance, in Sturm's original publication. This was later replaced by *validity* for all real closed fields. It should not be assumed that a similar replacement will be useful for every informal notion of proof!

[8] In contrast I do not see how to derive the axiom of dependent choices for elements of K, i.e.

$$\forall e(Ke \rightarrow \exists f[Kf \wedge X_{1,1}(e,f)]) \rightarrow \forall e(Ke \rightarrow \exists f[Kf \wedge f^0 \equiv e \wedge \forall x X_{1,1}(f^x, f^{x+1})])$$

where $f^0 \equiv e$ stands for $\forall m[f(\hat{0} * m) = em]$ although it is intuitionistically valid.

Theorem. $\forall e[Ke \to \forall a \exists m \exists x(em = x^+ \wedge \bar{a} j_1 m = m)]$.

To avoid assuming closure conditions on the species of constructive functions, we prove a slightly stronger theorem by induction on K:

$$\forall a \forall e\{Ke \to \forall n \exists m \exists x(\bar{a} j_1 n = n \to [em = x^+ \wedge \bar{a}(j_1 n + j_1 m) = n * m])\}.$$

Proof. If e is a starting function, i.e., $\dot{1}$, $\dot{2}$, $\cdots$, we take $m = 0$ and $x = 0, 1, \cdots$ respectively. Next suppose that $e0 = 0$, $j_1 n = p$ and $ap = \xi_{p+1}$ (here we use the fact that values of a are determined). By assumption, $\exists f \forall m[Kf \wedge fm = e(\hat{\xi}_{p+1} * m)]$. In the induction hypothesis, replace e by f, and n by $n * \xi_{p+1}$; if m is such that

$$fm = x^+ \wedge \bar{a}(p+1+j_1 m) = n * \hat{\xi}_{p+1} * m$$

we have $e(\hat{\xi}_{p+1} * m) = x^+$ and $\bar{a}[p+j_1(\hat{\xi}_{p+1} * m)] = n * \hat{\xi}_{p+1} * m$, as required.

The elementary condition

$$\forall e(Ke \to \forall m \forall n \forall x[em = x^+ \to e(m * n) = x^+])$$

follows by a straight induction on K. For further properties of K, see [10].

1.6 *Abbreviations.* We write

$$b(a) = u \text{ for } \exists m(bm = u^+ \wedge \bar{a} j_1 m = m)$$

and

$$(b|a)(m) = v \text{ for } \exists n(\forall y < j_1 m)[a(\hat{y} * m) = n_y^+ \wedge bn = v]$$
$$\vee (v = 0 \wedge \neg \exists n(\forall y < j_1 m)[a(\hat{y} * m) = n_y^+],$$

where n_y is the $(1+y)$th element of the sequence n.

The main theorem about K can be restated as

1.61 $$\forall e(Ke \to \forall a \exists x \forall u[e(a) = u \leftrightarrow x = u]).$$

Note that the expression $b(a)$ cannot be regarded as a function of the two variables a and b, since we have $\neg \forall a \forall b \exists x[b(a) = x]$. In fact, by [10], using only the assumption that not all disjoint constructively enumerable sets are constructively separable, we have even $\neg \forall a \forall b \exists x[Kb \to b(a) = x]$.

In contrast $(b|a)$ which, for a and b in K, is a *composition* operation, can be extended to all pairs of constructive functions. Formally, if $C(e, f, g)$ expresses that g is the composition obtained by applying (the functional with neighborhood function) e to f, we have not only

$$\forall e \forall f([Ke \wedge Kf] \rightarrow \exists g[Kg \wedge C(e, f, g)])$$

but also

$$\forall e \forall f \exists g([Ke \wedge Kf] \rightarrow [Kg \wedge C(e, f, g)]).$$

An *extension* of the theory is obtained by replacing the axioms $(AC-NF)$, $(DC-F)$ 1.52, by the corresponding *schemata* (for all definable relations, quantification over species included).

Discussion. We now review the main reasons for using the *primitive* notion of constructive function without making the identification (or, better, restriction): constructive = recursive. Here, for once, I seem to disagree with Kleene; see [5].

First, as far as formal elegance and generality are concerned, the primitive notion presents, I believe, *only* advantages. In particular, one sees what properties are *actually* used (and how few!); besides, why should one clutter up the exposition with recursion equations when one only uses such general facts as $\neg \forall a \exists b \forall x[bx = 0 \leftrightarrow \exists y(ay = x)]$? The verification that the species of recursive functions possesses these properties is quite an independent matter. Further, the general properties are often evident for the informal notion of constructive function when the identification, constructive = recursive, is dubious. Specifically, suppose we include Brouwer's "empirical" sequences defined by rules referring to the thinking subject (in [9] and particularly [12]). All our axioms for constructive functions are still valid, but the identification above is *refutable.* (This central point of [12] is not considered at all in Kleene's answer [5].)

Second, as far as a *development* of our subject is concerned, the need for the primitive notion is even greater. It is somewhat similar to the need, discussed in 1.1, for species variables instead of a restriction to an explicit list of species. Amusingly, some of the principal open problems in the subject of constructive functions also involve species variables, in particular certain forms of the *axiom of choice* associating numbers ("N") to functions ("F").

The usual form $(AC-FN)$ is

$$\forall X(\forall a \exists x X \rightarrow \exists Y[\forall a \exists ! x Y \wedge \forall a \forall x(Y \rightarrow X)]),$$

where both X and Y are of type $(1, 1)$ with variables a and x. $(AC-FN)$ is unproblematic for general species Y, but Y cannot be expected to be definable from X by the operations (1.1).

Note that $\forall Y[\forall a \exists ! x Y \rightarrow (Y \vee \neg Y)]$, (i.e., Y is decidable), since

$$\forall Y[\forall a \exists ! x Y \leftrightarrow \forall a \exists y \forall x(Y \leftrightarrow x = y)]$$

and $$\forall x \forall y (x = y \vee \neg\, x = y).$$

Now let us denote

$\forall b \exists y \forall a \forall x [\forall u (au = bu) \rightarrow (Y \leftrightarrow x = y)]$ by $\boldsymbol{E}(Y)$

("$\boldsymbol{E}$" for *extensional dependence* of y on b)

$\forall b \exists y \exists u \forall a \forall x [\bar{a}u = \bar{b}u \rightarrow (Y \leftrightarrow x = y)]$ by $\boldsymbol{C}(Y)$

("$\boldsymbol{C}$" for *continuous dependence* of y on b).

Clearly, $\forall Y [\boldsymbol{C}(Y) \rightarrow \boldsymbol{E}(Y)]$. Consider now the *problematic* forms of the axiom of choice

$$(\boldsymbol{E}-AC-FN) : \forall X (\forall a \exists x X \rightarrow \exists Y [\boldsymbol{E}(Y) \wedge \forall a \forall x (Y \rightarrow X)])$$
$$(\boldsymbol{C}-AC-FN) : \forall X (\forall a \exists x X \rightarrow \exists Y [\boldsymbol{C}(Y) \wedge \forall a \forall x (Y \rightarrow X)]),$$

and, for comparison with Section 2, with "$\boldsymbol{BO}$" for Brouwer-operation,

$$(\boldsymbol{BO}-AC-FN) : \forall X [\forall a \exists x X \rightarrow \exists e (Ke \wedge \forall a X[a, e(a)])].$$

As they stand, these forms are certainly not plausible. In the first place, *some* restriction on X is needed, e.g. that X be at least extensional, i.e., $\forall a \forall b \forall x (\forall u [au = bu] \rightarrow [X(a, x) \leftrightarrow X(b, x)])$. But even this is not enough for $(\boldsymbol{E}-AC-FN)$ by [12]. As to the identification, constructive = recursive, even $(AC-FN)$ is not valid for extensional X (if $Y \vee \neg\, Y$ is interpreted to mean that Y has a recursive characteristic function); though, for the identification, $\forall Y [\boldsymbol{E}(Y) \rightarrow \boldsymbol{C}(Y)]$ holds classically, I know of no intuitionistic proof; finally, Kleene gave a counterexample to

$$\forall X [\boldsymbol{C}(X) \rightarrow \exists e (Ke \wedge \forall a X[a, e(a)])].$$

In contrast, I know no evident properties of constructive functions which allow one to decide the corresponding questions, for suitable kinds of species X. In short, a premature identification is bad because it prejudges open problems.

What then is the motive behind the identification? I suspect that people are reluctant to use the primitive notion, not for formal or technical reasons, but simply because they feel ill at ease with it. But then they should feel ill at ease with any *constructive* theory of recursive functions too, since the same notion is already involved in the very definition of the notion of recursive function, as mentioned in the general discussion on page 000. In short, as is well known, the identification does not provide an *analysis* of the primitive notion.

What can be saved from work done under the heading of this identification (in connection with our present subject)? Above all we can get most useful results about the *proof theoretic strength* of our axiomatic theory of constructive functions. We have presented it as a second-order theory, but the additional axiom

$$\forall a \exists x \forall u \exists v [T(x, u, v) \wedge au = Uv]$$

reduces it to a first-order theory with K_R, the species of *recursive* Brouwer operations, i.e., a species of natural numbers, replacing K. I do *not* see how to establish the axioms for constructive functions when the variables $a, b, \cdots$ range over the recursive functions, and the logical particles are interpreted as by Heyting. But the axioms are satisfied for a recursive realizability interpretation if x realizes Ke only if $x = e \wedge K_R x$ ([8], p. 141, 2.6292). For related results see [5].

2. Lawless sequences of natural numbers

We modify the (two-sorted) formalism of Section 1 by adding variables $\alpha, \beta, \gamma, \cdots$ for lawless sequences and variables $X_{n,m,p}$ for species having n numerical, m function and p sequence arguments. The variables $X_{n,m}$ of Section 1 are replaced by $X_{n,m,0}$.

The different sorts of objects are regarded as *disjoint*, e.g. $\forall x \forall \alpha (\neg\, x = \alpha)$. The closure conditions on species in (1.1) and on $K_{0,1,0}$ (1.52) are extended in the obvious way; in particular, we have specialization of number and function arguments exactly as in (1.1), while "specialization" of sequence arguments takes the form of the axioms on p. 18. The only *terms* for sequences are the variables $\alpha, \beta, \gamma, \cdots$ themselves. The rule for forming *numerical* terms is extended so that αt is a numerical term if t is one.

To state the axioms it is convenient to use two *abbreviations*:

$\alpha \in n$ expresses that for $n \neq 0$, n codes the sequence $\alpha 0, \cdots, \alpha(j_1 n - 1)$ where $j_1 n$ is the length of n;

$\neq(\alpha, \alpha_1, \cdots, \alpha_p)$ stands for the conjunction of all formulae $\neg\, \alpha = \alpha_i$ for $1 \leqq i \leqq p$.

2.1 $\quad \forall n \exists \alpha (\alpha \in n)$ and, by convention, $\forall \alpha (\alpha \in 0)$.

This axiom expresses the fundamental requirement on lawless sequences that any finite initial segment may be prescribed. Note for reference that 2.1 is the only existential axiom on lawless sequences. (For α subject to *chance* only, 2.1 is absurd, and only $\forall n \neg\neg \exists \alpha (\alpha \in n)$ holds.)

2.2 $$\forall\alpha\forall\beta(\alpha = \beta \vee \neg\, \alpha = \beta).$$

To see the validity of 2.2 for our notion, recall first that a lawless sequence is not to be conceived as a "completed" extension, but rather as a die or, perhaps better, the idea of a die. Further, the meaning of $\neg\, \alpha = \beta$ is that it is absurd that $\alpha = \beta$ be provable. Now, for given α and β, we either intend them to denote the *same* object, e.g., the same die, or else it is absurd that we could prove them to be identical since no restriction is imposed except for a finite initial segment of values. What we have in mind is that the dice, or whatever lawless objects are considered, should be *given* or *presented* explicitly; we do not allow "descriptions" which are not explicit enough to identify the (intensional) objects described.

Axiom 2.2 allows us to argue by *cases*. Thus

$$\begin{aligned}&\forall\alpha_1\forall\alpha_2 A(\alpha_1, \alpha_2)\\ &\qquad \leftrightarrow [\forall\alpha_1 A(\alpha_1, \alpha_1) \wedge \forall\alpha_1\forall\alpha_2(\neq(\alpha_1, \alpha_2) \to A(\alpha_1, \alpha_2))]\\ &\forall\alpha_1\exists\alpha_2 A(\alpha_1, \alpha_2)\\ &\qquad \leftrightarrow \forall\alpha_1[A(\alpha_1, \alpha_1) \vee \exists\alpha_2(\neq(\alpha_1, \alpha_2) \wedge A(\alpha_1, \alpha_2))].\end{aligned}$$

Next, let $X_{0,0,p+1}$ be a species variable whose arguments are $\alpha, \alpha_1, \cdots, \alpha_p$ and let X^α_β be obtained from X by replacing α by β. Then

2.3 $$\begin{aligned}&\forall\alpha\forall\alpha_1 \cdots \forall\alpha_p\forall X_{0,0,p+1}([\neq(\alpha, \alpha_1, \cdots, \alpha_p) \wedge X]\\ &\qquad \to \exists n[\alpha \in n_\wedge\ \forall\beta([\beta \in n_\wedge \neq(\beta, \alpha_1, \cdots, \alpha_p)] \to X^\alpha_\beta)].\end{aligned}$$

To see this, consider any $\alpha, \alpha_1, \cdots, \alpha_p$. To be able to assert X we either have $\alpha = \alpha_{1\wedge} X^\alpha_{\alpha_1}$ or $\cdots$ $\alpha = \alpha_{p\wedge} X^\alpha_{\alpha_p}$ or else $\neq(\alpha, \alpha_1, \cdots, \alpha_p)$. But in this last case, since the only restriction allowed on α is that, for some n, $\alpha \in n$, we must also have X^α_β for $\beta \in n$ provided only that no relation is imposed between β and one of the $\alpha_1, \cdots, \alpha_p$. (The condition $\neq(\beta, \alpha_1, \cdots, \alpha_p)$ is needed; take $p = 1$ and $\alpha \neq \alpha_1$ for X.)

Note that, by the closure conditions on species, 2.3 also holds with numerical and function parameters, i.e., with $X_{n,m,p+1}$ in place of $X_{0,0,p+1}$. The following consequences 2.31–2.33 of 2.3 are useful.

2.31 $$\forall\alpha\forall\beta[\alpha = \beta \leftrightarrow \underset{\sim}{\forall} x(\alpha x = \beta x)].$$

The direction $\to$ is an instance of the equality axioms. For the converse, we argue by cases. If $\alpha = \beta$ there is nothing to prove. If $\neq(\alpha, \beta)_\wedge \forall x(\alpha x = \beta x)$, apply 2.3 with $\forall x(\alpha x = \beta x)$ for X;

we conclude $\exists n(\alpha \in n_{\wedge} \forall \gamma[(\neq(\gamma, \beta)_{\wedge} \gamma \in n) \rightarrow \forall x(\gamma x = \beta x)])$, which conflicts with 2.1. Thus $\neq(\alpha, \beta) \rightarrow \neg \forall x(\alpha x = \beta x)$.

The Theorem 2.31 is a strong *extensionality* principle for lawless sequences.

2.32
$$\forall X_{n,m,p+1}[\exists \alpha X \leftrightarrow (X^{\alpha}_{\alpha_1} \vee \cdots \vee X^{\alpha}_{\alpha_p} \vee \exists n \forall \alpha[(\alpha \in n_{\wedge} \neq(\alpha, \alpha_1, \cdots, \alpha_p)) \rightarrow X])].$$

This is immediate from 2.1 and 2.3; as above, 2.32 could be deduced from its special case with $n = m = 0$. Note that 2.32 allows us to "replace" existential operators "$\exists\alpha$" by universal ones.

2.33
$$\forall X_{0,0,1} \forall Y_{0,0,1}([\forall \alpha(X \rightarrow Y)] \leftrightarrow \forall n[(\forall \alpha \in n)X \rightarrow (\forall \alpha \in n)Y]).$$

The direction $\leftarrow$ is logical: take $n = 0$. Suppose then that $X \wedge \forall n[(\forall \alpha \in n)X \rightarrow (\forall \alpha \in n)Y]$. First, by 2.3,

$$X \rightarrow \exists m[\alpha \in m \wedge (\forall \beta \in m)X^{\alpha}_{\beta}];$$

next

$$(\forall n[(\forall \alpha \in n)X \rightarrow (\forall \alpha \in n)Y] \wedge \exists m[\alpha \in m \wedge (\forall \beta \in m)X^{\alpha}_{\beta}]) \rightarrow \exists m[\alpha \in m \wedge (\forall \beta \in m)Y^{\alpha}_{\beta}].$$

Since $\exists m[\alpha \in m_{\wedge}(\forall \beta \in m)Y^{\alpha}_{\beta}] \rightarrow Y$, we have Y.

Note that 2.33 allows us to distribute the universal quantifier $\forall\alpha$ over $\rightarrow$. More precisely, in this "distribution" we introduce the restricted quantifier $(\forall \alpha \in n)$, just as in 2.32 the quantifier $\exists\alpha$ was replaced by the restricted quantifier

$$\forall \alpha[(\neq(\alpha, \alpha_1, \cdots, \alpha_p)_{\wedge} \alpha \in n) \rightarrow.$$

Finally, observe that the distribution of $\forall\alpha$ over universal numerical and function quantifiers does not depend on special properties of lawless sequences since it is purely logical:

$$\forall X_{1,1,1}(\forall \alpha \forall n \forall a X \leftrightarrow \forall n \forall a \forall \alpha X).$$

We now come to the only problematic axiom.

2.4
$$\forall X_{1,0,p+1}[\forall \alpha_1 \cdots \forall \alpha_p\{\forall \alpha[\neq(\alpha, \alpha_1, \cdots, \alpha_p) \rightarrow \exists x X] \rightarrow \exists e(Ke \wedge \forall \alpha[\neq(\alpha, \alpha_1, \cdots, \alpha_p) \rightarrow X^{x}_{e(\alpha)}])\},$$

where we have used the abbreviation $e(\alpha)$ corresponding to 1.6. By the Theorem of Section 1 (set out so as to apply equally to $e(\alpha)$ as to $e(a)$) we have $\forall e(Ke \rightarrow \forall \alpha \exists ! n[e(\alpha) = n])$ and hence $\forall e[Ke \rightarrow (\forall \alpha X[\alpha, e(\alpha)] \leftrightarrow \forall m \forall n \forall \alpha[(em = n^{+}_{\wedge} \alpha \in m) \rightarrow X(\alpha, n)])]$, which permits us to *eliminate* systematically the notation $e(\alpha)$.

Concerning a *justification* of this axiom, which makes explicit Brouwer's dogma [1] about the nature of everywhere defined operations on choice sequences, observe that, by **2.3**, if $\forall\alpha\exists x X(\alpha, x)$, then, in the case of *lawless* sequences, x depends continuously and hence extensionally on α. Thus the doubts corresponding to $(E-AC-FN)$ and $(C-AC-FN)$ at the end of Section 1, do not apply in the case of lawless sequences: $(C-AC-LN)$ certainly holds. **2.4** asserts that *every continuous operation is a Brouwer operation.* We do not analyze **2.4** further in this paper (though its *consistency* is established by the remarks in the discussion at the end of the present section). The following fact, certainly known to Brouwer (and pointed out to me by Gödel) is perhaps worth noting. If the axioms for K are interpreted as a *classical* implicit definition, with function variables ranging over all number theoretic functions, then it is a *theorem* that all classically continuous operations on N^N with the product topology, have a neighborhood function in K. It is therefore not surprising that the most familiar continuous operations e are obtained by such an elementary use of the generation principles of K that Ke can be proved quite constructively; so one hardly expects to have a *simple* counterexample to Brouwer's dogma. More generally, many closure conditions on the class of continuous operations which are usually proved non-constructively (for the usual definition of continuity) can be proved intuitionistically for K. In other words, the "usual" proofs can be separated into, first, a non-constructive identification of K with the class of continuous operations, and second, intuitionistic proofs about K.

Note that **2.4** allows us to distribute universal sequence quantifiers over $\vee$. Recall that

$$\forall X_{0,0,1}\forall Y_{0,0,1}(\forall\alpha(Y\vee Y)\leftrightarrow\forall\alpha\exists x[(x=0\rightarrow X)_\wedge(x\neq 0\rightarrow Y)])$$

whence

2.41 $$\forall X_{0,0,1}\forall Y_{0,0,1}[\forall\alpha(Y\vee Y)\leftrightarrow\exists e(Ke_\wedge\forall\alpha[e(\alpha)=0\rightarrow X]_\wedge \forall\alpha[e(\alpha)\neq 0\rightarrow Y])].$$

The next axiom asserts that if a constructive function depends on a lawless sequence it depends on an initial segment.

2.5 $$\forall X_{0,1,1}[\forall\alpha\exists a X(\alpha, a)\rightarrow\exists b\forall\alpha\exists x X(\alpha, b_x)].$$

The basic theory above may be *extended* by replacing the axioms **2.3**, **2.4**, **2.5**, by *schemata* (without parameters for sequences), cf. Section 1, p. 12.

Similarly, for species (which are also conceived of as *definite* constructions given by a *law*):

2.6 $$\forall\alpha\exists X_{n,m,p}A(\alpha, X) \to \exists Y_{n+1,m,p}\forall\alpha\exists x A(\alpha, Y_x)$$

for all *formulae* A not containing free variables for sequences other than α. Note that 2.6 is formulated as a *schema* and not as a single axiom because we do not have variables for species of species in our formalism.

In the statement of the next theorem we adopt a *convention*. In the context of the basic system the formulae A do not contain any species variables; cf. the remark in (1.1). For the extended system, A may be arbitrary.

PRINCIPAL THEOREM. *If A is a closed formula of our systems there is a formula A' in the language of Section 1 such that $A \leftrightarrow A'$ follows from the axioms listed above.*

SKETCH OF THE PROOF. The theorem is established by an induction on the *complexity* of formulae A (containing lawless parameters). The measure is given by the *number* of *logical symbols* except that, by definition, the complexity of a formula of the form $([\neq(\alpha, \alpha_1, \cdots, \alpha_p)_\wedge \alpha \in m] \to B)$ is that of B itself. In other words, logical operations applied to decidable formulae need not be counted. Note also that any conjunction $\alpha \in n_1 \wedge \cdots \wedge \alpha \in n_k$ is equivalent to a single formula $\alpha \in n_i$ $(1 \leqq i \leqq k)$ or is contradictory. In the former case, the sequences n_i, n_j are pairwise extensions of one another and n_i is the longest among them.

Suppose A has all its free variables for (lawless) sequences among $\alpha, \alpha_1, \cdots, \alpha_k$, and its species variables of the type $X_{r,s,t+1}$ (i.e., containing sequence arguments) among X_{r_i,s_i,t_i+1} $(1 \leqq i \leqq l)$. We introduce *new* symbols $X'_{r_i+t_i+1,s_i,0}$, $1 \leqq i \leqq l$ with the axioms

$$\forall x_{\sigma(1)} \cdots \forall x_{\sigma(r_i)} \forall a_{\tau(1)} \cdots \forall a_{\tau(s_i)} \forall n_1 \cdots \forall n_{1+t_i}(X' \leftrightarrow (\forall \alpha_{\xi(1)} \in n_1) \cdots (\forall \alpha_{\xi(1+t_i)} \in n_{1+t_i})[\neq(\alpha_{\xi(1)}, \cdots, \alpha_{\xi(1+t_i)}) \to X])$$

where $x_{\sigma(j)}$, $1 \leqq j \leqq r_i$, are the numerical arguments of X, $a_{\tau(j)}$, $1 \leqq j \leqq s_i$, the functional arguments, and $\alpha_{\xi(j)}$, $1 \leqq j \leqq 1+t_i$, the sequence arguments, and $n_1, \cdots, n_{1+t_i}$ are new numerical variables. Then we introduce in the same way $X''_{r_i+t_i,s_i,0}$ where first, two sequence arguments in X have been identified to give the species Y; thus X'' is obtained from Y as X' was obtained from X, and so forth.

Then the *induction hypothesis* asserts: there is a formula A' such that its variables for sequences are all free and among the free sequence variables of A, its free species variables are those of A of the form $X_{r,s,0}$, and the X', X'', etc. introduced above and $A \leftrightarrow A'$ follows from the additional axioms above.

When the induction is completed, the additional axioms can be removed by the closure conditions on species. Specifically, to a given X there are X', X'', $\cdots$ satisfying the axioms above, and conversely. The converse uses the fact that $\alpha_i = \alpha_j$ is decidable, and so $X(\cdots, \alpha_i, \cdots, \alpha_j, \cdots)$ for $\alpha_i \neq \alpha_j$, and $X(\cdots, \alpha, \cdots, \alpha, \cdots)$ are quite independent of each other. To treat adjacent universal quantifiers replace $\forall\alpha\forall\beta\exists x A(\alpha, \beta, x)$ first by $\forall\alpha\exists x A(\alpha, \alpha, x) \wedge \forall\alpha\forall\beta[\alpha \neq \beta \to \exists x A(\alpha, \beta, x)]$ and then apply **2.4** and **2.5** to show that y depends continuously on both α and β.

Next the reduction of $\exists\alpha$ to $\forall\alpha$ in **2.32** and the distribution of sequence quantifiers $\forall\alpha$ over all logical operations (**2.33**, **2.4**, **2.41**, **2.5**, **2.6** and of course $\forall x$, $\forall a$, and $\wedge$) reduces the theorem to the case of formulae A of the form

$$(*) \qquad \forall\alpha([\neq (\alpha, \alpha_1, \cdots, \alpha_p) \wedge \alpha \in m] \to B)$$

where B is atomic, i.e., $\alpha = \alpha_i$, or $t = t'$ where t and t' are numerical terms containing α, or $X_{r,s,1+p}$ where α is one of the sequence arguments of B. We consider the cases separately.

If B is $\alpha = \alpha_i$, we refute $(*)$ by taking n in **2.1** so as to satisfy $j_1 n > j_1 m$ and $n \neq \bar{\alpha}_i j_1 n$.

If B is $t = t'$ we have to look at the rules of term formation (or, to be safe, we have to remember the property of the notion of lawless sequence which *led* to the rules). Only a finite number of arguments of α are involved, say $t_i^{(0)}$ $(1 \leqq i \leqq k_0)$, $t_j^{(1)}$ $(1 \leqq j \leqq k_1)$ where $t_j^{(1)}$ is obtained by applying symbols *other* than α to $\alpha t_i^{(0)}$ say $T_j(\alpha t_1^{(0)}, \cdots, \alpha t_{k_0}^{(0)})$, and so forth. Introduce the new individual variables $u_i^{(0)}, u_j^{(1)}, \cdots$ with the axioms $t_j^{(1)} = T_j(u_1^{(0)}, \cdots, u_{k_0}^{(0)})$ and equality axioms, $t_i^{(0)} = t_j^{(0)} \to u_i^{(0)} = u_j^{(0)}$ etc. Then B is reduced, by **2.1**, to a formula not containing α at all.

Finally suppose B is $X_{r,s,1+p}$. It was precisely this case which led us to introduce the (eliminable) species variables X', X'', $\cdots$.

Refinement. As mentioned in footnote 5 we can give a finitist reduction of our basic system to the system of Section 1. One verifies meticulously that all the axioms and rules of Section 2 become *theorems* of Section 1 when each closed formula A appearing in the axioms and rules is replaced by A' as constructed above. For a detailed exposition, in a slightly more complicated situation, see [10]. This reduction provides a *classical consistency proof* for the present Section because Section 1 is a subsystem of classical analysis. From the classical point of view, i.e., without understanding of the notion of lawless sequence, a consistency proof is needed since there is no obvious classical interpretation for our axioms. (Adding the law of the excluded middle certainly makes our system inconsistent since e.g. $\forall\alpha \neg\neg \exists x(\alpha x = 0)$ and $\neg \forall\alpha\exists x(\alpha x = 0)$ both hold; so the α cannot be regarded as classical "objects".) The intuitionistic consistency proof in terms of the intended interpretation is of course more elegant than the finitist reduction, but, as usual, the latter provides, as a byproduct, *conservative extension results* which do not seem to be obvious from the interpretation. Presumably corresponding results hold for the extensions in Sections 1 and 2 respectively.

Discussion. The problematic axiom 2.4; further conservative extension results. It is an easy exercise (cf. [8], p. 140, 2.623—2.6234, and, in detail, [10]) to show that 2.4 and (what Kleene calls) the "bar theorem" are equivalent in the presence of the other axioms of Section 2. In fact if K is *explicitly defined* by

$$\forall\alpha\exists m\forall n[\alpha \in m \wedge em \neq 0 \wedge em = e(m * n)] \quad ([8], 2.6233)$$

it can be proved to satisfy the basic axioms 1.52 for K by use of the bar theorem. This means of course that the addition of K is then a conservative extension.

What then is there to choose between 2.4 and the bar theorem (applied to lawless sequences)? The matter has been discussed in [12] and [5], footnote 3. It seems to me that, formally, there is little to choose in the sense that the tricks needed to infer one from the other are of much the same kind. For an elimination theorem and, as an application, for proof theoretic results about the strength of the systems concerned, 2.4 seems much superior. Personally, I think there is a real advantage in 2.4: not because it is particularly evident that 2.4 is *true*, but because it is really clear *what* 2.4 asserts.

3. Lawless binary sequences

denoted by variables α_B, β_B, $\cdots$; reduction to arithmetic. Instead of allowing sequences of arbitrary natural numbers, we *restrict* ourselves to sequences of 0 and 1 or, alternatively, lawless predicates of natural numbers. Beyond this restriction no further restriction is to be made except for fixing finite initial segments. (The work below extends to uniformly bounded sequences too, i.e., $\alpha n < an$, a fixed, instead of $\alpha_B n < 2$.) The present theory will be simpler than Section 2, roughly because binary sequences form a compact space and continuous functions on such spaces are uniformly continuous. More precisely, we shall show that *the action of an e in K can be coded into a natural number.* (This will lead to a reduction to arithmetic because we can now *separate out* axioms for constructive functions: 2.1—2.3 do not involve such objects, 2.4 is replaced by an arithmetic axiom, and 2.5, 2.6 are not needed if one drops functions and species altogether.)

Let $A(m, x)$ express that m is an *array* of length x, i.e., that m is a sequence of pairs (n_i, u_i) for $0 \leqq i < 2^x$, each n_i being a sequence of 0 or 1 of length x, and n_i, n_j distinct for $i \neq j$. Thus the n_i list all possible sequences of 0 and 1 of length x; the natural numbers u_i may be thought of as associated to n_i.

We write $m(\alpha_B) = u$ for $(\exists i < 2^x)(\bar{\alpha}_B x = n_i \wedge u = u_i)$. We may think of m as defining a (continuous) operation on all binary sequences, with modulus of continuity $\leqq x$, since clearly $\forall \alpha_B \exists ! u[m(\alpha_B) = u]$. Conversely we have the

LEMMA. $\forall e[Ke \rightarrow \exists m \exists x(A \wedge \forall \alpha_B[m(\alpha_B) = e(\alpha_B)])]$.

The proof uses a straightforward induction on K. If e is a constant with $e(m) = u^+$, take $x = 1$ and $m = ((0, u), (1, u))$. Suppose $e(0) = 0$ and suppose m_0 and m_1, of lengths x_0 and x_1 respectively, are the arrays corresponding to $\lambda ne(\hat{0} * n)$ and $\lambda ne(\hat{1} * n)$. Then there is clearly an array of length $1+\max(x_0, x_1)$ corresponding to e.

Consequently axiom 2.4 (with $p = 0$) takes the form

3.4′ $\forall X_{1,0,1}[\forall \alpha_B \exists x X(\alpha_B, x) \rightarrow \exists m \exists x(A \wedge \forall \alpha_B X[\alpha_B, m(\alpha_B)])]$.

The only other changes in the axioms of Section 2 are first the addition of

$$\forall \alpha_B \forall x(\alpha_B x = 0 \vee \alpha_B x = 1)$$

and, instead of 2.1, we have

3.1′ $$\forall x(\forall y < 2^x)\exists \alpha_B[y = [\sum_{z=0}^{z=x-1} (\alpha_B z)2^z].$$

THEOREM. *Let A be a closed formula in the language obtained by adjoining $\alpha_B, \beta_B, \cdots$ to first order arithmetic and extending the rules of term formation accordingly. Then there is a formula A' of first order arithmetic itself such that $A \leftrightarrow A'$ follows from our axioms for binary lawless sequences.*

It is understood that we now throw in the recursion equations for addition and multiplication since we do not have function variables and hence not $(AC{-}NF)$. Also since we do not have species variables, the axioms involving universal quantifiers over species are replaced by *schemata.*

Discussion. The theorem above improves Theorem 4 of [6] where only formulae A of the form $\forall \alpha_B B$ were treated, with α_B as the *only* sequence variable in B. Naturally axiom 2.2 was not used nor the restriction $\neq (\alpha, \alpha_1, \cdots, \alpha_p)$ in 2.3. The schema corresponding to 2.4′ was replaced by an equivalent form

$$\forall \alpha_B \exists x X(\alpha_B, x) \to \exists x \forall \alpha_B \exists y \forall \beta_B [\bar{\alpha}x = \bar{\beta}x \to X(\beta_B, y)].$$

It is worth noting that the need for *some* restriction such as in the present 2.3 was clear ([6], p. 373, Remark 5.1). But, as so often in logic, instead of facing the problem and getting an elegant solution, I was being "practical": for the immediate purpose of [6], the fragment of formulae $\forall \alpha_B B$ was enough!

As in Section 2, though more simply, one can show that *the present theory of binary lawless sequences is a conservative extension of Heyting's first order arithmetic.*

This improves a result of Kripke [11] who was the first to show that the *fragment* of the present theory considered in [6] (where it was called "FC") is a conservative extension of Heyting's arithmetic. It may be remarked that for the method here described ("elimination of lawless sequences") it is no easier to prove Kripke's result than the stronger result above.

4. Derived notions of choice sequence

or: how fundamental are lawless sequences? It is clear from the examples in the introduction that the sequences used in intuitionistic analysis are not lawless. More formally (at least in so far as Kleene's system or the slightly [9] different system in [8], p. 135 are adequate axiomatizations) the lawless sequences do

[9] *Correction.* In [8] I did not state a proper substitution rule or, equivalently, a proper restriction on the λ-symbols allowed. (Kleene had no restriction.) Since, by [8], p. 135, *l.*-3 and *l.*-2, a choice sequence is *given* with a spread law, i.e., we mean α^*_σ in 4.1 below and not α^*, the restriction on the λ-symbols t must ensure that t maps a spread into a spread. Troelstra [13] discovered that not every λ-term has this property. I owe the compound notion called α^* below, to conversations with Dr. Troelstra.

not satisfy the formal principles of this subject. For instance, for lawless α we have

$$\neg \forall X_{2,0,0}[\forall x \exists y X(x, y) \rightarrow \exists \alpha \forall x X(x, \alpha x)]$$

or, more simply if we use variables for constructive functions, $\neg \forall a \exists \alpha \forall x(\alpha x = ax)$ since, by **2.3**, even $\forall a \forall \alpha \neg \forall x(\alpha x = ax)$ holds. We shall now consider whether the notion of choice sequence involved in intuitionistic practice can be *reduced* to lawless sequences, for instance by use of *compound* or *defined* notions. To get some perspective we begin, in **4.1**, by studying properties of two such compound notions, and then, in **4.2**, we consider their use for the proposed reduction (was sie sind und was sie sollen).

4.1. Recall the informal distinctions in the introduction. Denote by α^* and α_σ^* the pairs (a, α) and (a_σ, α) respectively, where both a and $a_\sigma \in K$, and, further a_σ maps an arbitrary α into a spread σ as described in **2**. We define an application or function evaluation

$$\alpha^* y = z \text{ by } \exists m[\alpha \in m \wedge a(\hat{y} * m) = z^+]$$

and similarly $\alpha_\sigma^* y = z$. We denote by $X_{m,n}^*$ those species $X_{m,n,n}$ which depend only on the *compound* α^* and not on the components a and α, i.e., for $m = n = 1$

$$\forall x \forall \alpha^* \forall \beta^* (\forall y(\alpha^* y = \beta^* y) \rightarrow [X_{1,1,1}(x, a, \alpha) \leftrightarrow X_{1,1,1}(x, b, \beta)]).$$

In contrast to the case of lawless sequences we now have $\forall b \exists \alpha_\sigma^* \forall x(\alpha_\sigma^* x = bx)$ and *a fortiori* $\forall b \exists \alpha^* \forall x(\alpha^* x = bx)$. We simply take for σ the spread consisting of the single path given by b, and for a_σ (in (a_σ, α)) the function given by $a_\sigma(\hat{y} * m) = (by)^+$; clearly $a_\sigma \in K$.

For a property which distinguishes between α^* and α_σ^*, consider the closure condition

$$(\forall e \in K) \forall \alpha^* \exists \beta^* \forall y[\beta^* y = (e \overset{*}{|} \alpha^*)(y)]$$

where $(e \overset{*}{|} \alpha^*)(y) = z$ is defined by $\exists m[\alpha \in m \wedge (e|a)(\hat{y} * m) = z^+]$ (cf. **1.6**). We take for β^* the pair $(e|a, \alpha)$. In contrast, by [**13**] we have

$$\neg (\forall e \in K) \forall \alpha_\sigma^* \exists \beta_\sigma^* \forall y[\beta_\sigma^* y = (e \overset{*}{|} \alpha_\sigma^*)(y)].$$

Instead we must restrict ourselves to e which map spreads into spreads [10]. (To state a corresponding result without the use of

[10] Cf. footnote 9. This *kind* of restriction is of course very familiar; for instance in intuitionistic analysis one requires not only that operations map (freely chosen) real number generators again into real number generators, but that, in addition, equivalences be preserved.

function variables e, one considers K_R instead of K.)

More generally we have the following *axiomatization* problem which (often) leads to a neat way of stating properties of compound notions. Consider the language of Section 2, but with α^* (or α^*_σ) replacing α and $X^*_{m,n}$ replacing $X_{m,0,n}$ [11]. *Is there a simple set of axioms in the * language which implies exactly those formulae which are translations* (via our definitions above) *of theorems of Section* 2? (Or, if one shares Kleene's taste [5] one will drop variables $a, b, \cdots$ form the * language: of course they are translations of formulae in Section 2 which *do* contain $a, b, \cdots$.) I hope to come back to this axiomatization problem in another paper. Here only two remarks are to be made (besides the postscript).

First the analogue of the axiomatization problem is familiar from other branches of mathematics. Thus, even though real numbers are (often) *defined* in terms of natural numbers and sets of natural numbers, it has been useful to axiomatize the theory of addition and multiplication of real numbers without mentioning the *definientes* (in the theory of real closed fields).

Second, it is not particularly plausible (without significant restrictions on the species considered) that the current formal systems mentioned at the beginning of this Section already solve this axiomatization problem since, apparently, too much can be proved there. Let us consider the continuity axiom (which Kleene calls "Brouwer's principle"):

$$\forall \alpha^* \exists x X(\alpha^*, x) \to \forall \alpha^* \exists y \exists x \forall \beta^* [\bar{\beta}^* y = \bar{\alpha}^* y \to X^*(\beta^*, x)].$$

This is translated (with $a \in K$, $b \in K$) by

$$\forall a \forall \alpha \exists x X(a, \alpha, x) \to \forall a \forall \alpha \exists y \exists x \forall b \forall \beta [\bar{\beta}^* y = \bar{\alpha}^* y \to X(b, \beta, x)].$$

But all that is *obvious* is

$$\forall a \forall \alpha \exists x X(a, \alpha, x) \to \forall a \forall \alpha \exists z \exists x \forall \beta [\bar{\beta} z = \bar{\alpha} z \to X(a, \beta, x)],$$

in other words, the quantifier b is too much! Note incidentally that the species $[\bar{\beta} z = \bar{\alpha} z \to X(a, \beta, x)]$ is *not* an X^* at all.

If one uses function variables the situation can be stated more neatly. Current systems assert the strong axiom, which we may call

$$2.4^* \qquad \forall \alpha^* \exists x X^*(\alpha^*, x) \to \exists e \forall \alpha^* X^*[\alpha^*, (e \overset{*}{|} \alpha^*)]$$

while the only obvious "similar" principle is

[11] If one wants a language with constructive function variables (but no lawless sequences!) one uses $X^*_{m,n,p}$ corresponding to $X_{m,n+p,p}$ in an obvious way.

$$\forall\alpha^*\exists x X^*(\alpha^*, x) \to \forall a \exists e \forall \alpha X^*[\alpha^*, e(\alpha)].$$

4.2. *Reduction problems.* Assume for the moment that 2 4* is not valid, i.e., if we consider *arbitrary* proofs of $\forall\alpha^*\exists x X^*(\alpha^*, x)$, x need not depend only on α^*, but may depend on the components even though X^* does not. (Put differently, e may depend on a in (a, α).) What conclusion are we to draw?

4.21. The simplest conclusion is that 2.4* is simply not valid without additional conditions on the quantifier combination $\forall\alpha^*\exists x$. Here it is natural to consider the analogue to $\forall Y[E(Y) \to C(Y)]$ on p. 13, which comes to this (for the identity operator i, and Y^* obtained from Y as on p. 23): do we have

$$\forall Y_{1,1,1}\forall e'[\forall\alpha^*\exists x Y^* \to (\forall\alpha Y[e'(\alpha), i, \alpha] \to \forall\alpha^* Y[e'(\alpha^*), e, \alpha])]?$$

4.22. Another conclusion is to give up the attempt to *reduce* 2.4* to principles about lawless sequences only. This is considered in [12] and [13]. Troelstra [13] points out some formal defects of [12], and presents what seems to be a formally satisfactory treatment. Further the *general* idea of [13] is most interesting. The objects α^* involve constructive functions and hence the laws used to define them; Troelstra considers the limits on the *communication* of such a law from one thinking subject to another, and confines himself to intersubjective (communicable) laws. However the detailed execution *as it stands* is less satisfactory: instead of considering *objective* limits on communication, he considers a collection of subjects who (frivolously?) *withhold* information. If one is going to consider *this* game, one might as well consider a mere game with symbols à la Hilbert and be satisfied with a consistency proof. But Troelstra's idea may well *lead* to a significant analysis.

4.23. Finally we may wish to *combine* a reduction to lawless sequences with a special interpretation of the quantifier combination $\forall a\forall\alpha\exists x$ or, equivalently, with a restriction on the proofs of $\forall a\forall\alpha\exists x$ considered ([8], p. 133, 2.5; p. 136, 2.531, also taken up in [13]). In terms of footnote 9, the formalization of the intuitive situation is not done merely by considering a new kind of *object*, (and thus only implicit restrictions on methods of proof) but also retains explicit restrictions on proofs. Despite the intrinsic interest of analyzing the notion of thinking subject as in [12] or [13], it seems to me that the most natural situation to consider here is nothing more recondite than the *geometric notion of continuity*. Let us remember that the classical rendering, i.e., arithmetization, of the geometric idea of "continuity at *each* point of an interval" is not immediately convincing, and is justified more or less by consequences such as uniform continuity (in the case of closed intervals). It is simply a fact that if we *formally* transfer the classical version to the intuitionistic theory and use a universal quantifier over *constructive* points, we cannot expect uniform continuity to follow (and we know that it does not follow if we identify constructive and recursive). In any case, if we use α^* to represent points, the component a in (a, α) is a kind of coordinate, always alien to geometric conceptions. So for geometric purposes one has to express a strong independence of these coordinates and it seems plausible that 2.4* does this correctly. (Of course, instead of restrictions on proofs one can also have restrictions on definitions of functionals $F(\alpha^*)$, and then 2.4* expresses the consequences of such restrictions made *outside* the * language, as far as statements in the * language are concerned.) [12]

Postscript concerning the *axiomatization* problem of Section 4.

Troelstra has pointed out that CS of [13] is certainly *not* satisfied by the objects (e, α): $\neg\exists\alpha_{CS}\forall\alpha\neg\forall x(\alpha x = ax)$ holds in CS, but, by Section 2, we have $\exists\alpha^*\forall a\neg\forall x(\alpha^* x = ax)$; take $\alpha^* = (i, \alpha)$ where i is the identity operator, that is $i \in K \wedge \forall\alpha\forall x[(i|\alpha)(x) = \alpha(x)]$. As I see it, in terms of footnote 3, the idea behind CS is that the objects α_{CS} are *not* pairs (e, α) with e fixed once for all, but "approximated" by sequences $e_0, e_1, e_2, \cdots$ in the sence that the *ranges* of $\lambda\alpha e_0|\alpha$, $\lambda\alpha e_0|e_1|\alpha$, etc. contract. (Of course the sequences $e_0|\alpha$, $e_0|e_1|\alpha$, $\cdots$ themselves have little to do with one another, except under special conditions [on $e_0, e_1, \cdots$]: e.g., $e_0^0(\alpha)$ constant for all α, [for given e_0] $e_0^1(e_1|\alpha)$ constant, etc.) Further work is needed to show whether the objects α_{CS} on the pairs (e, α) are more suitable for the formulation of intuitionistic practice.

We do have $\forall\alpha_\sigma^*\neg\neg\exists a\forall x(\alpha_\sigma^* x = ax)$ whenever σ is not the universal spread since we cannot be sure that any lawless α lies wholly inside σ.

[12] *Warning.* This presents a change of view from [8], p. 134. At that time I simply did not see this justification, and so I had to be content with analyzing the objects "away".

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