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Compositio Mathematica, tome 20 (1968), p. 107-124

<http://www.numdam.org/item?id=CM_1968__20__107_0>

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Dedicated to A. Heyting on the occasion of his 70^{th} birthday

by

W. A. Howard

Introduction

By means of his functional interpretation, Gödel [1] gave a consistency proof of classical first order arithmetic relative to the free variable theory T of primitive recursive functionals of finite type. Spector [7] extended Gödel's method to classical analysis. The crucial step in [7] is the construction of a functional interpretation of the negative version of the axiom of choice. For this purpose Spector introduces the notion of *bar induction of finite type*, which generalizes Kleene's [3] formulation of Brouwer's bar theorem. The corresponding notion which Spector adds to T is *bar recursion of finite type*. Actually he uses the schema of bar recursion of finite type for his consistency proof, though apparently he had also intended to give a consistency proof based on bar induction of finite type.

The purpose of the following is to give a functional interpretation of bar induction of finite type by means of bar recursion of (the same) finite type and to show how this can be used to give an alternative derivation of Spector's result; i.e., a functional interpretation of the negative version of the axiom of choice. It is also shown that the *axiom* of bar induction of finite type can be derived from the *rule* of bar induction of an associated finite type (of higher type level); and the corresponding result is obtained for bar recursion of finite type. Finally, we give a consistency proof for analysis by means of the rule of bar induction of finite type (applied intuitionistically) plus the general axiom of choice.

1. Notation and definitions

Let T denote the free variable formal system of primitive recursive functionals of finite type [1]. For the purpose of the present paper it is not necessary to give a precise formulation of $\boldsymbol{T}$. It is understood that the terms of $\boldsymbol{T}$ are classified into *types*, and that, for terms s and t, the term st (interpreted: s applied to t) is well-formed when the proper conditions on the types of s and t are satisfied ([7], p. 5). The type symbols are generated as follows: 0 is a type symbol; if σ and τ are type symbols, so is $(\sigma)\tau$; the latter is the type of a functional which takes arguments of type σ and has values of type τ (this notation is due to K. Schütte). Every type symbol has a *level* defined as follows: the level of 0 is zero; the level of $(\sigma)\tau$ is the maximum of $1+\text{level }(\sigma)$ and level (τ). It is easily seen that every type symbol has the form $(\sigma_1)\cdots(\sigma_n)0$ and that the level of the latter is the maximum of $1+\text{level }(\sigma_1), \cdots, 1+\text{level }(\sigma_n)$.

Notation: if $t_1, \cdots, t_k$ are terms, $t_1 t_2 \cdots t_k$ denotes $(\cdots((t_1 t_2)t_3)\cdots)t_k$.

There are two possible formulations of $\boldsymbol{T}$: the *intensional* formulation of Gödel's paper [1] and the *extensional* formulation of Spector's paper [7]. In Gödel's formulation, equality is interpreted as a decidable intensional equality; equations between terms of finite type are allowed, and such equations are combined in the usual way by means of the propositional connectives; classical propositional logic is used (which we can regard as arising from intuitionistic propositional logic together with the axioms $E \vee \neg E$ for equations E). The equality axioms and rules for $\boldsymbol{T}$ are:

$$s = s \tag{1.1}$$

$$\{s = t, A\} \vdash A^*, \tag{1.2}$$

where A is any formula and A^* arises from A by the replacement of one occurrence of s by t. When $\boldsymbol{T}$ is extended by addition of the schema of bar recursion it is necessary to replace (1.2) by the more general rule (1.3), below.

In Spector's formulation of $\boldsymbol{T}$ the atomic formulae are equations between terms of type *zero*; and an equation $s = t$ between terms of higher type is regarded as an abbreviation for $sx_1 \cdots x_n = tx_1 \cdots x_n$ where $x_1, \cdots, x_n$ are variables, not contained in s or t, of types such that $sx_1 \cdots x_n$ and $tx_1 \cdots x_n$ are terms of type zero.

The treatment of this paper is valid for both formulations of $\boldsymbol{T}$.

Notation: in the following, the variable c ranges over sequences $\langle c_0, \cdots, c_{k-1}\rangle$ of some finite type σ; $lh(c)$ denotes the length k of c; $\langle\ \rangle$ denotes the empty sequence (which has length zero); $c * u$ denotes $\langle c_0, \cdots, c_{k-1}, u\rangle$; for a functional α with numerical argument, $\bar{\alpha}k$ denotes $\langle \alpha 0, \alpha 1, \cdots, \alpha(k-1)\rangle$; $[c]$ denotes a function α associated with c in some systematic way (by primitive recursion) such that $\bar{\alpha}(lh(c)) = c$.

The functional φ of bar recursion of type σ (where σ is the type of the components $c_0, \cdots, c_{k-1}$ of the sequence c) is introduced by the following schema:

$$BR_\sigma \begin{cases} Y[c] < lh(c) \rightarrow \varphi YGHc = Gc \\ Y[c] \geqq lh(c) \rightarrow \varphi YGHc = H(\lambda u \cdot \varphi YGH(c * u))c. \end{cases}$$

The type of $\varphi YGHc$ is arbitrary. By $\boldsymbol{T}+BR_\sigma$ is meant the formal system obtained from $\boldsymbol{T}$ by adjoining constants φ of suitable types together with the schemata BR_σ, the rule (1.2) of equality being replaced by

$$\text{(1.3)} \qquad \{P \rightarrow s = t, A\} \vdash P \rightarrow A^*$$

where P is a propositional combination of equations between terms of type zero. (Actually we need (1.3) only for the case in which P has the form $Y[c] < lh(c)$ or $Y[c] \geqq lh(c)$.)

$\boldsymbol{H}_\omega$ denotes Heyting arithmetic of finite type; namely, the formal system obtained by adding quantifiers to $\boldsymbol{T}$ together with the usual rules of formula formation, the axioms and rules of the intuitionistic predicate calculus, and, of course, mathematical induction. By $\boldsymbol{H}_\omega+BR_\sigma$ is meant the system obtained by extending $\boldsymbol{T}+BR_\sigma$ in the same way.

The schema BI_σ of bar induction of type σ, applied to the formulae $P(c)$ and $Q(c)$ of $\boldsymbol{H}_\omega$ is:

$$BI_\sigma \qquad [(\text{Hyp } 1) \wedge \cdots \wedge (\text{Hyp } 4)] \rightarrow Q(\langle\ \rangle),$$

where (Hyp 1) denotes $\bigwedge \alpha \bigvee n P(\bar{\alpha}n)$, (Hyp 2) denotes

$$\bigwedge c (\bigwedge m \leqq lh(c)) [P(\langle c_0, \cdots, c_{m-1}\rangle) \rightarrow P(c)],$$

(Hyp 3) denotes $\bigwedge c[P(c) \rightarrow Q(c)]$, and (Hyp 4) denotes

$$\bigwedge c[\bigwedge u Q(c * u) \rightarrow Q(c)].$$

2. Gödel's functional interpretation

For formulae P of $\boldsymbol{H}_\omega$, let P' denote the formula $\bigvee y \bigwedge z A(y, z)$ which Gödel [1] associates with P, where $A(y, z)$ is quantifier free. (Gödel considers only P in Heyting arithmetic of lowest type but his procedure obviously extends to formulae of $\boldsymbol{H}_\omega$.) In Gödel's paper y and z stand for *finite sequences* of functionals.

Although we could work with finite sequences in the present paper it is perhaps more convenient to consider finite sequences of functionals to be coded as single functionals. After such coding, and the corresponding change in the formula $A(y, z)$, we can consider y and z to denote single functionals. If the reader prefers to work with finite sequences in the following, he can easily supply the necessary changes of phraseology and interpretation.

Let $\bigvee y \bigwedge z A(y, z)$ be Gödel's translation of P, as just described. By a *functional interpretation* of P in the free variable system $\boldsymbol{T}$ is meant a term t together with a proof in $\boldsymbol{T}$ of $A(t, z)$. Gödel shows that every theorem of ordinary Heyting arithmetic has a functional interpretation in $\boldsymbol{T}$, and his procedure obviously extends to $\boldsymbol{H}_\omega$. His procedure consists in giving a functional interpretation of the axioms of $\boldsymbol{H}_\omega$ and showing how a functional interpretation is transformed by the rules of inference of $\boldsymbol{H}_\omega$.

Let $\boldsymbol{H}_\omega^*$ denote the formal system obtained from $\boldsymbol{H}_\omega$ by adjoining, as axioms, all formulae of the form $P \leftrightarrow P'$. It is easy to give a functional interpretation of $P \leftrightarrow P'$: merely observe that $(P \leftrightarrow P')'$ is identical $(P \leftrightarrow P)'$, and that $P \leftrightarrow P$ has a functional interpretation because it is a theorem of $\boldsymbol{H}_\omega$. Thus *every theorem of $\boldsymbol{H}_\omega^*$ has a functional interpretation in $\boldsymbol{T}$.* Moreover, since Gödel's translation of a quantifier free formula is the formula itself, *every theorem of $\boldsymbol{H}_\omega^* + BR_\sigma$ has a functional interpretation in $\boldsymbol{T} + BR_\sigma$.*

3. Functional interpretation of bar induction

The existence of a functional interpretation of each instance of BI_σ in $\boldsymbol{T} + BR_\sigma$ follows from:

THEOREM 3A. Every theorem of $\boldsymbol{H}_\omega^* + BI_\sigma$ has a functional interpretation in $\boldsymbol{T} + BR_\sigma$.

As was shown in § 2, every theorem of $\boldsymbol{H}_\omega^* + BR_\sigma$ has a functional interpretation in $\boldsymbol{T} + BR_\sigma$. Hence to prove Theorem 3A it is sufficient to prove:

THEOREM 3B. Every instance of BI_σ is a theorem of $\boldsymbol{H}_\omega^* + BR_\sigma$.

The purpose of the present section is to prove Theorem 3B. This will be accomplished by means of Lemmas 3A-3C, below.

We shall make use of the following property of $\boldsymbol{H}_\omega^*$: for all formulae $E(y)$ and $F(z)$ of $\boldsymbol{H}_\omega$,

$$(3.1) \quad [\bigvee y E(y) \rightarrow \bigvee z F(z)] \rightarrow \bigvee Z \bigwedge y [E(y) \rightarrow F(Zy)]$$

is a theorem of $\boldsymbol{H}^*_\omega$. Actually, in the following, we only need (3.1) for the case in which $E(y)$ and $F(z)$ are purely universal formulae; so let us consider this case. Temporarily taking the viewpoint in which y and z denote finite sequences, we observe that $[\bigvee yE(y) \to \bigvee zF(z)]'$ is identical to $(\bigvee Z \bigwedge y[E(y) \to F(Zy)])'$, so (3.1) follows from axioms of the form $P \leftrightarrow P'$.

Considering now the schema BI_σ of bar induction (§ 1), our task is to prove $Q(\langle\ \rangle)$ from (Hyp 1), $\cdots$, (Hyp 4) in $\boldsymbol{H}^*_\omega + BR_\sigma$. We shall reason informally in $\boldsymbol{H}^*_\omega + BR_\sigma$. Denote $P(c)'$ and $Q(c)'$ by $\bigvee rA(r, c)$ and $\bigvee yB(y, c)$, where $A(r, c)$ and $B(y, c)$ are purely universal formulae. Note: the formulae $P(c)$ and $Q(c)$, and hence $\bigvee rA(r, c)$ and $\bigvee yB(y, c)$, may contain free variables (other than c). By virtue of the axioms $D \leftrightarrow D'$ of $\boldsymbol{H}^*_\omega$ we may replace $P(c)$ and $Q(c)$ by $P(c)'$ and $Q(c)'$ in (Hyp 2), $\cdots$, (Hyp 4); we may replace $\bigwedge uQ(c * u)$ by $\bigvee Y \bigwedge uB(Yu, c * u)$ in (Hyp 4), and we may replace (Hyp 1) by $\bigvee N \bigvee R \bigwedge \alpha A(R\alpha, \bar{\alpha}(N\alpha))$. Hence from (Hyp 1), $\cdots$, (Hyp 4) we conclude, with the help of (3.1), that there exist N, R, L, S and X such that:

(3.2) $\bigwedge \alpha A(R\alpha, \bar{\alpha}(N\alpha))$

(3.3) $\bigwedge c(\bigvee m \leqq lh(c)) \bigwedge r[A(r, \langle c_0, \cdots, c_{m-1}\rangle) \to A(Lcmr, c)]$

(3.4) $\bigwedge c \bigwedge r[A(r, c) \to B(Scr, c)]$

(3.5) $\bigwedge c \bigwedge Y[\bigwedge uB(Yu, c * u) \to B(XcY, c)]$.

From (3.2)—(3.5) we shall prove the existence of w such that $B(w, \langle\ \rangle)$; then $Q(\langle\ \rangle)$ follows in $\boldsymbol{H}^*_\omega$. Let W be defined in terms of N, R, L, S and X by the equations

(3.6) $\quad N[c] < lh(c) \to Wc = Sc(Lc(N[c])(R[c]))$

(3.7) $\quad N(c) \geqq lh(c) \to Wc = Xc(\lambda u \cdot W(c * u))$,

which can easily be reduced to the schema BR_σ of bar recursion (§ 1). It will be shown that $W\langle\ \rangle$ is the desired functional w.

Lemma 3A. $N[c] < lh(c) \to B(Wc, c)$ for all c.

Proof. Substituting $[c]$ for α in (3.2), we conclude

$$A(R[c], \bar{g}(N[c])),$$

where g denotes $[c]$. Assume $N[c] < lh(c)$. Then

$$\bar{g}(N[c]) = \langle c_0, \cdots, c_{m-1}\rangle,$$

where $m = N[c]$. Hence $A(Lc(N[c])(R[c], c)$ by (3.3). Hence $B(Wc, c)$ by (3.4) and the clause (3.6) in the definition of W.

Lemma 3B. If $N[c] \geqq lh(c)$, then

$$\bigwedge uB(W(c * u), c * u) \rightarrow B(Wc, c),$$

for all c.

Proof. Take Y in (3.5) to be $\lambda u \cdot W(c * u)$ and apply the definition (3.7) of W for the case $N[c] \geqq lh(c)$.

From Lemmas 3A and 3B we conclude

$$\text{(3.8)} \qquad \bigwedge c[\bigwedge uB(W(c * u), c * u) \rightarrow B(Wc, c)].$$

Recall that $B(Wc, c)$ is a purely universal formula. Thus $B(Wc, c)$ is of the form $\bigwedge xD(c, x)$, where $D(c, x)$ is quantifier free; and (3.8) becomes

$$\text{(3.9)} \qquad \bigwedge c[\bigwedge u \bigwedge xD(c * u, x) \rightarrow \bigwedge xD(c, x)].$$

Also, by Lemma 1,

$$\text{(3.10)} \qquad N[c] < lh(c) \rightarrow \bigwedge xD(c, x), \text{ for all } c.$$

Our problem is to prove $\bigwedge xD(\langle\ \rangle, x)$. By (3.9) and (3.10) our problem has been reduced, essentially, to finding a functional interpretation of a bar induction in which $Q(c)$ is a purely universal formula $\bigwedge xD(c, x)$: the remainder of the present section provides a solution of that problem. Let C denote (3.9). From (3.9) and the axiom $C \leftrightarrow C'$ we conclude that there exist U and Z such that

$$\text{(3.11)} \qquad D(c * (Ucx), Zcx) \rightarrow D(c, x),$$

for all c, x. Taking c and x to be free variables, formula (3.11) can be regarded as the inductive clause of a free variable bar induction. We shall reduce this to bar *recursion* by using Kreisel's trick [5] for reducing free variable transfinite induction to ordinary induction plus transfinite recursion.

First we shall define a pair of functionals G and H by primitive recursion. Both functionals will take arguments k and x, where k is a number (upon which the primitive recursion is performed) and x has the type required in (3.11). By convention $\langle G, H\rangle$ will denote a functional such that $\langle G, H\rangle kx = \langle Gkx, Hkx\rangle$ for all k, x. The types of Gkx and Hkx will be those of c and x, respectively. Let Fcx denote $c * (Ucx)$ and define the pair $\langle G, H\rangle$ by the primitive recursion equations

$$\langle G, H\rangle Ox = \langle\langle\ \rangle, x\rangle$$
$$\langle G, H\rangle(k+1)x = \langle F(Gkx)(Hkx),\ Z(Gkx)(Hkx)\rangle.$$

Using (3.11), we obtain, by ordinary induction on k (which is allowed in $\boldsymbol{H}^*_\omega$),

$$D(Gkx, Hkx) \to D(\langle\ \rangle, x), \tag{3.12}$$

for all x, k.

Given x, we wish to show $D(\langle\ \rangle, x)$. We shall do this by showing the existence of k such that $N[Gkx] < lh(Gkx)$. The required result $D(\langle\ \rangle, x)$ then follows from (3.10) and (3.12) with $c = Gkx$.

Denote $U(Gkx)x$ by gk (where x is not indicated as an argument of g because x remains constant in the following discussion). From the defining equations for G and H it is easy to prove, by induction on k, that $\bar{g}k = Gkx$ holds for all k. We must show the existence of k such that $N[\bar{g}k] < k$. This is done by the following lemma (Kreisel's trick [5]).

Lemma 3C. By bar recursion of type σ plus primitive recursion we can define θ (as a function of N) such that

$$(\bigvee k \leqq \theta\alpha\langle\ \rangle)(N[\bar{\alpha}k] < k)$$

for all α.

Proof. Define $\theta\alpha c$ to be 0 if $(\bigvee k \leqq lh(c))(N[\langle c_0, \cdots, c_{k-1}\rangle] < k)$ and $\theta\alpha c$ equal to $1 + \theta\alpha(c * v)$ otherwise, where v denotes $\alpha(lh(c))$. Clearly θ is obtainable from bar recursion of type σ plus primitive recursion: because, if $N[c] < lh(c)$ then $\theta\alpha c$ is defined outright (as 0) whereas if $N[c] \geqq lh(c)$ then $\theta\alpha c$ can easily be expressed as a certain primitive recursive functional of $\lambda u \cdot \theta\alpha(c * u)$. It remains to show $(\bigvee k \leqq \theta\alpha\langle\ \rangle)(N[\bar{\alpha}k] < k)$.

Denote $\theta\alpha(\bar{\alpha}i)$ by βi. Putting $\bar{\alpha}i$ for c in the definition of θ, and observing that $(\bar{\alpha}i) * (\alpha i)$ is equal to $\bar{\alpha}(i+1)$, we get

$$\beta i = 0 \text{ if } (\bigvee k \leqq i)(N[\bar{\alpha}k] < k) \tag{3.13}$$

$$\beta i = 1 + \beta(i+1) \text{ otherwise.} \tag{3.14}$$

By (3.13) and (3.14),

$$(\beta i \neq 0 \wedge j \leqq i) \to \beta j = 1 + \beta(j+1). \tag{3.15}$$

Using (3.15) we easily prove, by induction on j, that if $\beta i \neq 0$ and $j \leqq i$ then $\beta 0 = j + \beta j$. Putting $i = j$ we conclude

$$\beta i \neq 0 \to \beta 0 = i + \beta i. \tag{3.16}$$

Putting $\beta 0$ for i in (3.16) we conclude $\beta(\beta 0) = 0$. Hence, by (3.13) and (3.14), $(\bigvee k \leqq \beta 0)(N[\bar{\alpha}k] < k)$. Since $\beta 0$ is $\theta\alpha\langle\ \rangle$, Lemma 3C is proved.

As explained in the discussion preceding Lemma 3C, $\bigwedge xD(\langle\,\rangle, x)$ follows from Lemma 3C. Thus Theorem 3B has been proved.

I wish to acknowledge Professor Kreisel's help with the present section, in particular his suggestion to use Lemma 3C.

4. Proof of Spector's result

Let $\boldsymbol{Z}_\omega$ be the system $\boldsymbol{H}_\omega$ provided with classical logic. The axiom of choice treated by Spector [7] is:

$$AC_{0\tau} \qquad \bigwedge n \bigvee YS(n, Y) \rightarrow \bigvee F \bigwedge nS(n, Fn),$$

where n is a number variable, Y has arbitrary finite type τ, and $S(n, Y)$ is an arbitrary formula of $\boldsymbol{Z}_\omega$. For any formula P of $\boldsymbol{Z}_\omega$, let P^- denote the "negative version" of P obtained by prefixing all disjunctions and existential quantifiers by double negations. Spector [7] capitalizes on the well known result that the mapping of each formula P into its negative version P^- sends the axioms and rules of inference of $\boldsymbol{Z}_\omega$ into theorems and derived rules of inference of $\boldsymbol{H}_\omega$. Thus Spector reduces the consistency problem for $\boldsymbol{Z}_\omega + AC_{0\tau}$ to the consistency problem for $\boldsymbol{H}_\omega + AC_{0\tau}^-$. He then constructs a functional interpretation of $AC_{0\tau}^-$ in $\boldsymbol{T} + BR_\sigma$ (for suitably chosen σ), thereby obtaining his reduction of the consistency problem for $\boldsymbol{Z}_\omega + AC_{0\tau}$ to the consistency problem for $\boldsymbol{T} + BR_\sigma$.

The purpose of the present section is to show the existence of a functional interpretation of $AC_{0\tau}^-$ in $\boldsymbol{T} + BR_\sigma$ by using Theorem 3A of § 3 and a result of [2]. Indeed, we shall obtain a functional interpretation of the negative version of

$$(4.1) \quad \bigwedge n \bigwedge X \bigvee YA(n, X, Y) \rightarrow \bigvee F \bigwedge nA(n, Fn, F(n+1)),$$

where $A(n, X, Y)$ is an arbitrary formula of $\boldsymbol{Z}_\omega$. As shown in [2], pp. 351—352, the axiom (4.1) implies both $AC_{0\tau}$ and an (apparently) stronger axiom DC_τ of "dependent choices". Thus our purpose is to prove

THEOREM 4A. The negative version of (4.1) has a functional interpretation in $\boldsymbol{T} + BR_\sigma$ for suitable σ.

Theorem 4A follows immediately from Theorem 3A and the following theorem.

THEOREM 4B. The negative version of (4.1) is provable in $\boldsymbol{H}_\omega^* + BI_\sigma$ for suitable σ.

The proof of Theorem 4B will be accomplished by means of four lemmas.

Let $Z_\omega^{\#}$ be the formal system obtained from Z_ω by adjoining, as axioms, all formulae of the form $D \leftrightarrow (D^-)'$.

Lemma 4A. For every formula P, if P is a theorem of $Z_\omega^{\#}$ then P^- is a theorem of H_ω^*.

Proof. Since the mapping of each formula into its negative version sends the axioms and rules of inference of Z_ω into theorems and derived rules of inference of H_ω, it is sufficient to show that the negative version of $D \leftrightarrow (D^-)'$ is a theorem of H_ω^*. Thus we must prove $D^- \leftrightarrow ((D^-)')^-$ in H_ω^*. By taking D^- to be P in the axiom $P \leftrightarrow P'$ of H_ω^*, we obtain $D^- \leftrightarrow (D^-)'$. As is well known, $D^- \leftrightarrow \neg\neg D^-$ is a theorem of H_ω. Hence $D^- \leftrightarrow \neg\neg (D^-)'$. But $(D^-)'$ is of the form $\bigvee y \bigwedge z B(y, z)$, where $B(y, z)$ is quantifier free. Hence $\neg\neg (D^-)'$ is $((D^-)')^-$. Thus $D \leftrightarrow ((D^-)')^-$, which was to be proved.

Discussion. Kreisel has shown in § 5.1 of [4] that by means of the "quantifier free" axiom of choice

$$(QF\text{—}AC_{\sigma\tau}) \qquad \bigwedge X \bigvee Y A(X, Y) \rightarrow \bigvee F \bigwedge X A(X, FX),$$

where $A(X, Y)$ is quantifier free, all statements of the form $D^- \leftrightarrow (D^-)'$ can be derived in Z_ω. On the other hand, if D is taken to be $\bigwedge X \bigvee Y A(X, Y)$ with $A(X, Y)$ quantifier free, then $(D^-)'$ is just $\bigvee F \bigwedge X A(X, FX)$; so $(QF\text{—}AC_{\sigma\tau})$ is a theorem of $Z_\omega^{\#}$. Thus $Z_\omega^{\#}$ is identical to $Z_\omega + (QF\text{—}AC_{\sigma\tau})$. Of course Lemma 4A could have been proved by an appeal to this result plus the observation that $(QF\text{—}AC_{\sigma\tau})^-$ is a theorem of H_ω^*.

The fact that in $Z_\omega^{\#}$ every formula D is equivalent to a formula of the form $\bigvee y \bigwedge z B(y, z)$, with quantifier free $B(y, z)$, allows us to prove the following lemma.

Lemma 4B. Each instance of (4.1) can be derived in $Z_\omega^{\#}$ from another instance of (4.1) with purely universal $A(n, X, Y)$.

Proof. For arbitrary $A(n, X, Y)$, let $(A(n, X, Y)^-)'$ be $\bigvee W \bigwedge Z B(n, X, Y, W, Z)$. Then

$$(4.2) \qquad A(n, X, Y) \leftrightarrow \bigvee W \bigwedge Z B(n, X, Y, W, Z)$$

by the axiom schema $D \leftrightarrow (D^-)'$ of $Z_\omega^{\#}$. Assume

$$\bigwedge n \bigwedge X \bigvee Y A(n, X, Y).$$

It is required to prove

$$\bigvee F \bigwedge nA(n, Fn, F(n+1))$$

by an appeal to (4.1) in which $A(n, X, Y)$ has been replaced by some purely universal formula. The reasoning from now on is intuitionistic, so is certainly valid in $\boldsymbol{Z}_\omega^\#$. By (4.2) our assumption is $\bigwedge n \bigwedge X \bigvee Y \bigvee W \bigwedge ZB(n, X, Y, W, Z)$. This can be written as $\bigwedge n \bigwedge X \bigwedge U \bigvee Y \bigvee WA_1(n, X, U, Y, W)$, where U is a variable of the same type as W, and $A_1(n, X, U, Y, W)$ denotes $\bigwedge ZB(n, X, Y, W)$. By coding the pairs $\langle X, U\rangle$ and $\langle Y, W\rangle$ and an appeal to (4.1) applied to a purely universal formula, we conclude that there exist F and G such that

$$\bigwedge nA_1(n, Fn, Gn, F(n+1), G(n+1)).$$

But the latter formula implies

$$\bigwedge n \bigvee W \bigwedge ZB(N, Fn, F(n+1), W, Z),$$

which is equivalent to $\bigwedge nA(n, Fn, F(n+1))$. Thus Lemma 4B is proved.

Notation. $(EX-BI_\sigma)$ denotes the schema BI_σ of § 1 applied to purely existential formula $P(c)$.

From the statement of Theorem (ii), page 352 of [2], we conclude that (4.1) is derivable from BI_σ in $\boldsymbol{Z}_\omega$; but from a glance at the proof of Theorem (ii) we see that to derive (4.1) applied to a purely universal formula $A(n, X, Y)$, the special form $(EX-BI_\sigma)$ is used [1]. From this and Lemma 4B we conclude:

Lemma 4C. Each instance of (4.1) can be derived in $\boldsymbol{Z}_\omega^\#$ from some instance of $(EX-BI_\sigma)$ with appropriate σ.

Finally, we prove:

Lemma 4D. Each instance of $(EX-BI_\sigma)^-$ is a theorem of $\boldsymbol{H}_\omega^*+BI_\sigma$.

Proof. Denote $P(c)$ in $(EX-BI_\sigma)$ by $\bigvee ZB(Z, c)$, where $B(Z, c)$ is quantifier free. Clearly $(EX-BI_\sigma)^-$ applied to this $P(c)$ and arbitrary $Q(c)$ is just BI_σ applied to $P(c)^-$ and $Q(c)^-$ except that (Hyp 1) is replaced by

$$\bigwedge \alpha \neg\neg \bigvee nP(\bar{\alpha}n)^-;\ \text{i.e.,} \bigwedge \alpha \neg\neg \bigvee n \neg\neg \bigvee ZB(Z, \bar{\alpha}n).$$

Gödel's translation of the latter formula is the same as Gödel's translation of $\bigwedge \alpha \bigvee n \neg\neg \bigvee ZB(Z, \bar{\alpha}n)$ because $B(Z, \bar{\alpha}n)$ is

[1] A new, improved proof of Theorem (ii) is given in the Appendix to the present paper.

quantifier free. Thus $\bigwedge \alpha \neg\neg \bigvee nP(\bar{\alpha}n)^-$ is equivalent in $\boldsymbol{H}_\omega^*$ to $\bigwedge \alpha \bigvee nP(\bar{\alpha}n)^-$, so (Hyp 1) has been restored.

PROOF OF THEOREM 4B. Taking P in Lemma 4A to be of the form $D \to E$, we conclude that if E is derivable from D in $\boldsymbol{Z}_\omega^\#$ then E^- is derivable from D^- in $\boldsymbol{H}_\omega^*$. Hence, by Lemma 4C, each instance of $(4.1)^-$ can be derived in $\boldsymbol{H}_\omega^*$ from some instance of $(EX-BI_\sigma)^-$ with appropriate σ. Hence, by Lemma 4D, $(4.1)^-$ is a theorem of $\boldsymbol{H}_\omega^*+BI_\sigma$. Thus Theorem 4B is proved.

COMMENT 4.1. Since it is the negative version of $(EX-BI_\sigma)$, and not the negative version of BI_σ in general, which is shown to be equivalent in $\boldsymbol{H}_\omega^*$ to (another instance) of BI_σ in Lemma 4D, a crucial step in the proof of Theorem 4A consists in the reduction of (4.1) to (4.1) applied to purely universal $A(n, X, Y)$ (Lemma 4B). This reduction is accomplished at the price of raising the type level of X and Y in (4.1). It is for this reason that the consistency proof for $\boldsymbol{Z}_\omega+(4.1)$ even for X and Y of type 0 (i.e., numerical X and Y) requires BR_σ for higher types σ.

COMMENT 4.2. By Theorem (vii), page 353 of [2], (4.1) is equivalent to bar induction of finite type in $\boldsymbol{Z}_\omega$. Thus Theorem 4A provides a reduction of the consistency of $\boldsymbol{Z}_\omega+BI_\tau$ to the consistency of $\boldsymbol{T}+BR_\sigma$ (where σ may differ from τ for the reason explained in Comment 4.1).

5. The rules of bar induction and bar recursion

The purpose of the present section is to introduce the *rule* of bar induction (Rule-BI_σ) and the corresponding recursion schema (Rule-BR_σ), and to show that they yield the axiom BI_τ and the schema BR_τ, respectively, where σ depends on τ. The discussion uses intuitionistic propositional and quantifier logic.

By (Rule-BI_σ) is meant the rule of inference (in some formal system, say $\boldsymbol{H}_\omega$) which says that if (Hyp 1), $\cdots$, (Hyp 4) have been proved, then infer $Q(\langle\ \rangle)$ — the notation being as in § 1. Since this rule is perhaps most natural in the case in which $P(c)$ and $Q(c)$ contain no free variables other than c, it will be shown below that (Rule-BI_σ) can be reduced to this case. Indeed, we shall show that the axiom BI_σ follows from the rule (Rule-BI_σ), of the same type σ, plus the axiom of choice, when (Rule-BI_σ) is applied to $P(c)$ and $Q(c)$ containing a free variable Y.

REMARK 5.1. By applying (Rule-BI_σ) to new formulae $P_1(c)$

and $Q_1(c)$, we can replace the conclusion $Q(\langle\ \rangle)$ by the stronger conclusion $\bigwedge dQ(d)$. Namely, for

$$c = \langle c_0, \cdots, c_{m-1}\rangle \text{ and } d = \langle d_0, \cdots, d_{k-1}\rangle,$$

let $c \square d$ denote $\langle c_0, \cdots, c_{m-1}, d_0, \cdots, d_{k-1}\rangle$, and take $P_1(c)$ and $Q_1(c)$ to be $\bigwedge dP(c \square d)$ and $\bigwedge dQ(c \square d)$, respectively.

At first sight (Rule-BI_σ) appears weaker than the *strong* rule of bar induction which says that if (Hyp 1) has been proved, infer (Hyp 2) $\wedge \cdots \wedge$ (Hyp 4) $\rightarrow Q(\langle\ \rangle)$. However, the strong rule is easily derived from (Rule-BI_σ) by applying (Rule-BI_σ) to the following $P_1(c)$ and $Q_1(c)$: take $P_1(c)$ to be (Hyp 2) $\rightarrow P(c)$ and take $Q_1(c)$ to be (Hyp 2) $\wedge \cdots \wedge$ (Hyp 4) $\rightarrow Q(c)$.

The recursion schema corresponding to (Rule-BI_σ) is as follows. For any given closed terms Y of type $((0)\sigma)0$ and G, H of proper types, introduce a constant θ and the schema

$$\text{(Rule-}BR_\sigma\text{)} \quad \begin{aligned} &Y[c] < lh(c) \rightarrow \theta c = Gc \\ &Y[c] \geqq lh(c) \rightarrow \theta c = H(\lambda u \cdot \theta(c * u))c. \end{aligned}$$

The schema (Rule-BR_σ) is stated for the particular closed terms y, G and H with which θ is associated; whereas the schema BR_σ of § 1, for given φ, applies to *all* terms Y, G and H of the proper types.

The recursion schema corresponding to the strong rule of bar induction is as follows. For each closed term Y of type $((0)\sigma)0$, introduce a constant ξ_Y and the schema

$$(5.1) \quad \begin{aligned} &Y[c] < lh(c) \rightarrow \xi_Y GHc = Gc \\ &Y[c] \geqq lh(c) \rightarrow \xi_Y GHc = H(\lambda u \cdot \xi_Y GH(c * u))c, \end{aligned}$$

understood to apply to all terms G and H of the proper types.

For a given closed term Y, in order to obtain a term ξ_Y by use of (Rule-BR_σ) and λ-abstraction, such that ξ_Y satisfies (5.1) for all G and H, proceed as follows. Let G_1 and H_1 be $\lambda c \cdot \lambda G \cdot \lambda H \cdot Gc$ and $\lambda X \cdot \lambda c \cdot \lambda G \cdot \lambda H \cdot H(\lambda u \cdot XuGH)c$, respectively, and let θ_1 be the constant associated with Y, G_1 and H_1 by (Rule-BI_σ). Define ξ_Y to be $\lambda G \cdot \lambda H \cdot \lambda c \cdot \theta_1 cGH$. From the schema (Rule-$BR_\sigma$) applied to Y, G_1, H_1 and associated θ_1, it is easy to verify by λ-conversions that $\theta_1 cGH$ equals Gc if $Y[c] < lh(c)$ and equals $H(\lambda u \cdot \theta_1(c * u)GH)c$ otherwise. Hence (5.1), since $\xi_Y GHc = \theta_1 cGH$ for all c, G and H.

Coding. If t and u are functionals of types $\tau = (\tau_1) \cdots (\tau_n)0$

and $\sigma = (\sigma_1) \cdots (\sigma_k)0$, respectively, then t and u can both be coded as functionals of type $\nu = (\tau_1) \cdots (\tau_n)(\sigma_1) \cdots (\sigma_k)0$. Namely, let At and Bu denote $\lambda X_1 \cdot \cdots \cdot \lambda X_{n+k} \cdot tX_1 \cdots X_n$ and $\lambda X_1 \cdot \cdots \cdot \lambda X_n \cdot u$ respectively, where $X_1, \cdots, X_n$ are variables of type $\tau_1, \cdots, \tau_n$, and $X_{n+1}, \cdots, X_{n+k}$ are variables of type $\sigma_1, \cdots, \sigma_k$. It is easy to define functionals $A^{\#}$ and $B^{\#}$ such that $A^{\#}(At) = t$ and $B^{\#}(Bu) = u$ for all t and u of types τ and σ, respectively. Thus t and u have been coded as At and Bu, respectively. We can now code the infinitely proceeding sequence $\langle t, u_0, \cdots, u_m, \cdots \rangle$, t of type τ and u_m of type σ for all m, as a functional of type $(0)\nu$. Namely, the sequence just mentioned is coded as $\langle At, Bu_0, \cdots, Bu_m, \cdots \rangle$. Correspondingly, finite initial segments $\langle t, u_0, \cdots, u_{m-1} \rangle$ are coded as $\langle At, Bu_0, \cdots, Bu_{m-1} \rangle$. In the following, for clarity of notation, we shall sometimes omit the functionals A and B in the coded sequences.

Elimination of free variables. To eliminate a free variable t (other than c) from $P(c)$ and $Q(c)$ in (Rule-BI_σ), at the expense of a possible change in σ, proceed as follows. We indicate the presence of t by the notation $P(c, t)$ and $Q(c, t)$. Code infinitely proceeding sequences $\langle t, c_0, \cdots, c_m, \cdots \rangle$ as functionals of type $(0)\nu$ as described in the preceding paragraph, with finite initial segments coded correspondingly. Define P_1 and Q_1 as follows. $\neg P_1(\langle\ \rangle)$ and $Q_1(\langle\ \rangle)$ are defined to be true. $P_1(\langle t, c_0, \cdots, c_{m+1} \rangle)$ and $Q_1(\langle t, c_0, \cdots, c_{m-1} \rangle)$ are defined to be $P(\langle c_0, \cdots, c_{m-1} \rangle, t)$ and $Q(\langle c_0, \cdots, c_{m-1} \rangle, t)$ respectively. If (Hyp 1), $\cdots$, (Hyp 4) for P and Q have been proved as formulae with the free variable t, then (Hyp 1), $\cdots$, (Hyp 4) preceded by the quantifier $\bigwedge t$ can also be proved. But the latter formulae imply (Hyp 1), $\cdots$, (Hyp 4) for P_1 and Q_1. Hence, by (Rule-BI_ν) applied to P_1 and Q_1 we conclude $\bigwedge t Q_1(\langle t \rangle)$ by Remark 5.1, above. Thus $\bigwedge t Q(\langle\ \rangle, t)$.

Proof of BI_σ from (Rule-BI_ν), where $\nu = ((0)\sigma)\sigma$, by use of the axiom of choice. By the axiom of choice, $\bigwedge \alpha \bigvee n P(\bar{\alpha} n)$ is equivalent to $\bigvee Y \bigwedge \alpha P(\bar{\alpha}(Y\alpha))$, so BI_σ is equivalent to

$$\bigwedge Y[\bigwedge \alpha P(\bar{\alpha}(Y\alpha)) \wedge (\text{Hyp } 2) \wedge \cdots \wedge (\text{Hyp } 4) \cdot \rightarrow Q(\langle\ \rangle)].$$

Define $P_1(c, Y)$ and $Q_1(c, Y)$ to be

$$\bigwedge \alpha P(\bar{\alpha}(Y\alpha)) \wedge (\text{Hyp } 2) \cdot \rightarrow P(c)$$

and

$$\bigwedge \alpha P(\bar{\alpha}(Y\alpha)) \wedge (\text{Hyp } 2) \wedge \cdots \wedge (\text{Hyp } 4) \cdot \rightarrow Q(c),$$

respectively. It is easy to prove (Hyp 1), $\cdots$, (Hyp 4) for P_1

and Q_1 in $\boldsymbol{H}_\omega$, treating Y as a free variable. Hence $\bigwedge YQ_1(\langle\ \rangle, Y)$ by (Rule-BI_σ) applied to P_1 and Q_1 with free variable Y. This is the desired conclusion BI_σ. Finally, the free variable Y can be eliminated from (Rule-BI_σ) by use of (Rule-BI_ν) as in the preceding paragraph.

Proof of BR_σ from (Rule-BR_ν), where $\nu = ((0)\sigma)\sigma$. In the following discussion the strong version (5.1) of (Rule-BR_ν) is needed. We shall construct primitive recursive terms Z, D, E and F such that if φ denotes $\lambda Y \cdot \lambda G \cdot \lambda H \cdot \lambda c \cdot \xi_Z(EG)(FH)(DYc)$ then φ can be proved in T to satisfy the schema BR_σ, assuming the schema (5.1) of the strong version of (Rule-BR_ν) for ξ_Z.

Let 0^σ denote $\lambda X_1 \cdot \cdots \cdot \lambda X_k \cdot 0$, where $X_1, \cdots, X_k$ are variables of types $\sigma_1, \cdots, \sigma_k$ respectively, where $\sigma = (\sigma_1)\cdots(\sigma_k)0$. In the rest of this paper, $[c]$ denotes any primitive recursive functional of c such that $[c]i = c_i$ for all $i < lh(c)$. In the present discussion we make $[c]$ specific by requiring $[c]i = 0^\sigma$ for all $i \geqq lh(c)$. Let A, $A^\#$, B and $B^\#$ be the functionals discussed in the paragraph on coding, above. It is easy to define primitive recursive functionals such that $DYc = \langle AY, Bc_0, \cdots, Bc_{m-1}\rangle$ and $D^\#(DYc) = c$, for all Y and $c = \langle c_0, \cdots, c_{m-1}\rangle$. Define Z to $\lambda X \cdot \{1 + A^\#(X0)(\lambda n \cdot B^\#(X(n+1)))\}$, where X is a variable of type $(0)\nu$. Using the fact that $B^\# 0^\nu = 0^\sigma$ (because $B0^\sigma = 0^\nu$), it is easy to verify $Z[DYc] = 1 + Y[c]$ in $\boldsymbol{T}$. Hence

$$Z[DYc] \geqq lh(DYc) \leftrightarrow Y[c] \geqq lh(c).$$

Thus *the mapping of each Y and c into DYc maps the tree of Y (i.e., the set of c such that $Y[c] \geqq lh(c)$) isomorphically into the tree of Z.* Bar recursion of type σ (resp. ν) is, of course, just a kind of transfinite recursion over the tree of Y (resp. Z).

Define E to be $\lambda G \cdot \lambda W \cdot G(D^\# W)$, where W is a variable of type ν. Define F to be $\lambda H \cdot \lambda S \cdot \lambda W \cdot H(\lambda u \cdot S(Bu))(D^\# W)$, where u, S and W are variables of types σ, ν and $(\sigma)\rho$, respectively, where ρ is the type of Gc. If φ is defined as above, it is easy to prove in $\boldsymbol{T}$ (essentially by λ-conversions) that φ satisfies the schema BR_σ, assuming the schema (5.1) for ξ_Z.

6. Functional interpretation by use of (Rule-BR_σ)

From a glance at § 3 we see that the proof of Theorem 3A remains valid when BI_σ and BR_σ are replaced by (Rule-BI_σ) and (Rule-BR_σ), respectively. (The strong version (5.1) of

(Rule-BR_σ) is needed because the variable α of Lemma 3C is contained in the term H of BR_σ.) Thus we obtain:

THEOREM 6A. Every theorem of $\boldsymbol{H}_\omega^*$+(Rule-BI_σ) has a functional interpretation in $\boldsymbol{T}$+(Rule-BR_σ).

Corresponding to Theorem 4A we have:

THEOREM 6B. The negative version of (4.1) has a functional interpretation in $\boldsymbol{T}$+(Rule-BR_ν) for suitable ν.

PROOF. Theorem 6B follows from Theorem 4A plus the reduction, given in § 5, BR_σ to (Rule-BR_ν) in $\boldsymbol{T}$.

There are two other methods of proving Theorem 6B which bring out some points of interest. By Theorem 6A it is sufficient to prove:

THEOREM 6C. The negative version of (4.1) is provable in $\boldsymbol{H}_\omega^*$+(Rule-BI_ν) for suitable ν.

FIRST PROOF OF THEOREM 6C. It is easily seen that the axiom of choice is provable in $\boldsymbol{H}_\omega^*$. Theorem 6C now follows from the proof, given in § 5, of BI_σ in $\boldsymbol{H}_\omega$ from (Rule-BI_ν) and the axiom of choice.

SECOND PROOF OF THEOREM 6C. This proof is based on the idea (suggested to the author by G. Kreisel) of making the passage from an axiom to a rule in $\boldsymbol{Z}_\omega$. Namely, consider the contrapositive of (4.1):

$$(6.1) \quad \bigwedge F \bigvee n \neg A(n, Fn, F(n+1)) \to \bigvee n \bigvee X \bigwedge Y \neg A(n, X, Y),$$

which is equivalent to (4.1) in $\boldsymbol{Z}_\omega$. By using the trick which Kreisel uses in [6] (Technical notes, III) we can derive (6.1), regarded as an axiom, from the corresponding rule in $\boldsymbol{Z}_\omega$. Namely, let (Rule-6.1) denote the rule which says that if the premise of (6.1) has been proved then the conclusion may be inferred. Take $\neg A_1(n, X, Y)$ to be $\bigwedge F \bigvee n \neg A(n, Fn, F(n+1)) \to \neg A(n, X, Y)$. Then $\bigwedge F \bigvee n \neg A_1(n, Fn, F(n+1))$ is easily proved in $\boldsymbol{Z}_\omega$. Hence by (Rule-6.1) we obtain $\bigvee n \bigvee X \bigwedge Y \neg A_1(n, X, Y)$, which implies (6.1).

It is easy to eliminate a free variable Z from the formula A in (Rule-6.1). Namely, suppose $\bigwedge F \bigvee n \neg A(n, Fn, F(n+1), Z)$ has been proved. Then $\bigwedge Z \bigwedge F \bigvee n \neg A(n, Fn, F(n+1), Z)$ is provable. Take $\neg B(n, U, X, Z, Y)$ to be $\neg A(n, X, Y)$, where U is a variable of the same type as Z. Then

$$\bigwedge W \bigwedge F \bigvee n \neg B(n, Wn, Fn, W(n+1), F(n+1)).$$

By considering the pair $\langle W, F\rangle$ to be a new functional G such that $Gn = \langle Wn, Fn\rangle$, and applying (Rule-6.1), we infer

$$\bigvee n \bigvee U \bigvee X \bigwedge Z \bigwedge Y \neg B(n, U, X, Y),$$

from which the desired conclusion

$$\bigwedge Z \bigvee n \bigvee X \bigwedge Y \neg A(n, X, Y, Z)$$

follows.

Theorem 6C can now be proved by paralleling the discussion of § 4, after first replacing (4.1) by (6.1) and then replacing the axioms (6.1), $(6.1)^-$, BI_σ and $(BI_\sigma)^-$ by the corresponding rules.

7. Functional interpretation in $\boldsymbol{H}_\omega$+(Rule-BI_ν)+$AC_{\nu\tau}$

The purpose of the present section is to show that if a formula D is a theorem of $\boldsymbol{H}^*_\omega$+(Rule-BI_ν) then Gödel's translation D' can be proved in $\boldsymbol{H}_\omega$+(Rule-BI_ν) with the help of the axiom of choice.

$AC_{\nu\tau}$ $\quad \bigwedge X \bigvee YB(X, Y) \rightarrow \bigvee F \bigwedge XB(X, FX)$

From this and Theorem 6C together with the remarks at the beginning of § 4, we will have a consistency proof of $\boldsymbol{Z}_\omega$+(4.1) relative to $\boldsymbol{H}_\omega$+(Rule-BI_ν)+$AC_{\nu\tau}$.

By Gödel's paper [1] and the remarks made in § 2, the mapping of each formula D into D' sends the axioms and rules of inference of $\boldsymbol{H}^*_\omega$ into theorems and derived rules of inference of $\boldsymbol{T}$ and hence of $\boldsymbol{H}_\omega$ (since $\boldsymbol{H}_\omega$ contains $\boldsymbol{T}$). Thus it remains to show that this mapping sends (Rule-BI_ν) into a derived rule of $\boldsymbol{H}_\omega$+(Rule-BI_ν)+$AC_{\nu\tau}$. Hence suppose (Hyp 1)$'$, $\cdots$, (Hyp 4)$'$ have been proved, the notation being as in § 1. We must show how to infer $Q(\langle\ \rangle)'$ in $\boldsymbol{H}_\omega$+(Rule-BI_ν)+$AC_{\nu\tau}$.

It is easy to verify, for all formulae D and E, that $(D \rightarrow E)' \rightarrow (D' \rightarrow E')$ and $[\bigwedge xE(x)]' \rightarrow \bigwedge x[E(x)]'$ are theorems of $\boldsymbol{H}_\omega$; of course $[\bigvee xE(x)]'$ *is* essentially $\bigvee x[E(x)]'$. Using these facts we conclude, in $\boldsymbol{H}_\omega$,

(7.1) $\quad \bigwedge \alpha \bigvee nP(\bar{\alpha}n)'$

(7.2) $\quad \bigwedge c(\bigwedge m \leqq lh(c))[P(\langle c_0, \cdots, c_{m-1}\rangle) \rightarrow P(c)]'$

(7.3) $\quad \bigwedge c[P(c)' \rightarrow Q(c)']$

(7.4) $\quad \bigwedge c[(\bigwedge uQ(c * u))' \rightarrow Q(c)'].$

Observe that (7.1)—(7.3) are just (Hyp 1)—(Hyp 3) applied to the formulae $P(c)'$ and $Q(c)'$. But (7.4) is not in the proper form of (Hyp 4). However, by use of $AC_{\nu\tau}$ we can prove

$$\bigwedge uQ(c * u)' \rightarrow (\bigwedge uQ(c * u))'$$

with free variable c. From this and (7.4) we get

$$\bigwedge c[\bigwedge uQ(c * u)' \rightarrow Q(c)'],$$

which is (Hyp 4). Hence $Q(\langle\ \rangle)'$ by (Rule-BI_ν).

Appendix

The purpose of this appendix is to give a new, improved proof of Theorem (ii) of Howard and Kreisel [2], page **351**, which says that

(1) $\bigwedge n \bigwedge X \bigvee YA(n, X, Y) \rightarrow \bigvee F \bigwedge nA(n, Fn, F(n+1))$

is derivable from BI_σ in $\boldsymbol{Z}_\omega$, where σ is the type of X and Y. Since (2), below, is equivalent to (1) in $\boldsymbol{Z}_\omega$, and since $\boldsymbol{Z}_\omega$ contains $\boldsymbol{H}_\omega$, it is sufficient to prove:

Theorem 1. In $\boldsymbol{H}_\omega + BI_\sigma$ we can derive

(2) $\bigwedge F \bigvee n \neg A(n, Fn, F(n+1)) \rightarrow \neg \bigwedge n \bigwedge X \bigvee YA(n, X, Y)$.

Proof. We shall reason informally in $\boldsymbol{H}_\omega + BI_\sigma$. For sequences $\langle c_0, \cdots, c_{k-1}\rangle$ with components c_i of type σ, where $k = lh(c)$, define $P(c)$ to be $(\bigvee i < lh(c)) \neg A(i, c_{i-1}, c_i)$, with the understanding that $P(c)$ is false if $lh(c) < 1$. Take $Q(c)$ to be $P(c)$. To prove Theorem 1 it suffices to derive a contradiction from the assumptions

(3) $$\bigwedge F \bigvee n \neg A(n, Fn, F(n+1))$$

and

(4) $$\bigwedge n \bigwedge X \bigvee Y \neg A(n, X, Y)$$

by applying BI_σ to the $P(c)$ and $Q(c)$ just defined. In the notation of § 1: (Hyp 1) follows from (3); (Hyp 2) and (Hyp 3) are automatic since $Q(c)$ is the same as $P(c)$. To verify (Hyp 4), assume $\bigwedge uQ(c * u)$. Thus

(5) $\bigwedge u[(\bigvee i < lh(c)) \neg A(i, c_{i-1}, c_i) \vee \neg A(k, c_{k-1}, u)]$,

where $k = lh(c)$. But, by (4), there exists u such that

$\neg A(k, c_{k-1}, u)$. Taking this value for u in (5), we conclude $(\bigvee i < lh(c)) \neg A(i, c_{i-1}, c_i)$, which is just $Q(c)$.

Thus (Hyp 1), · · ·, (Hyp 4) have been verified. Hence $Q(\langle\ \rangle)$ by BI_σ. But $\neg Q(\langle\ \rangle)$ by the definition of $Q(c)$. Thus we have obtained the desired contradiction.

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(Oblatum 3–1–'68)