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Constructive mathematics as a philosophical problem

Dedicated to A. Heyting on the occasion of his 70^{th} birthday

by

P. Lorenzen

Let me begin with a historical remark about what I will call "Constructive Mathematics". As distinct from other forms of mathematics, namely as distinct from the Cantorian (or set-theoretical) mathematics, it started with Kronecker. It started as a reaction against Cantorianism — and this was considered as a deplorable outcome of a long history; beginning with Greek mathematical theories about infinity, merging with the Indian-Arabic arithmetic of decimal-fractions since the 16th century, and leading to the mathematical treatment of "arbitrary" functions by Fourier and Dirichlet.

Constructivism means nothing else than criticizing this so-called "classical" tradition — and trying to save its achievements as far as this can be done by reconstructing systematically the historically given.

The name "Constructive" Mathematics is irrelevant; it could be called as well "critical" — as opposed to "traditional" — or simply "Mathematics" as it claims to be all that which can be *justified* from our mathematical heritage.

"Justification" — in my use of this word — means two different things:

1. Constructive Mathematics has to be shown as a possible human activity.

2. Constructive Mathematics has to be shown as a *good* possibility, at least as a *better* possibility than its rivals, i.e. set-theoretical mathematics in naive or axiomatic forms.

The first problem of whether constructive mathematics is possible at all, is a logical, more specific an *epistemological* question. The second problem of whether, if possible, it is a *good* possibility, is a question of evaluating. A thorough discussion of value-judgments will lead us inevitably to broader issues of *moral-philosophy.*

In order to deal with our epistemological and moral questions, I propose first of all to look for a common basis from where to start. There is one thing in common to mathematicians of all philosophical denominations: assertions of the type that a certain formula X is derivable ($\vdash_F X$) according to the rules of a formal system F. There is no difficulty in interpreting "$\vdash$" as "derivable", because in spite of the modal flavour of the word the assertion $\vdash X$ may be understood in the following sense: if *you* assert $\vdash X$ *I* may ask you to write down a derivation of X. Only after this has been done — and this is a finite affair — *I* have to agree to *your* assertion.

This is the simple basis which in spite of all philosophical controversies still unites the mathematicians all over the world into a family-like group which enjoys a perfect mutual understanding. But, alas, this basis is too small to decide our problems of constructivism. No mathematical results whatsoever in this narrow sense of derivability-results will settle our problems.

This means that we have to look for a common basis beyond simple derivations. It seems as though we are at a loss, if we try to find out, what a group of mathematicians discussing foundational problems may have in common — besides their skill in formal derivations. Do all of them share some beliefs in reason?, or in the existence of some objects? The very moment you start to formulate such possibly common basis, you will be at a loss. Because you will have to formulate it in traditional philosophical language — and, as long as there is no agreement about justifying such elementary logical or arithmetical truths, as e.g. "*a* or *b and* not *a implies b*" or "for all m, n: $m+n = n+m$", you should not expect any agreement on philosophical statements such as "Only concrete entities exist", "your are justified to use the term, if there is a notion, clearly presented to your mind, which you denote by the term". "If society honors some kind of traditional mathematics, you are justified to do it." "You are justified to do whatever you like as long as it is legal."

No matter how you feel about such statements, you should not expect any agreement, because they all belong to our philosophical traditions, originating mostly in ancient Greece and being pretty much deteriorated by such intellectual adventures as Christian theology and Modern Science.

But there is, nevertheless, one thing in common to all serious participants of a discussion, namely the simple fact, that they want to discuss their subject. They want to discuss, in our case, which possibilities there are to do mathematics — and they want to discuss which possibilities are good, and there may even be a best one.

In order to discuss something, you need a language. Either you have got one or you must be able quickly to introduce one. Even if we dismiss from our so-called natural language, English in our case, all terms and complicated syntactical features which may be suspected to belong to a philosophical tradition, there remains a level of simple talk — I will call it the "practical level" of English — common to all participants of a discussion. "What do you want to say?" "I propose so and so." "I would like to say ···" are examples of this practical level. There is no mystery about this practical level. A foreigner easily can pick up this basic English, as the use of all its words can be demonstrated by examples. Of course, on this practical level, we don't have a "precise" language; it makes no sense to find out "exactly" whether such statements as "I like coffee" are true — on the contrary, just this is the common understanding of the practical level that it is practical. To treat practical statements as theoretical ones is nothing but boring and silly.

Once we realize this common basis, the practical level which is contained in every natural language, we have the task to work upwards, extending our language step by step, so that we continue to understand each other while describing possible ways of doing mathematics — and while comparing different possibilities. Or we may work downwards. This means to jump into a traditional piece of elaborated language, e.g. into the expression "to grasp the idea of an actual infinite totality" and then to try to reduce the number of theoretical terms in it. Working downward we may provisionally use other theoretical terms, say "concept" or "set" in combination with the words of our basic English. But the "analysis" — as this method of working downwards is traditionally called — is never finished unless every last theoretical term is analyzed. Just *one* term left unanalyzed — and the whole job is spoiled: the different philosophical denominations will manage to interpret this last unanalyzed term in many different and mutually incompatible ways.

Since for the last ten years I have deliberately only worked upwards (what is traditionally called "synthesis") — I would like to mention here my paper on "Methodical Thinking" in

Ratio and my book on "Differential und Integral" both published in 1965 — I feel inclined to join the analytic game for a while, at least for this paper.

Constructivism in the spirit of Kronecker, Borel, Poincaré, Brouwer, Weyl and Heyting — I consider my own work most near to Weyl — is today attacked on both levels, epistemologically and morally. There are the finitists (as Tarski and Abraham Robinson) and the nominalists (as Goodman) who hold that constructive mathematics is impossible: There exist only concrete individuals, no abstract entities, say the nominalists. There exist only finite sets as abstract entities, say the finitists. These dogmas are directed, e.g. against such arithmetical truths as "There exist infinitely many prime numbers". This theorem occurs already in Euclid in a formulation very near to the following: "For a finite sequence of prime numbers there always exists another one". If we consider that the proof of this theorem is given by using the following term "the smallest divisor (> 1) of $p_1 \cdots p_n + 1$" (where $p_1, \cdots, p_n$ is the given finite sequence) we come down to such philosophical questions as whether the use of such a term is justified. Is this perhaps meant if some say that the smallest divisor (> 1) of $p_1 \cdots p_n + 1$ *exists*?

Does existence in arithmetic perhaps mean nothing more than a possible — and good — use of some term? If we e.g. say that for all numerals p and q their product $p \cdot q$ exists — do we necessarily mean more than that the term "$p \cdot q$" is a term which may be substituted for any number-variable. The nominalists are quite right in saying that "the product $p \cdot q$" does not "exist" as concrete objects do, i.e. that the product is not a concrete object. And they are also right that a finite set is not a concrete object. But if we say that a finite set exists, we could mean that it is possible — and good — to abstract from some differences between sequences of individuals, e.g. between $a, b, c,$ and b, c, a and a, b, b, c. We say that the sets $\{a, b, c\}$, $\{b, c, a\}$, $\{a, b, b, c\}$ are the same, though the sequences are different. This comes down to nothing more than an equivalence relation between finite sequences — and to the restriction of asserting for sets only such statements for sequences which hold for all equivalent sequences simultaneously. Infinite sets do not "exist" in the same sense as finite ones, because we cannot write down an infinite sequence. But it is possible — and this is a good possibility — to introduce e.g. variables $m, n \cdots$ for numerals which are constructed according to the following rules

$$\Rightarrow 1$$

$$n \Rightarrow n1$$

With variables we easily get "sentence-forms" such as

$$m = 1, m+n > 2, \cdots$$

If you allow the unanalyzed use of logical particles you get arbitrarily complicated formulas (sentences or sentence-forms) e.g.

$$\neg V_{m,n}(m > 1 \wedge n > 1 \wedge m \cdot n = p)$$

From this particular sentence form $A(p)$ you may "abstract" $\varepsilon_p A(p)$, the set of prime numbers. Here we abstract from differences between equivalent formulas

$$\varepsilon_p A(p) \doteqdot \varepsilon_p B(p) \leftrightarrows \bigwedge p \cdot A(p) \leftrightarrow B(p).$$

To speak of the infinite set of prime numbers means now to speak of the formula "$A(p)$" only in such ways as hold simultaneously for all equivalent formulas. This new possibility of speaking of sets is quite different from the earlier possibility for finite sets only. But the new possibility includes for each sequence $a_1, \cdots, a_n$ the set $\varepsilon_p(p = a_1 \vee p = a_2 \vee \cdots \vee p = a_n)$ — and this may be symbolized as $\{a_1, \cdots, a_n\}$. This shows how Constructivism may avoid the nominalist and finitist criticism.

Much more difficult, it seems to me, will it be to meet the objections raised on the moral level by the formalists, namely that Constructivists should not waste their time, especially that they should not try to persuade other people to waste their time too, with cumbersome and perhaps unusual attempts to reconstruct the achievements of the traditionally given higher parts of mathematics. Instead, they should join the big game of axiomatic set-theory: "You will become famous if you please famous people — and all famous mathematicians like axiomatic set-theory".

It seems to me that the difference of opinion concerning whether or not axiomatic set-theory is a good thing to do, should be argued much more carefully than it is usually done. For the following analysis of the moral claims of the formalists, I will assume that constructive mathematics is possible. There is of course no question that each axiomatic theory (whether known to be consistent or not) is a possible thing to do. The problem is whether or not we can find out that we can spend our time *better* if we do axiomatic instead of constructive mathematics. As long as axiomatic theories are justified because they have at least one model in constructive mathematics there is no moral difficulty in principle. For number theoretic problems e.g. it has to be left to the ingenuity, perhaps the taste, of each mathematician how much of purely axiomatic theory he is going to use as an additional tool in his immediate constructive reasoning.

The moral problem arises with axiomatic theories for which there is no constructive model and for which there is no constructive consistency proof providing a constructive interpretation of that theory no matter how indirect it may be. This is the case with e.g. Zermelo-Fraenkel set theory. For the Peano-arithmetic there is a constructive model (and the use of classical logical calculi can constructively be interpreted). But if we add axioms for the real numbers, $x, y, \cdots$, especially the classical completeness axiom that every (non-empty but bounded) *set* of reals has a real as its least upper bound, we have once more a theory R with no model, and no constructive consistency proof. In this completeness axiom we could use sentence-forms $A(x)$ of the theory instead of sets. The point is that a sentence-form $A(x)$ with one free variable x for reals may contain bound real variables too. If we use a *restricted completeness axiom*, the restriction being that $A(x)$ may contain *no* bound real variables, we get a theory R_0 for which a constructive model easily may be found. Why now shall it be *better* to use R instead of R_0? The formalists say that at least since Dedekind and Cantor mathematicians have naively — or intuitively, as one says — used R instead of R_0. But this merely begs the question whether they would not have done better with R_0 from the very beginning.

Of course, then we would have no text-books with theorems about the hierarchy of transfinite cardinals. But we have also no text-books any more about the hierarchy of angels. No one seriously regrets this — though of course scholastic theology could be formalized.

In considering the moral issue: "Which is better to adopt, R or R_0?" let me summarize the assumptions I shall make.

1. R_0 has a constructive model and, constructive mathematics being granted as a possible human activity, there is no doubt that R_0 has some merits.

2. R is not known to be consistent. No derived formula allows any inference to any theorem in constructive arithmetic or analysis.

Stating the initial conditions for the discussion in this way leaves the burden of proof on the defenders of R, i.e. to the formalists.

But before the arguments begin the formalists may say that R is the way the so-called working-mathematicians *do* their jobs and have done them for nearly a hundred years. This is a matter of fact — and, so they might say, everyone who is proposing a new way such as R_0, has the burden of proof that the new way is better. That is the argument of conservatism: R has worked well, at least well enough that we have survived with it up to the present — no rival theory has stood this test of history.

Very often this conservative argument is put in the following form: R has proved useful for physics. The answer can be given in general: no historical success with some specific activity can prove that another way of doing things could not have led to a still greater success. This means in our case: though R has been found useful for physics, nobody can predict that in the future, when working mathematicians may have switched to R_0, physics will flourish less.

Only if in the argument, instead of referring in general to the usefulness of R for physics, a particular result of physics would be pointed out which has been derived with the help of R, but which could *not* be established with R_0, would this argument be strong. I have asked as many formalists as possible whether they could point out such a result to me — unfortunately I have never come across even one piece of mathematical reasoning which — as far as it was relevant for physical theory — could not easily be reconstructed on the basis of R_0 instead of R. If from the world of mathematics all higher transfinite cardinalities would disappear, no change whatsoever in the world of physics would be observable, at least not to the best of my knowledge.

If the formalists would agree that at present R has no systematic justification, but that they would like to continue with R, because the constructivists may probably in the future prove at least the consistency of R, then the situation would be different. The probability for finding a constructive consistency proof is very difficult to evaluate. I am not going to argue about that. The very moment, it seems to me, the desirability of at least a constructive consistency-proof is admitted, the moral issue is settled in principle. The rest would be organizational in character: how much labour should be put in developing R, how much in developing R_0, how much in finding a constructive consistency proof of R?

To my experience the present situation is not very favourable to such a solution of mutual understanding, because most of the formalists are at the same time finitists (in the sense of not seeing constructive mathematics as a possible alternative at all).

This state of affairs is of course no accident. Just because the naive (or intuitive) mathematics lost its credibility, it was an understandable reaction to lose faith even in the simplest cases of reasoning about infinity. Once being reduced to finitistic scepticism, there is only the formalist solution left, if one is to do higher mathematics at all, namely to do it as deriving formulas in a formal theory. It is the constructive position that even this "solution" is not really a way out — because there is no way of reasonably choosing the formal systems.

Because of this connection between finitism and formalism the epistemological discussion about the possibility of constructive mathematics seems to me vital for the moral issue. The constructivist position epistemologically is a medium position — the golden mean, it seems to me — between the finitistic scepticism (Thou shalt not use the word "infinity" at all) and Cantorian dogmatism (There exist infinitely many infinities, cardinally different). I hold with Aristotle that there is a possible way of speaking about infinity, namely by setting up rules for constructing symbols — this is what traditionally called a potential infinity. Only if this epistemological issue has been settled so that at least this so-called "potential infinity" is accepted as a possible term in our language, does the moral issue namely the question of whether it is still better to pursue the axiomatic theories, which have emerged from mathematics historically than to *restrict* ourselves to constructive mathematics, make sense.

On the other hand the epistemological quarrel about the "existence" of potentially infinite sets gets its strongest impulse just from the formalists, who try to defend their predilection for axiomatic theories by arguing that — unless you play with formal systems — there is no way to deal with infinity at all.

The interest for the epistemological question derives from the interest in the practical question: we formalists want to continue our activities as usual; we do not want to be disturbed by superscrupulous philosophers.

I would like therefore to analyze the moral issue a little bit further. Let us now set aside the argument for R as being useful for physics — very often formalists try to avoid the moral question involved by claiming that R is a *beautiful* theory. With this move of course all the difficulties connected with a serious understanding of art, especially modern art, enters the discussion. The ideology of modern art is indeed that it claims public interest for no other reason than that it pleases certain types of people. Obviously the art of perfume-making could make the same claim. But, as a matter of fact, it doesn't. People, especially women who like perfumes, have to pay all the expenses of the perfume-industry. There is no perfume-making instruction in public schools, or universities for the general public. Though there are far more people who like to smell perfume than there are people who like to derive set-theoretical theorems — only the latter is financially supported by the government, i.e. by *all* tax-payers.

It seems to me that very often the following two things are mixed up: the question why a particular mathematician has chosen his subject may be answered quite correctly by such phrases as "I just happen to like it" or: "I personally find mathematical theories more beautiful than say musical symphonies. Though I like music, I am fascinated by the beauty of mathematics much more — and there is nothing more to be said."

I think this kind of answer is sufficient for justifying the personal choice, say between music and axiomatic set-theory. But this kind of answer is sufficient only for the choice between human activities, if it is taken for granted that each of the different activities one chooses from has some merits anyhow, if each of the activities is taken as a reasonable possibility, as a possibility which is a good possibility in some sense. If e.g. you would specialize in making perfumes no one likes — but you would insist on being accepted as a good member of the society — then your argument that you happen to like just this kind of stinking stuff, would surely not be discussed seriously in journals, meetings, etc. But curiously enough, personal confessions of famous mathematicians that they just like the beauty of axiomatic set theory are printed and treated as valuable information in meetings, etc. It seems to me easy to understand the present tendency to treat all value-judgments as strictly personal as an outcome of positivism. Only scientific, value-free results are accepted as objective, the rest — and this includes morality and the arts — is left to subjective arbitrariness, called individual freedom.

I shall not try to go to the bottom of the question here. I shall not deal in general with the question to what extent objective moral reasoning is possible; instead I shall restrict myself to pointing out that in the discussion of our special moral question, namely the choice between R_0 and R, it is of no help to refer to this modern dogma (actually it is very old and comes from the Greek sophists) that all value-judgments are subjective.

Epistemological scepticism, which leads to finitism, and moral subjectivism, which leads to ignoring the problem of justification of axiomatic theories and leads therefore to formalism, both these old philosophical doctrines are combining their power in fighting constructive mathematics. First it shall be shown to be impossible — and second, if it should prove to be possible nevertheless it shall be shown to be undesirable.

Both philosophical doctrines, epistemological scepticism and moral subjectivism, seem to provide a most comfortable position: epistemologically you have nothing to defend, you just keep on doubting — and morally you have nothing to justify, you just ignore all justification talk altogether.

A very comfortable position indeed — but nevertheless it seems to me, mistaken. Against the comfortableness of doing nothing, there is a possibility of constructing a language in which "infinity" enters — and there is a possibility to justify this language.

The philosophy of constructive mathematics which takes up these tasks, thereby contributes in its own way to overcome such general weaknesses of our age as scepticism and subjectivism.

(Oblatum 3–1–'68)