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ALAN ROSE

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Binary generators for the m-valued and $\aleph_0$-valued Łukasiewicz propositional calculi

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Alan Rose

It has been shown [1] that, if $m-1$ is not divisible by 3, the implication and negation functors of Łukasiewicz [2] are denoted by C, N respectively and

$$SPQ =_T CPNQ \; (=_T SQP)$$

then, in the m-valued propositional calculus with S as the only primitive functor, the functors C and N are definable. The result fails whenever $m-1$ is divisible by 3, though functors of more than two arguments having properties similar to those of S and truth-tables constructed in a similar manner have been considered [3]. In order to establish the failure we have only to note that if, in general, we denote the truth-values by the rational numbers $i/(m-1)$ $(i = 0, \cdots, m-1)$ then, in the case considered, $\frac{2}{3}$ is a truth-value and, if P, Q both take the truth-value $\frac{2}{3}$, so does SPQ. A non-commutative solution to the binary generator problem was given earlier, for all m $(m < \aleph_0)$ by McKinsey [4].

We shall now, in the case where $m-1$ is divisible by 3, consider the binary functor F whose truth-table is such that

$$FPQ =_T SPQ$$

[1] Alan Rose, "Some generalized Sheffer functions", Proc. Cambridge Phil. Soc., vol. 48 (1952), pp. 369–373, especially pp. 370–371.

[2] See, for example, J. B. Rosser and A. R. Turquette, Many-valued logics, Amsterdam 1952, pp. 15–18.

[3] See footnote 1.

[4] J. C. C. McKinsey, "On the generation of the functions Cpq and Np of Łukasiewicz and Tarski by means of a single binary operation", Bull. Amer. Math. Soc., vol. 42 (1936), pp. 849–851. The author was not aware of the existence of this paper when the paper referred to in footnote 1 was published, but the functor E_{n-2} considered by McKinsey was not, except in the 2-valued case, any of the functors considered by the author in either paper.

except when P takes the truth-value $\frac{2}{3}$ and Q takes one of the truth-values $\frac{1}{3}$, $\frac{2}{3}$. In both the latter cases we assign to FPQ the truth-value 0. We shall then consider a commutative functor closely related to F.

THEOREM 1. *In the m-valued propositional calculus with F as the only primitive functor we may define C and N and, in the m-valued Łukasiewicz propositional calculus, we may define F* $(m = 4, 7, \cdots)$.

Since the truth-value of FPQ is equal to 0 whenever it differs from that of SPQ it follows at once from a theorem of McNaughton[5] that we may define F in terms of C and N.

Let the truth-tables[6] of the functors J_i be such that J_iP takes the truth-value 1 when P takes the truth-value i and J_iP takes the truth-value 0 in all other cases $(i = \frac{1}{3}, \frac{2}{3})$. Let V be a functor such that VP always takes the truth-value 1 and B, L be functors[7] such that if P, Q, BPQ, LPQ take the truth-values x, y, $b(x, y)$, $l(x, y)$ respectively then

$$b(x, y) = \min(1, x+y),\ l(x, y) = \max(0, x+y-1).$$

Let B', L' be functors such that if P, Q, $B'PQ$, $L'PQ$ take the truth-values x, y, $b'(x, y)$, $l'(x, y)$ respectively then

$$b'(\tfrac{1}{3}, \tfrac{1}{3}) = b'(\tfrac{1}{3}, \tfrac{2}{3}) = 0,\ l'(\tfrac{2}{3}, \tfrac{1}{3}) = l'(\tfrac{2}{3}, \tfrac{2}{3}) = 1$$

and, in all other cases,

$$b'(x, y) = b(x, y),\ l'(x, y) = l(x, y).$$

We shall consider now[8] the following definitions:

$$DP =_{df} FFPPFPP,$$

$$VP =_{df} FD^{m-2}PFD^{m-2}PD^{m-2}P$$

(D^{m-2} denoting $m-2$ symbols D),

[5] Robert McNaughton, "A theorem about infinite-valued sentential logic", Journal of Symbolic Logic, vol. 16 (1951), pp. 1–13, especially pp. 12–13.

[6] If integer truth-values are used these functors become the functors $J_{(m+2)/3}(\ )$, $J_{(2m+1)/3}(\ )$ of Rosser and Turquette. See, for example, pp. 18–22 of the book referred to in footnote 2.

[7] *Cf.* Alan Rose and J. Barkley Rosser, "Fragments of many-valued statement calculi", Trans. Amer. Math. Soc., vol. 87 (1958), pp. 1–53, especially, pp. 2–3.

[8] The functor B' will not be considered further until the proof of Theorem 2.

$$NP =_{df} FVPP,\ L'PQ =_{df} NFPQ,\ B'PQ =_{df} FNPNQ,$$
$$J_{\frac{2}{3}}P =_{df} NFPNP,\ J_{\frac{1}{3}}P =_{df} NFNPP,$$
$$LPQ =_{df} L'L'L'PQNL'L'J_{\frac{2}{3}}PJ_{\frac{2}{3}}QQNL'J_{\frac{2}{3}}PJ_{\frac{1}{3}}Q,$$
$$CPQ =_{df} NLPNQ.$$

Since, if P, Q, FPQ take the truth-values x, y, $f(x, y)$ respectively,

$$f(\tfrac{2}{3}, \tfrac{1}{3}) = f(\tfrac{2}{3}, \tfrac{2}{3}) = 0$$

and, in all other cases,

$$f(x, y) = \min(1, 2-x-y),$$

it follows at once that, if $x \neq \frac{2}{3}$,

$$\begin{aligned} f(f(x, x), f(x, x)) &= \min\big(1, 2-2\min(1, 2-2x)\big) \\ &= \min\big(1, \max(0, 4x-2)\big) \end{aligned}$$

and that

$$f(f(\tfrac{2}{3}, \tfrac{2}{3}), f(\tfrac{2}{3}, \tfrac{2}{3})) = f(0, 0) = 1.$$

If $D^i P$ takes the truth-value $d_i(x)$ when P takes the truth-value x $(i = 0, 1, \cdots)$ it follows at once that

$$d_1(\tfrac{2}{3}) = 1,\ d_1(x) = \min\big(1, \max(0, 4x-2)\big)\ (x \neq \tfrac{2}{3}).$$

Since

$$d_1(0) = 0$$

and

$$d_1(1) = 1$$

we deduce that

$$d_i(\tfrac{2}{3}) \in \{0, 1\} \qquad (i = 1, 2, \cdots)$$

and that, for all truth-values x, if

$$d_i(x) \in \{0, 1\}$$

then

(A) $$d_{i+1}(x) \in \{0, 1\} \qquad (i = 0, 1, \cdots).$$

Since, when $x \neq \frac{2}{3}$,

$$d_1(x) = \min\big(1, \max(0, 4x-2)\big)$$

it follows at once that, unless $d_i(x) \in \{0, 1\}$,

$$d_{i+1}(x) > d_i(x) \quad \text{or} \quad d_{i+1}(x) < d_i(x)$$

according as

$$x > \tfrac{2}{3} \quad \text{or} \quad x < \tfrac{2}{3} \qquad (i = 0, 1, \cdots).$$

Hence, if $x > \frac{2}{3}$, it follows, using (A), that either

$$d_{m-3}(x) \in \{0, 1\}$$

or

$$d_{m-2}(x) > \tfrac{2}{3}+(m-2)/(m-1) > 1.$$

It follows at once that

$$d_{m-3}(x) \in \{0, 1\}$$

and hence, by (A), that

$$d_{m-2}(x) \in \{0, 1\}.$$

If $x < \frac{2}{3}$ then, by a similar argument, either

$$d_{m-3}(x) \in \{0, 1\}$$

or

$$d_{m-2}(x) < \tfrac{2}{3}-(m-2)/(m-1) \leqq 0.$$

In the first case it follows from (A) that

$$d_{m-2}(x) \in \{0, 1\}$$

and, in the second case, we have again a contradiction.

Since we have already established that

$$d_i(\tfrac{2}{3}) \in \{0, 1\} \qquad (i = 1, 2, \cdots)$$

it follows at once that, for all truth-values x,

$$d_{m-2}(x) \in \{0, 1\}.$$

Since

$$f(1, 1) = 0, f(0, 0) = 1$$

it then follows immediately that the truth-value of the formula

$$FD^{m-2}PD^{m-2}P$$

is 1 or 0 according as that of $D^{m-2}P$ is 0 or 1. Hence, since

$$f(0, 1) = f(1, 0) = 1$$

our definition of the functor V is appropriate. Since, for all truth-values x,

$$f(1, x) = 1-x,$$

our definition of the functor N is appropriate.

We note next that, unless

$$x = \tfrac{2}{3} \quad \text{and} \quad y \in \{\tfrac{1}{3}, \tfrac{2}{3}\},$$
$$1-f(x, y) = 1-\min(1, 2-x-y)$$
$$= \max(0, x+y-1)$$

and that

$$1-f(\tfrac{2}{3}, \tfrac{1}{3}) = 1-f(\tfrac{2}{3}, \tfrac{2}{3}) = 1-0 = 1.$$

Thus our definition of the functor L' is appropriate.

Except when $x = \frac{1}{3}$ and $y \in \{\frac{1}{3}, \frac{2}{3}\}$,

$$1-x \neq \tfrac{2}{3} \quad \text{or} \quad 1-y \notin \{\tfrac{1}{3}, \tfrac{2}{3}\}$$

and it follows at once that

$$\begin{aligned} f(1-x, 1-y) &= \min\big(1, 2-(1-x)-(1-y)\big) \\ &= \min(1, x+y). \end{aligned}$$

Since, further,

$$f(1-\tfrac{1}{3}, 1-\tfrac{1}{3}) = f(\tfrac{2}{3}, \tfrac{2}{3}) = 0$$

and

$$f(1-\tfrac{1}{3}, 1-\tfrac{2}{3}) = f(\tfrac{2}{3}, \tfrac{1}{3}) = 0$$

our definition of the functor B' is appropriate.

If $x \neq \frac{2}{3}$,

$$\begin{aligned} 1-f(x, 1-x) &= 1-\min\big(1, 2-x-(1-x)\big) \\ &= 1-\min(1, 1) \\ &= 0. \end{aligned}$$

But

$$\begin{aligned} 1-f(\tfrac{2}{3}, 1-\tfrac{2}{3}) &= 1-f(\tfrac{2}{3}, \tfrac{1}{3}) \\ &= 1-0 \\ &= 1. \end{aligned}$$

Thus our definition of the functor $J_{\frac{2}{3}}$ is justified. Since

$$NFNPP =_T NFNPNNP$$

it follows, by definition, that

$$NFNPP =_T J_{\frac{2}{3}}NP.$$

Hence $NFNPP$ takes the truth-value 1 or the truth-value 0 according as NP does or does not take the truth-value $\frac{2}{3}$, i.e. according as P does or does not take the truth-value $\frac{1}{3}$. Thus our definition of the functor $J_{\frac{1}{3}}$ is justified.

In order to justify our definition of the functor L we note first that, for all truth-values x,

$$l'(0, x) = 0, \; l'(1, x) = x.$$

Hence the formula

$$NL'L'J_{\frac{2}{3}}PJ_{\frac{2}{3}}QQ$$

takes the truth-value $\frac{1}{3}$ when P, Q both take the truth-value $\frac{2}{3}$ and, in all other cases, it takes the truth-value 1. For the same reasons the formula

$$NL'J_{\frac{2}{3}}PJ_{\frac{1}{3}}Q$$

takes the truth-value 0 when P, Q take the truth-values $\frac{2}{3}$, $\frac{1}{3}$ respectively and, in all other cases, it takes the truth-value 1.

Unless P takes the truth-value $\frac{2}{3}$ and Q takes one of the truth-values $\frac{1}{3}$, $\frac{2}{3}$, the truth-values of the formulae

$$L'PQ,\ NL'L'J_{\frac{2}{3}}PJ_{\frac{2}{3}}QQ,\ NL'J_{\frac{2}{3}}PJ_{\frac{1}{3}}Q$$

are equal to $l(x, y)$, 1, 1 respectively. Since, for all truth-values x,

$$l'(x, 1) = x$$

it follows at once that the truth-value of the formula

$$(1) \qquad L'L'L'PQNL'L'J_{\frac{2}{3}}PJ_{\frac{2}{3}}QQNL'J_{\frac{2}{3}}PJ_{\frac{1}{3}}Q$$

is equal to $l(x, y)$. If P, Q both take the truth-value $\frac{2}{3}$ then $L'PQ$, $NL'L'J_{\frac{2}{3}}PJ_{\frac{2}{3}}QQ$, $NL'J_{\frac{2}{3}}PJ_{\frac{1}{3}}Q$ take the truth-values 1, $\frac{1}{3}$, 1 respectively and, since

$$l'(1, \tfrac{1}{3}) = l'(\tfrac{1}{3}, 1) = \tfrac{1}{3}$$

the formula (1) takes the truth-value $\frac{1}{3}$. If P, Q take the truth-values $\frac{2}{3}$, $\frac{1}{3}$ respectively then $NL'J_{\frac{2}{3}}PJ_{\frac{1}{3}}Q$ takes the truth-value 0 and since, for all truth-values x,

$$l'(x, 0) = 0$$

the formula (1) takes the truth-value 0. Thus our definition of the functor L is justified. Finally, since

$$\begin{aligned} 1-l(x, 1-y) &= 1-\max(0, x+1-y-1) \\ &= 1-\max(0, x-y) \\ &= \min(1, 1-x+y), \end{aligned}$$

our definition of the functor C is justified.

Thus Theorem 1 is proved. The solution to the problem provided thereby is not a commutative functor since

$$f(\tfrac{2}{3}, \tfrac{1}{3}) = 0,\ f(\tfrac{1}{3}, \tfrac{2}{3}) = 1$$

although, in all other cases,

$$f(x, y) = f(y, x).$$

It is not difficult, however, to obtain a commutative solution as a corollary of Theorem 1. Let G be a binary functor such that, if P, Q, GPQ take the truth-values x, y, $g(x, y)$ respectively then

$$g(\tfrac{1}{3}, \tfrac{2}{3}) = 0$$

and, in all other cases,

$$g(x, y) = f(x, y).$$

Thus

$$GPQ =_T GQP.$$

THEOREM 2. *In the m-valued propositional calculus with G as the only primitive functor we may define C and N and, in the m-valued Łukasiewicz propositional calculus, we may define G* ($m = 4, 7, \cdots$).

Since $g(x, y) = 0$ in the only case where $g(x, y) \neq f(x, y)$, it follows at once from the theorem of McNaughton referred to above [9] that we may define G in terms of C and N. In order to define C and N in terms of G we note first that, by arguments strictly analogous to those given in the proof of Theorem 1, we make the definitions

$$DP =_{df} GGPPGPP,\ VP =_{df} GD^{m-2}PGD^{m-2}PD^{m-2}P,$$
$$NP =_{df} GVPP.$$

Similarly, if we make the definitions

$$L''PQ =_{df} NGPQ,\ B''PQ =_{df} GNPNQ,$$

the formula $L''PQ$ will take the truth-value 1 when P, Q take the truth-values $\frac{1}{3}$, $\frac{2}{3}$ respectively, the formula $B''PQ$ will take the truth-value 0 when P, Q take the truth-values $\frac{2}{3}$, $\frac{1}{3}$ respectively and, in all other cases,

$$L''PQ =_T L'PQ,\ B''PQ =_T B'PQ.$$

We consider next the definitions

$$HP =_{df} L''PP,\ MP =_{df} NGPNP,\ J_{\frac{2}{3}}P =_{df} L''HPMP,$$
$$J_{\frac{1}{3}}P =_{df} J_{\frac{2}{3}}NP,\ FPQ =_{df} B''GPQL''J_{\frac{1}{3}}PJ_{\frac{2}{3}}Q.$$

We note first that if P, Q, $L''PQ$ take the truth-values x, y, $l''(x, y)$ respectively then

$$l''(1, 1) = 1$$

and, for all truth-values x,

$$l''(0, x) = l''(x, 0) = 0.$$

[9] See footnote 5.

If P takes the truth-value $\frac{2}{3}$ then HP, MP both take the truth-value 1, as does $L''HPMP$. If P takes the truth-value $\frac{1}{3}$ then HP takes the truth-value 0, as does $L''HPMP$. If P takes a truth-value other than $\frac{1}{3}$ or $\frac{2}{3}$ then MP takes the truth-value 0, as does $L''HPMP$. Thus our definition of the functor $J_{\frac{2}{3}}$ is justified. Since NP takes the truth-value $\frac{2}{3}$ if and only if P takes the truth-value $\frac{1}{3}$, our definition of the functor $J_{\frac{1}{3}}$ is justified.

In order to justify our last definition we note first that if P, Q, $B''PQ$ take the truth-values x, y, $b''(x, y)$ respectively then

$$b''(0, 1) = 1$$

and, for all truth-values x,

$$b''(x, 0) = x.$$

If P, Q take the truth-values $\frac{1}{3}$, $\frac{2}{3}$ respectively then the formula

$$L''J_{\frac{1}{3}}PJ_{\frac{2}{3}}Q$$

takes the truth-value 1, as does the formula

$$B''GPQL''J_{\frac{1}{3}}PJ_{\frac{2}{3}}Q.$$

In all other cases the formula

$$L''J_{\frac{1}{3}}PJ_{\frac{2}{3}}Q$$

takes the truth-value 0 and

$$B''GPQL''J_{\frac{1}{3}}PJ\ Q =_T GPQ.$$

Thus our definition of the functor F is justified. Since F is definable in terms of G it follows at once from Theorem 1 that C and N are definable in terms of G.

It has been shown [10] that, in the $\aleph_0$-valued case, there are no solutions, but that, if a certain third primitive functor is adjoined to those of Łukasiewicz [11], a quaternary generator exists. We shall show now that another extension of the Łukasiewicz system possesses a binary generator and, in Theorem 4, that the resulting system is less extensive than that of the previous paper. Let us consider the functors J, F of the $\aleph_0$-valued propositional calculus such that if P, Q, JP, FPQ take the truth-values x, y, $j(x)$, $f(x, y)$ respectively then

[10] See the paper referred to in footnote 1, especially pp. 371—372.

[11] See, for example, the paper referred to in footnote 7, especially pp. 1—5.

$$j(x) = 1\ (\tfrac{5}{8} < x < \tfrac{3}{4}),$$
$$f(x, y) = x+y-1\ (\tfrac{5}{8} < x < \tfrac{3}{4}, \tfrac{5}{8} < y < \tfrac{3}{4}),$$
$$f(x, y) = 0\ (\tfrac{5}{8} < x < \tfrac{3}{4}, \tfrac{1}{4} < y < \tfrac{3}{8})$$

and, in all other cases,

$$j(x) = 0,\ f(x, y) = \min(1, 2-x-y).$$

Theorem 3. *In the $\aleph_0$-valued propositional calculus we may define F in terms of C, N and J and we may define C, N and J in terms of F.*

In the system obtained from that of Łukasiewicz by taking J as a third primitive functor let us consider the definition

$$FPQ =_{df} LLNLLSPQJPJQBSPQLJPJQBNJPNJNQ,$$

where

$$SPQ =_{df} CPNQ,\ LPQ =_{df} NCPNQ,\ BPQ =_{df} CNPQ.$$

Let us denote the truth-values of P, Q by x, y respectively.

If $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{5}{8} < y < \frac{3}{4}$ then

$$j(x) = j(y) = 1$$

and

$$NLLSPQJPJQ =_T NSPQ.$$

Since

$$j(x) = j(y) = 1$$

it follows also that the formula

$$BSPQLJPJQ$$

takes the truth-value 1. Hence

$$LNLLSPQJPJQBSPQLJPJQ =_T NSPQ.$$

Since $\frac{5}{8} < y < \frac{3}{4}$ it follows at once that

$$\tfrac{1}{4} < 1-y < \tfrac{3}{8}$$

and JNQ takes the truth-value 0. Hence the formula

$$BNJPNJNQ$$

takes the truth-value 1 and

$$LLNLLSPQJPJQBSPQLJPJQBNJPNJNQ =_T NSPQ.$$

Thus the formula $LLNLLSPQJPJQBSPQLJPJQBNJPNJNQ$ takes the truth-value

$$1-\min(1, 2-x-y) = 1-(2-x-y)$$
$$= x+y-1.$$

If $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{1}{4} < y < \frac{3}{8}$ then $\frac{5}{8} < 1-y < \frac{3}{4}$ and the formulae JP, JNQ both take the truth-value 1. Hence the formula

$$BNJPNJNQ$$

takes the truth-value 0 as does the formula

$$LLNLLSPQJPJQBSPQLJPJQBNJPNJNQ.$$

In all other cases JP takes the truth-value 0 or JQ, JNQ both take the truth-value 0. Hence the formulae

$$NLLSPQJPJQ, BNJPNJNQ, LJPJQ$$

take the truth-values 1, 1, 0 respectively and

$$LLNLLSPQJPJQBSPQLJPJQBNJPNJNQ$$
$$=_T BSPQLJPJQ =_T SPQ.$$

Thus the formula $LLNLLSPQJPJQBSPQLJPJQBNJPNJNQ$ always takes the truth-value $f(x, y)$ and our definition of the functor F is justified.

In the system with F as the only primitive functor let us consider the definitions

$$DP =_{df} FFPPFPP, VP =_{df} FDDPFDDPDDP,$$
$$NP =_{df} FVPP,$$
$$J^*P =_{df} FPNP, JP =_{df} NJ^*P, L'PQ =_{df} NFPQ,$$
$$B'PQ =_{df} FNPNQ,$$
$$LPQ =_{df} L'L'NL'L'L'PQJPJQB'L'PQL'JPJQB'B'L'QPJ^*$$
$$PJ^*NQ,$$
$$CPQ =_{df} NLPNQ.$$

Let us again denote the truth-values of P, Q by x, y respectively.

If $x \leqq \frac{1}{2}$ then

$$f(f(x, x), f(x, x)) = f(1, 1)$$
$$= 0.$$

If $\frac{1}{2} < x \leqq \frac{5}{8}$ then

$$f(f(x, x), f(x, x)) = f(2-2x, 2-2x)$$
$$= (\text{since } 2-2x \geqq \tfrac{3}{4})\ 4x-2.$$

If $\frac{5}{8} < x < \frac{3}{4}$ then

$$\begin{aligned} f(f(x, x), f(x, x)) &= f(2x-1, 2x-1) \\ &= (\text{since } 2x-1 < \tfrac{1}{2})\ 1. \end{aligned}$$

If $x \geqq \frac{3}{4}$ then

$$\begin{aligned} f(f(x, x), f(x, x)) &= f(2-2x, 2-2x) \\ &= (\text{since } 2-2x \leqq \tfrac{1}{2})\ 1. \end{aligned}$$

Thus, if DP (defined as in the previous paragraph) takes the truth-value $d(x)$,

$$\begin{aligned} d(x) &= 0\ (x \leqq \tfrac{1}{2}), \\ d(x) &= 4x-2\ (\tfrac{1}{2} < x \leqq \tfrac{5}{8}), \\ d(x) &= 1\ (x > \tfrac{5}{8}). \end{aligned}$$

Since

$$4x-2 \leqq \tfrac{1}{2}\ (\tfrac{1}{2} < x \leqq \tfrac{5}{8})$$

it follows that, for all truth-values x,

$$d(d(x)) \in \{0, 1\}.$$

Hence our definition of N may be justified exactly as in the proof of Theorem 1.

Since $\frac{1}{4} < 1-x < \frac{3}{8}$ whenever $\frac{5}{8} < x < \frac{3}{4}$,

$$f(x, 1-x) = 0\ (\tfrac{5}{8} < x < \tfrac{3}{4}).$$

In all other cases

$$\begin{aligned} f(x, 1-x) &= \min\big(1, 2-x-(1-x)\big) \\ &= 1. \end{aligned}$$

Since $1-1 = 0$ and $1-0 = 1$, our definition of the functor J is justified.

Let LPQ, BPQ, as defined in terms of C and N, take the truth-values $l(x, y)$, $b(x, y)$ respectively.

If $L'PQ$, $B'PQ$ take the truth-values $l'(x, y)$, $b'(x, y)$ respectively when defined by the method now under consideration then, if $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{5}{8} < y < \frac{3}{4}$,

$$\begin{aligned} l'(x, y) &= 1-f(x, y) \\ &= 1-(x+y-1) \\ &= 2-x-y. \end{aligned}$$

If $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{1}{4} < y < \frac{3}{8}$ then

$$\begin{aligned} l'(x, y) &= 1-0 \\ &= 1 \end{aligned}$$

and, in all other cases,

$$\begin{aligned} l'(x, y) &= 1-\min(1, 2-x-y) \\ &= \max(0, x+y-1) \\ &= l(x, y). \end{aligned}$$

If $\frac{1}{4} < x < \frac{3}{8}$ and $\frac{1}{4} < y < \frac{3}{8}$ then $\frac{5}{8} < 1-x < \frac{3}{4}$ and $\frac{5}{8} < 1-y < \frac{3}{4}$. Hence

$$\begin{aligned} b'(x, y) &= 1-x+1-y-1 \\ &= 1-x-y. \end{aligned}$$

If $\frac{1}{4} < x < \frac{3}{8}$ and $\frac{5}{8} < y < \frac{3}{4}$ then $\frac{5}{8} < 1-x < \frac{3}{4}$ and $\frac{1}{4} < 1-y < \frac{3}{8}$. Hence

$$b'(x, y) = 0.$$

In all other cases

$$\begin{aligned} b'(x, y) &= \min\big(1, 2-(1-x)-(1-y)\big) \\ &= \min(1, x+y) \\ &= b(x, y). \end{aligned}$$

It now follows at once that, for all truth-values x,

$l'(0, x) = l'(x, 0) = 0$, $l'(1, x) = l'(x, 1) = b'(0, x) = b'(x, 0) = x$, $b'(1, x) = b'(x, 1) = 1$.

If $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{5}{8} < y < \frac{3}{4}$ the formulae JP, JQ both take the truth-value 1. Hence the formula

$$NL'L'L'PQJPJQ$$

takes the truth-value

$$\begin{aligned} 1-l'(x, y) &= 1-(2-x-y) \\ &= x+y-1 \\ &= (\text{since } x+y-1 > 0)\ l(x, y). \end{aligned}$$

Since JP, JQ both take the truth-value 1 the formula

$$B'L'PQL'JPJQ$$

takes the truth-value 1 also. Since $\frac{1}{4} < 1-y < \frac{3}{8}$ the formula J^*NQ takes the truth-value 1 as does the formula

$$B'B'L'QPJ^*PJ^*NQ.$$

Since the formulae

$$NL'L'L'PQJPJQ,\ B'L'PQL'JPJQ,\ B'B'L'QPJ^*PJ^*NQ$$

take the truth-values $l(x, y)$, 1, 1 respectively it follows at once that the formula

$$L'L'NL'L'L'PQJPJQB'L'PQL'JPJQB'B'L'QPJ^*PJ^*NQ \tag{B}$$

takes the truth-value $l(x, y)$. If $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{1}{4} < y < \frac{3}{8}$ then JP, J^*P, JQ, J^*NQ take the truth-values 1, 0, 0, 0 respectively. Hence the formulae

$$NL'L'L'PQJPJQ,\ B'L'PQL'JPJQ,\ B'B'L'QPJ^*PJ^*NQ$$

take the truth-values 1, $l'(x, y)$, $l'(y, x)$ respectively. But, in this case,

$$l'(x, y) = 1$$

and

$$l'(y, x) = l(y, x)$$
$$= l(x, y).$$

Hence the formula (B) takes the truth-value $l(x, y)$. In all other cases at least one of the formulae JP, JQ takes the truth-value 0 and at least one of the formulae J^*P, J^*NQ takes the truth-value 1. Hence the formulae $NL'L'L'PQJPJQ$, $B'L'PQL'JPJQ$, $B'B'L'QPJ^*PJ^*NQ$ take the truth-values 1, $l'(x, y)$, 1 respectively. Since, in these cases,

$$l'(x, y) = l(x, y)$$

the formula (B) takes the truth-value $l(x, y)$.

We have now established that, for all truth-values x, y, the formula (B) takes the truth-value $l(x, y)$. Thus our definition of the functor L is justified and we may justify our definition of the functor C as in the proof of Theorem 1.

THEOREM 4. *If EPQ takes the truth-value* 1 *when the truth-values of P, Q are equal and it takes the truth-value* 0 *in all other cases then the functor J is definable in terms of C, N and E but the functor E is not definable in terms of C, N and J.*

Let

$$\phi(x) = \min(1, 11-16x) \; (0 \leqq x \leqq \tfrac{11}{16}),$$
$$\phi(x) = \min(1, 16x-11) \; (\tfrac{11}{16} < x \leqq 1).$$

It follows at once from a theorem of McNaughton [12] that, in terms of C and N, we may define a functor J' such that, if P takes the truth-value x, then $J'P$ takes the truth-value $\phi(x)$. We may then, in the system with C, N and E as primitive functors, make the definition

$$JP =_{df} NEJ'PEPP.$$

In the system with C, N and J as primitive functors it follows easily, by strong induction on the number of (not necessarily distinct) symbols occurring in P that, if P contains no propositional variables other than p and p, P take the truth-values x, y $(= y(x))$ respectively, then there exist a positive number ε $(= \varepsilon(P))$ and an integer n $(= n(P))$ such that, whenever $x < \varepsilon$,

$$y(x) - y(0) = nx.$$

Clearly no positive number ε and integer n correspond to the formula $EpNEpp$. Hence E cannot be defined in terms of C, N and J.

By a similar argument it can be shown that the $\aleph_0$-valued generalisation of E_{n-2} (in terms of which C, N and J can obviously be defined) cannot be defined in terms of C, N and J.

Added in proof.

The $\aleph_0$-valued propositional calculus considered in Theorem 3 may be generated by another binary functor. The generalised truth-table of this functor is constructed by a slightly more complicated rule, but the new table is commutative. Let us consider the functor G such that if P, Q, GPQ take the truth-values x, y, $g(x, y)$ respectively then

$$g(x, y) = g(y, x) = \max(0, x+y-1) \; (\tfrac{1}{4} < x < \tfrac{3}{8}, \tfrac{5}{8} < y < \tfrac{3}{4}),$$
$$g(x, y) = x+y-1 \; (\tfrac{5}{8} < x < \tfrac{3}{4}, \tfrac{5}{8} < y < \tfrac{3}{4})$$

and, in all other cases,

$$g(x, y) = \min(1, 2-x-y).$$

THEOREM 5. *The functor G is commutative and may be defined in*

[12] See the paper referred to in footnote 5, especially pp. 1–9.

terms of the functors C, N and J and the functors C, N and J may be defined in terms of G.

It follows at once from the definition of the function $g(,)$ that

$$GPQ =_T GQP.$$

In the propositional calculus with C, N and J as primitives we may make the definition

$$GPQ =_{df} BLLPQBBLJPJQLJPJNQLJNPJQLSPQN \\ BBLJPJQLJPJNQLJNPJQ,$$

the functors L, B, S being defined as in the first part of the proof of Theorem 3. Let P, Q take the truth-values x, y respectively.

If $\frac{1}{4} < x < \frac{3}{8}$ and $\frac{5}{8} < y < \frac{3}{4}$ then the formula $LJNPJQ$ takes the truth-value 1 as does the formula

$$BBLJPJQLJPJNQLJNPJQ.$$

For similar reasons the latter formula takes the truth-value 1 if $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{1}{4} < y < \frac{3}{8}$ and also if $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{5}{8} < y < \frac{3}{4}$. Thus, in all three cases, the formula

$$LLPQBBLJPJQLJPJNQLJNPJQ$$

takes the same truth-value as LPQ, i.e. its truth-value is equal to the value of $g(x, y)$. Since, in these three cases, the formula

$$NBBLJPJQLJPJNQLJNPJQ$$

takes the truth-value 0 it follows easily that the formula GPQ, defined as above, takes the truth-value equal to the value of $g(x, y)$.

In all the remaining cases the formulae

$$LJPJQ, LJPJNQ, LJNPJQ$$

all take the truth-value 0. Thus the formula

$$BBLJPJQLJPJNQLJNPJQ$$

and its negation take the truth-values 0, 1 respectively and the formula GPQ, defined as above, takes the same truth-value as the formula SPQ, i.e. its truth-value is equal to the value of $g(x, y)$.

In the propositional calculus with G as the only primitive functor let us consider the definitions

$$
\begin{aligned}
DP &=_{df} GGPPGPP,\\
VP &=_{df} GDDPGDDPDDP,\\
NP &=_{df} GVPP,\\
L'PQ &=_{df} NGPQ,\\
B'PQ &=_{df} GNPNQ,\\
JP &=_{df} L'DDPNGPNP,\\
LPQ &=_{df} L'NL'L'PQB'B'L'JPJQL'JPJNQL'JNPJQB'L'\\
&\qquad PQB'B'L'JPJQL'JPJNQL'JNPJQ,\\
CPQ &=_{df} NLPNQ.
\end{aligned}
$$

Our definitions of the functors D, V, N may be justified exactly as in the proof of Theorem 3, if these functors are regarded as having the same generalised truth-tables now as then. If the formula[13] $L'PQ$ takes the truth-value $l'(x, y)$ when P, Q take the truth-values x, y respectively then

$$l'(x, y) = 1-g(x, y).$$

Thus

$$l'(x, y) = l'(y, x) = \min(1, 2-x-y)\ (\tfrac{1}{4} < x < \tfrac{3}{8}, \tfrac{5}{8} < y < \tfrac{3}{4}),$$
$$l'(x, y) = 2-x-y\ (\tfrac{5}{8} < x < \tfrac{3}{4}, \tfrac{5}{8} < y < \tfrac{3}{4})$$

and, in all other cases,

$$l'(x, y) = \max(0, x+y-1).$$

If $B'PQ$ takes the truth-value $b'(x, y)$ when P, Q take the truth-values x, y respectively then

$$b'(x, y) = g(1-x, 1-y).$$

Hence

$$b'(x, y) = b'(y, x) = \max(0, 1-x-y)\ (\tfrac{1}{4} < x < \tfrac{3}{8}, \tfrac{5}{8} < y < \tfrac{3}{4}),$$
$$b'(x, y) = 1-x-y\ (\tfrac{1}{4} < x < \tfrac{3}{8}, \tfrac{1}{4} < y < \tfrac{3}{8})$$

and, in all other cases,

$$b'(x, y) = \min(1, x+y).$$

Hence

$$l'(0, x) = l'(x, 0) = 0,\ b'(1, x) = b'(x, 1) = 1,$$
$$l'(1, x) = l'(x, 1) = b'(0, x) = b'(x, 0) = x.$$

[13] The generalised truth-tables of the functors defined above are the same as in previous cases, except for the functors L', B'. The functions corresponding to the generalised truth-tables of these two functors will now be determined.

We shall make use of these latter eight equations without comment.

If $x \leqq \frac{5}{8}$ then DDP takes the truth-value 0 as does the formula $L'DDPNGPNP$. If $\frac{5}{8} < x < \frac{3}{4}$ then DDP takes the truth-value 1 and $GPNP$ takes the truth-value equal to the value of

$$g(x, 1-x).$$

Since

$$\tfrac{1}{4} < 1-x < \tfrac{3}{8},$$

$$g(x, 1-x) = \max(0, x+1-x-1) = 0.$$

Thus the formulae $NGPNP$, $L'DDPNGPNP$ take the truth-value 1. If $x \geqq \frac{3}{4}$ then

$$g(x, 1-x) = \min\big(1, 2-x-(1-x)\big) = 1.$$

Thus the formulae $NGPNP$, $L'DDPNGPNP$ take the truth-value 0 and our definition of the functor J is justified.

If $\frac{1}{4} < x < \frac{3}{8}$ and $\frac{5}{8} < y < \frac{3}{4}$ then the formula $L'JNPJQ$ takes the truth-value 1 as does the formula

$$B'B'L'JPJQL'JPJNQL'JNPJQ.$$

Thus the formula LPQ, defined as above, takes the same truth-value as the formula $NL'PQ$, i.e. its truth-value is given by

$$1-\min(1, 2-x-y) = \max(0, x+y-1).$$

The justification of our definition in the case where $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{1}{4} < y < \frac{3}{8}$ is similar, as is the justification when $\frac{5}{8} < x < \frac{3}{4}$ and $\frac{5}{8} < y < \frac{3}{4}$. In all the remaining cases the formula

$$B'B'L'JPJQL'JPJNQL'JNPJQ$$

takes the truth-value 0 and the formula LPQ, defined as above, takes the same truth-value as $L'PQ$, i.e. its truth-value is equal to the value of $\max(0, x+y-1)$. Thus our definition of L is justified and the corresponding justification for C is trivial.

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The University, Nottingham
England